Appendix A: Exact Analytical Solutions of Straight Bars and Beams

A1 Introduction

Chapters 4 though 6 contain discussions beam problems using the Bernoulli–Euler beam theory and the Timoshenko beam theory. Having analytical solutions of straight beams for some standard boundary conditions and loads is useful for comparison and verification purposes. In addition, problems involving Betti's and Maxwell's reciprocity theorems require the expressions for the transverse deflections and slopes (or rotations). For easy reference and use, analytical solutions are developed in this appendix.

For linear analysis of beams, the extensional (or axial) deformation is not coupled to the bending deformation. In other words, the displacement u can be determined independent of the transverse deflection w and vice versa. Thus, when a beam is subjected to both axial and bending loads, the beam can be analyzed separately for axial deformation and bending deformation. In the absence of axial forces, beams can be analyzed for bending deflections only. The principle of superposition is valid. The geometry and coordinate system of a typical beam and the sign convention for shear force and bending moment are shown in Fig. A1.1.

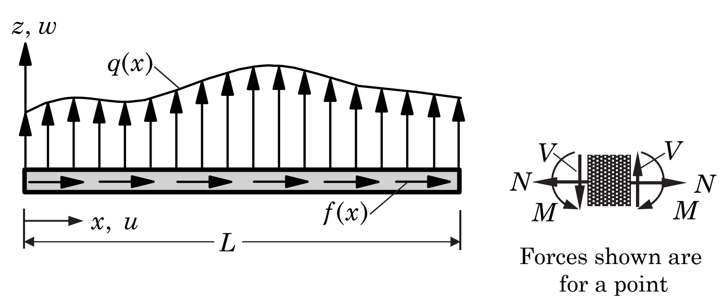


Fig. A.1 The geometry and loads on a beam.

A2 Analytical Solutions of Bars

The axial displacement of a straight beam (or bar) is determined by solving the equation

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x), \quad 0 < x < L. \tag{A2.1}$$

Integration of the equation gives

$$EA\frac{du}{dx} = -\int f(x)\, dx + b_1. \tag{A2.2}$$

If we assume that EA is constant, then the second integration yields

$$EAu(x) = -\int \left(\int f(x)\, dx\right) dx + b_1 x + b_2. \tag{A2.3}$$

The constants of integration, b_1 and b_2, are determined using boundary conditions on u (geometric boundary condition) or on $N = EA(du/dx)$ (force boundary condition).

For example, when $f = 0$ and the bar (with constant EA) is fixed at $x = 0$ and subjected to a tensile point load P at $x = L$, we have $b_2 = 0$ and $b_1 = P$. Hence, the solution for constant EA becomes

$$u(x) = \frac{Px}{EA}. \tag{A2.4}$$

When $f = f_0$, a constant, and the bar is subjected to a point load P at $x = L$, we have

$$EAu(x) = -f_0\frac{x^2}{2} + b_1 x + b_2. \tag{A2.5}$$

Then the constants of integration become: $b_2 = 0$ and $b_1 = f_1 = P + f_0 L$. The solution in this case is

$$u(x) = \frac{P + f_0 L}{EA}\, x - f_0\frac{x^2}{2EA}, \tag{A2.6}$$

which contains the solution in Eq. (A2.4) as a subset.

Example A2.1:

Develop the analytical solution for the composite bar shown in Fig. A.2. Use the following data:

$$\begin{aligned} &E_s = 30\times 10^6 \text{ psi},\ A_s = (a_1 + a_2 x)^2,\ a_1 = 1.5,\ a_2 = -1/192,\ h_1 = 96 \text{ in},\\ &E_a = 10\times 10^6 \text{ psi},\ A_a = 1 \text{ in}^2,\ L = 216 \text{ in},\ P_0 = 10,000 \text{ lb} \end{aligned} \tag{1}$$

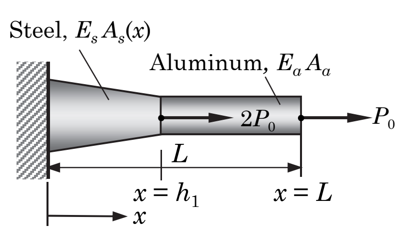


Fig. A.2 The bar discussed in Example A2.1.

Solution: The governing equations are given by

$$-\frac{d}{dx}\big(E_sA_s\frac{du_s}{dx}\big) = 0, \quad 0 < x < 96 \tag{2}$$

$$-\frac{d}{dx}\big(E_aA_a\frac{du_a}{dx}\big) = 0, \quad 96 < x < 216 \tag{3}$$

where the subscript 's' refers to steel and 'a' to aluminum. The solutions of Eqs. (2) and (3) are

$$E_sA_s\frac{du_s}{dx} = c_1, \quad 0 < x < 96 \tag{4}$$

$$E_aA_a\frac{du_a}{dx} = c_3, \quad 96 < x < 216 \tag{5}$$

$$u_s(x) = -\frac{c_1}{a_2}\frac{1}{E_s(a_1 + a_2x)} + c_2, \quad 0 < x < 96 \tag{6}$$

$$u_a(x) = c_3\frac{x}{E_aA_a} + c_4, \quad 96 < x < 216 \tag{7}$$

The constants of integration, c_1, c_2, c_3, and c_4 are determined subject to the boundary conditions

$$u_s(0) = 0, \quad \left(E_sA_s\frac{du_s}{dx}\right)_{x=96^-} - \left(E_aA_a\frac{du_a}{dx}\right)_{x=96^+} = 2P_0$$

$$u_s(96) = u_a(96), \quad \left(E_aA_a\frac{du_a}{dx}\right)_{x=216} = P_0. \tag{8}$$

We obtain

$$c_1 = 3P_0, \quad c_2 = -0.128 \times 10^{-4}P_0, \quad c_3 = P_0, \quad c_4 = -0.32 \times 10^{-5}P_0 \tag{9}$$

Hence, the solution is given by

$$u(x) = \begin{cases} \left(\frac{576}{156250}\right)\left(\frac{1}{288-x}\right)P_0 - 0.128 \times 10^{-4}P_0, & 0 \le x \le 96 \\ (x-32)10^{-7}P_0, & 96 \le x \le 216 \end{cases} \tag{10}$$

A3 Analytical Solutions of Bernoulli–Euler Beams

A3.1 General Solution

For linear static bending analysis in the absence of axial force f, the governing equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q \tag{A3.1}$$

Equation (A3.1) can be integrated, given the distributed load $q(x)$, to obtain $w(x)$. We have

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = \int^x q(\xi)\,d\xi + c_1 = -V(x), \tag{A3.2}$$

$$EI\frac{d^2w}{dx^2} = \int^x\int^\xi q(\eta)\,d\eta\,d\xi + c_1x + c_2 = -M(x). \tag{A3.3}$$

If EI is constant, then

$$EI\frac{dw}{dx} = \int^x \int^\xi \int^\eta q(\zeta)\, d\zeta\ d\eta\ d\xi + c_1\frac{x^2}{2} + c_2 x + c_3 \tag{A3.4}$$

$$EIw(x) = \int^x \int^\xi \int^\eta \int^\zeta q(\mu)\, d\mu\ d\zeta\ d\eta\ d\xi + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4, \tag{A3.5}$$

where c_1 through c_4 are constants of integration to be determined using the boundary conditions (see Table A3.1).

Table A3.1: Conventional boundary conditions for the Bernoulli–Euler beam theory.

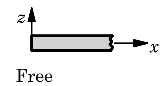
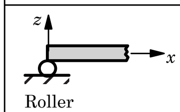
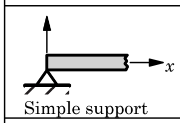
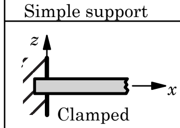
Type of support	Geometric B.C.	Force B.C.
 Free	None	Must know: bending moment M and shear force V
 Roller	$w = 0$	Must know: bending moment M
 Simple support	$w = 0$ $(u = 0)$	Must know: bending moment M
 Clamped	$w = 0$ $\frac{dw}{dx} = 0$ $(u = 0)$	None

Fig. A.3

When a beam is subjected to a boundary condition at a point intermediate to the ends, then the solutions in Eqs. (A3.2) and (A3.3) are valid for each segment. The number of constants in such cases will be equal to $4n$ where n is the number of segments. The additional conditions at the interface of any two segments of the beam are $[\theta_x = -(dw/dx)]$ are provided by the continuity of the deflection and slope and balance of shear force and bending moment.

$$w_1 = w_2, \quad \theta_{x1} = \theta_{x2}, \quad V_1 = V_2,, \quad M_1 = M_2, \tag{A3.6}$$

where the subscripts '1' and '2' refer the segment number (see Fig. A3.1).

A3.2 Examples

Here we present two examples that illustrate the procedure of obtaining analytical solutions to beams with constant EI. In both examples, vertical linear elastic support is considered so that the solutions obtained are valid for two different problems: one without the spring support and another with rigid support. Also, the second example is designed to illustrate how beams with discontinuous data (i.e., load or geometry are not continuous throughout the span of the beam).

Example A3.1:

Determine the expressions for the deflection, slope, bending moment, and shear force for a beam simply-supported at both ends and subjected to uniformly distributed load of intensity q_0.

Solution: For this problem, EI and $q = q_0$ are constant. Hence, Eqs. (A3.2)–(A3.5) become

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = q_0 x + c_1 = -V(x), \tag{1}$$

$$EI\frac{d^2w}{dx^2} = \frac{q_0 x^2}{2} + c_1 x + c_2 = -M(x). \tag{2}$$

$$EI\frac{dw}{dx} = \frac{q_0 x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \tag{3}$$

$$EIw(x) = \frac{q_0 x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4. \tag{4}$$

The boundary conditions of the beam are

$$w(0) = 0, \quad M(0) = 0, \quad w(L) = 0, \quad M(L) = 0. \tag{5}$$

One can also use the half beam because of the symmetry about $x = 0.5L$ and use the boundary conditions

$$w(0) = 0, \quad M(0) = 0, \quad \theta_x(0.5L) = 0, \quad V(0.5L) = 0. \tag{6}$$

Using the first two boundary conditions from Eq. (5) in Eqs. (2) and (4) gives $c_2 = c_4 = 0$. The last two boundary conmditions in Eq. (5) give

$$\frac{q_0 L^3}{24} + c_1\frac{L^2}{6} + c_3 = 0, \quad c_1 = -\frac{q_0 L}{2}. \tag{7}$$

Solving for c_3 we obtain

$$c_3 = -\frac{q_0 L^3}{24} + \frac{q_0 L^3}{12} = \frac{q_0 L^3}{24}. \tag{8}$$

Thus, we have $[\theta_x = -(dw/dx)]$

$$\begin{aligned}
w(x) &= \frac{q_0 L^4}{24EI}\left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right),\\
\theta_x(x) &= -\frac{q_0 L^3}{24EI}\left(1 - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3}\right),\\
M(x) &= \frac{q_0 L^2}{2}\left(\frac{x}{L} - \frac{x^2}{L^2}\right),\\
V(x) &= \frac{q_0 L}{2}\left(1 - 2\frac{x}{L}\right).
\end{aligned} \tag{9}$$

Example A3.2:

Determine the expressions for the deflection, slope, bending moment, and shear force for a beam clamped at both ends and subjected to uniformly distributed load of intensity q_0 and point load F_0 (upward) at the center.

Solution: For this problem, EI is a constant and $q = q_0 + F_0\delta(x - 0.5L)$, where $\delta(\cdot)$ is the Diract delta function. Hence, Eqs. (A3.2)–(A3.5) become

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = q_0x + F_0\,\delta(x-0.5L) + c_1 = -V(x), \tag{1}$$

$$EI\frac{d^2w}{dx^2} = \frac{q_0x^2}{2} + F_0\langle x-0.5L\rangle + c_1x + c_2 = -M(x). \tag{2}$$

$$EI\frac{dw}{dx} = \frac{q_0x^3}{6} + \frac{F_0}{2}\langle x-0.5L\rangle^2 + c_1\frac{x^2}{2} + c_2x + c_3 \tag{3}$$

$$EIw(x) = \frac{q_0x^4}{24} + \frac{F_0}{6}\langle x-0.5L\rangle^3 + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4. \tag{4}$$

Here the bracket $\langle\ \rangle$ has a special meaning. If the bracketed quantity is negative, the function is zero; but once the bracketed expression becomes positive or zero the brackets are the usual parentheses. The boundary conditions are $[\theta_x = -(dw/dx)]$

$$w(0) = 0, \quad \theta_x(0) = 0, \quad w(L) = 0, \quad \theta_x(L) = 0. \tag{5}$$

Using the first two boundary conditions in Eqs. (3) and (4), we obtain $c_3 = c_4 = 0$. Using the boundary condition $\theta_x(L) = 0$ in Eq. (3), we obtain

$$\frac{q_0L^3}{6} + \frac{F_0L^2}{8} + c_1\frac{L^2}{2} + c_2L = 0. \tag{6}$$

The use of $w(L) = 0$ in Eq. (4) gives

$$\frac{q_0L^4}{24} + \frac{F_0L^3}{48} + c_1\frac{L^3}{6} + c_2\frac{L^2}{2} = 0. \tag{7}$$

The solution to these two equations is

$$c_1 = -\frac{q_0L}{2} - \frac{F_0}{2}, \quad c_2 = \frac{q_0L^2}{12} + \frac{F_0L}{8}. \tag{8}$$

Thus, we have

$$\begin{aligned}
w(x) &= \frac{q_0L^4}{24EI}\frac{x^2}{L^2}\left(1-\frac{x}{L}\right)^2 + \frac{F_0L^3}{48}\left(3\frac{x^2}{L^2} - 4\frac{x^3}{L^3} + 8\langle x-0.5L\rangle^3\right), \\
\theta_x(x) &= -\frac{q_0L^3}{24EI}\left(2\frac{x}{L} - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3}\right) - \frac{F_0L^2}{8}\left(\frac{x}{L} - 2\frac{x^2}{L^2} + 4\langle x-0.5L\rangle^2\right), \\
M(x) &= -\frac{q_0L^2}{24}\left(1 - 12\frac{x}{L} + 12\frac{x^2}{L^2}\right) - \frac{F_0L}{8}\left(1 - 4\frac{x}{L} + 8\langle x-0.5L\rangle\right), \\
V(x) &= \frac{q_0L}{2}\left(1-2\frac{x}{L}\right) + \frac{F_0L}{2}\left[1 - 2\delta(x-0.5L)\right],
\end{aligned} \tag{8}$$

Example A3.3:

Determine the expressions for the deflection, slope, bending moment, and shear force for the beam shown in Fig. A.4(a). Specialize the results to cases (a) $k = 0$ [cantilever beam with uniformly distributed load; Fig. A.4(b)] and (b) $k \to \infty$ [a beam clamped at the left end and simply-supported at the right end and with uniformly distributed load; Fig. A.4(c)].

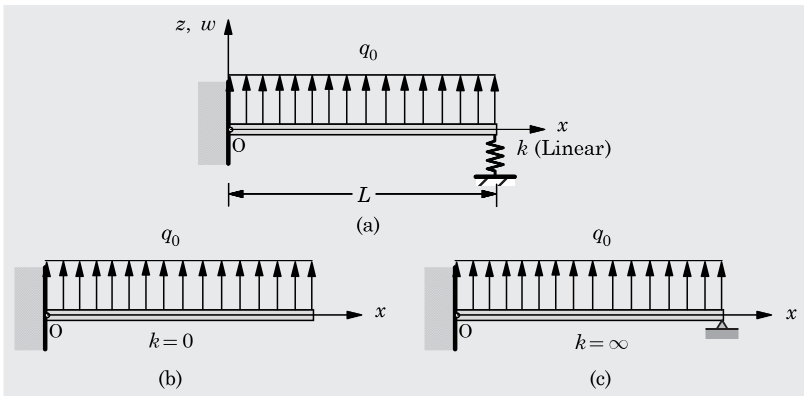


Fig. A.4 The beam discussed in Example A3.1.

Solution: For this problem, EI is a constant and $q = q_0$ is a constant. Hence, Eqs. (A3.2)–(A3.5) become

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = q_0 x + c_1 = -V(x), \tag{1}$$

$$EI\frac{d^2w}{dx^2} = q_0\frac{x^2}{2} + c_1 x + c_2 = -M(x). \tag{2}$$

$$EI\frac{dw}{dx} = \frac{q_0 x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \tag{3}$$

$$EIw(x) = \frac{q_0 x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4. \tag{4}$$

The boundary conditions of the beam are

$$w(0) = 0, \quad \left.\frac{dw}{dx}\right|_{x=0} = 0, \quad M(L) = 0, \quad V(L) = -F_s = -kw(L). \tag{5}$$

From Eqs. (3) and (4), we have $c_3 = c_4 = 0$. Use of the remaining boundary conditions in Eqs. (1) and (2) give

$$q_0 L + c_1 = kw(L), \quad \frac{q_0 L^2}{2} + c_1 L + c_2 = 0 \ \Rightarrow\ c_1 = kw(L) - q_0 L \text{ and } c_2 = -kLw(L) + q_0\frac{L^2}{2}.$$

Hence, the deflection becomes

$$w(x) = \frac{1}{EI}\left[\frac{q_0 x^4}{24} + (kw(L) - q_0 L)\frac{x^3}{6} + \left(-kLw(L) + \frac{q_0 L^2}{2}\right)\frac{x^2}{2}\right] \tag{6}$$

Evaluating at $x = L$, we find that

$$\left(1+\frac{kL^3}{3EI}\right)w(L)=\frac{q_0L^4}{8EI}\quad\text{or}\quad w(L)=\frac{q_0L^4}{8EI}\left(1+\frac{kL^3}{3EI}\right)^{-1}. \tag{7}$$

Finally, we have

$$w(x)=\frac{q_0x^4}{24EI}-\frac{q_0Lx^3}{6EI}+\frac{q_0L^2x^2}{4EI}+\frac{k}{6EI}\left(x^3-3Lx^2\right)\frac{q_0L^4}{8EI}\left(1+\frac{kL^3}{3EI}\right)^{-1} \tag{8}$$

or

$$w(x)=\frac{q_0x^4}{24EI}-\frac{q_0Lx^3}{6EI}+\frac{q_0L^2x^2}{4EI}+\left(x^3-3Lx^2\right)\frac{q_0L^4}{48EI}\left(\frac{EI}{k}+\frac{L^3}{3}\right)^{-1} \tag{9}$$

The slope is given by

$$\frac{dw}{dx}=\frac{q_0x^3}{6EI}-\frac{q_0Lx^2}{2EI}+\frac{q_0L^2x}{2EI}+\left(x^2-2Lx\right)\frac{q_0L^4}{16EI}\left(\frac{EI}{k}+\frac{L^3}{3}\right)^{-1} \tag{10}$$

(a) **Case**, $k = 0$ [see Fig. A.4(b)]. For this case, we obtain from Eqs. (8) and (10)

$$w=\frac{q_0x^4}{24EI}-\frac{q_0Lx^3}{6EI}+\frac{q_0L^2x^2}{4EI}=\frac{q_0x^2}{24EI}\left(6L^2-4Lx+x^2\right), \tag{11}$$

$$\frac{dw}{dx}=\frac{q_0x}{6EI}\left(3L^2-3Lx+x^2\right). \tag{12}$$

$$M=-\frac{q_0}{2}\left(L^2-2Lx+x^2\right)=-\frac{q_0}{2}\left(L-x\right)^2. \tag{13}$$

$$V=q_0(L-x). \tag{14}$$

(b) **Case**, $k = \infty$ [see Fig. A.4(c)]. For this case, we obtain from Eqs. (9) and (10) the results

$$\begin{aligned} w &= \frac{q_0x^4}{24EI}-\frac{q_0Lx^3}{6EI}+\frac{q_0L^2x^2}{4EI}+\left(x^3-3Lx^2\right)\frac{q_0L}{16EI}\\ &= \frac{q_0x^4}{24EI}-\frac{5q_0Lx^3}{48EI}+\frac{q_0L^2x^2}{16EI}=\frac{q_0x^2}{48EI}\left(3L^2-5Lx+2x^2\right), \end{aligned} \tag{15}$$

$$\frac{dw}{dx}=\frac{q_0x}{48EI}\left(2L^2-15Lx+8x^2\right). \tag{16}$$

$$M=-\frac{q_0}{24EI}\left(L^2-15Lx+12x^2\right). \tag{17}$$

$$V=-\frac{q_0}{8EI}\left(5L+8x\right). \tag{18}$$

Example A3.4:

Determine the expressions for the deflection, slope, bending moment, and shear force for the beam shown in Fig. A.5(a). Specialize the results to cases (a) $k = 0$ [cantilever beam; Fig. A.5(b)] and (b) $k \to \infty$ [a beam clamped at the left end and simply-supported at the right end; Fig. A.5(c)].

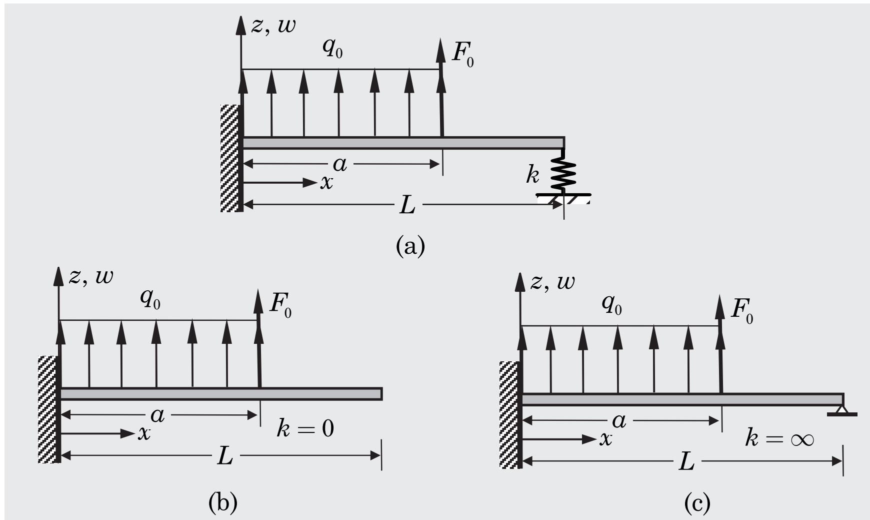


Fig. A.5 The beam discussed in Example A3.2.

Solution: There are two segments of the beam: $0 \leq x \leq a$ and $a \leq x \leq L$. The solutions in each segment are given by

$$\frac{d}{dx}\left(EI\frac{d^2w_1}{dx^2}\right) = q_0 x + c_1 = -V_1(x), \tag{1}$$

$$EI\frac{d^2w_1}{dx^2} = q_0\frac{x^2}{2} + c_1 x + c_2 = -M_1(x). \tag{2}$$

$$EI\frac{dw_1}{dx} = q_0\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \tag{3}$$

$$EIw_1(x) = q_0\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4. \tag{4}$$

and

$$\frac{d}{dx}\left(EI\frac{d^2w_2}{dx^2}\right) = d_1 = -V_2(x), \tag{5}$$

$$EI\frac{d^2w_2}{dx^2} = d_1 x + d_2 = -M_2(x). \tag{6}$$

$$EI\frac{dw_2}{dx} = d_1\frac{x^2}{2} + d_2 x + d_3 \tag{7}$$

$$EIw_2(x) = d_1\frac{x^3}{6} + d_2\frac{x^2}{2} + d_3 x + d_4. \tag{8}$$

The boundary conditions are

$$w_1(0) = 0, \quad \left.\frac{dw_1}{dx}\right|_{x=0} = 0, \quad V_2(L) = -kw_2(L), \quad M_2(L) = 0. \tag{9}$$

The first two boundary conditions give $c_3 = c_4 = 0$. The last two boundary conditions give

$$d_1 = kw_2(L), \quad d_1 L + d_2 = 0 \;\rightarrow\; d_2 = -kLw_2(L). \tag{10}$$

We need to determine c_1, c_2, d_3, and d_4. The continuity of the deflection, $w_1(a) = w_2(a)$, gives

$$q_0\frac{a^4}{24} + c_1\frac{a^3}{6} + c_2\frac{a^2}{2} = d_1\frac{a^3}{6} + d_2\frac{a^2}{2} + d_3 a + d_4. \tag{11}$$

The continuity of the slope, $(dw_1/dx)(a) = (dw_2/dx)(a)$, gives

$$q_0\frac{a^3}{6} + c_1\frac{a^2}{2} + c_2 a = d_1\frac{a^2}{2} + d_2 a + d_3. \tag{12}$$

The balance of shear force and bending moment at $x = a$, $V_1(a) = V_2(a) + F_0$ and $M_1(a) = M_2(a)$, gives

$$-q_0 a - c_1 = -d_1 + F_0, \quad q_0\frac{a^2}{2} + c_1 a + c_2 = d_1 a + d_2. \tag{13}$$

Solving Eqs. (10)–(13), we obtain

$$\begin{aligned} &c_1 = kw_2(L) - F_0 - q_0 a, \quad c_2 = F_0 a + q_0\frac{a^2}{2} - kLw_2(L), \quad d_1 = kw_2(L), \\ &d_2 = -kLw_2(L), \quad d_3 = \frac{F_0 a^2}{2} + \frac{q_0 a^3}{6}, \quad d_4 = -\frac{F_0 a^3}{6} - \frac{q_0 a^4}{24}, \\ &w_2(L) = . \left[\frac{F_0 a^2}{6EI}(3L - a) + \frac{q_0 a^3}{24}(4L - a)\right]\left(1 + \frac{kL^3}{3EI}\right)^{-1}. \end{aligned} \tag{14}$$

The deflections, slopes, moments, and shear forces in the two segments of the beam are

$$w = \begin{cases} \frac{x^2}{24EI}\left[4(3a - x)F_0 + q_0(x^2 - 4ax + 6a^2) - 4(3L - x)kw_2(L)\right], & 0 \le x \le a, \\ \frac{1}{24EI}\left[4(3x - a)a^2F_0 + q_0 a^3(4x - a) - 4(3L - x)x^2kw_2(L)\right], & a \le x \le L, \end{cases} \tag{15}$$

$$\frac{dw}{dx} = \begin{cases} \frac{x}{6EI}\left[3(2a - x)F_0 + q_0(3a^2 - 3ax + x^2) - 3(2L - x)kw_2(L)\right], & 0 \le x \le a, \\ \frac{1}{6EI}\left[3a^2F_0 + q_0 a^3 + 3(x - 2L)xkw_2(L)\right], & a \le x \le L, \end{cases} \tag{16}$$

$$M = \begin{cases} -F_0(a - x) - \frac{q_0}{2}(a - x)^2 + (L - x)kw_2(L), & 0 \le x \le a, \\ (L - x)kw_2(L), & a \le x \le L, \end{cases} \tag{17}$$

$$V = \begin{cases} F_0 + q_0(a - x) - kw_2(L), & 0 \le x \le a, \\ -kw_2(L), & a \le x \le L, \end{cases} \tag{18}$$

When $k = 0$, we obtain the solution for a beam clamped at the left end and free at the right end, and subjected to point load F_0 at a distance $x = a$ from the left end and uniformly distributed load of intensity q_0 over the span $0 \le x \le a$. For $k = \infty$, set $kw_2(L)$ to $(F_0 a^2/2L^3)(3L - a)$ in the above solution.

A4 Analytical Solutions of Timoshenko Beams

A4.1 General Solution

The governing equations for the Timoshenko beam theory are the same as for the Bernoulli–Euler beam theory and they were presented in the text book at several places [see Eqs. (7.4.3) and (7.4.4)]

$$-\frac{dV}{dx} = q, \quad -\frac{dM}{dx} + V = 0, \tag{A4.1}$$

but M and V are related to the generalized displacements (w, ϕ_x) by

$$M = EI\frac{d\phi_x}{dx}, \quad V = GAK_s\left(\phi_x + \frac{dw}{dx}\right). \tag{A4.2}$$

Substitution of Eq. (A4.2) in to Eq. (A4.1), we obtain

$$\begin{aligned} -\frac{d}{dx}\left(EI\frac{d\phi_x}{dx}\right) + GAK_s\left(\phi_x + \frac{dw}{dx}\right) &= 0 \\ -\frac{d}{dx}\left[GAK_s\left(\phi_x + \frac{dw}{dx}\right)\right] &= q \end{aligned} \tag{A4.3}$$

Now suppose that EI and GAK_s are constant. Then we have from the second equation in (A4.3)

$$GAK_s\left(\phi_x + \frac{dw}{dx}\right) = -\int q\,dx + c_1 = V(x). \tag{A4.4}$$

Substituting the result into the first equation of (A4.3) and integrating

$$EI\frac{d\phi_x}{dx} = -\int\int q\,dxdx + c_1 x + c_2 = M(x). \tag{A4.5}$$

Integrating the last expression, we arrive at

$$EI\phi_x(x) = -\int\int\int q\,dxdxdx + c_1\frac{x^2}{2} + c_2 x + c_3. \tag{A4.6}$$

Substituting for $\phi(x)$ from Eq. (A4.6) into Eq. (A4.4) and solving for $w(x)$, we obtain

$$EIw(x) = \Omega L^2\left(-\int\int q\,dxdx + c_1 x\right) - \left(-\int\int\int\int q\,dxdxdxdx + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4\right). \tag{A4.7}$$

where c_1 through c_4 are the constants of integration and

$$\Omega = \frac{EI}{GAK_s L^2}. \tag{A4.8}$$

Example A4.1:

Use the Timoshenko beam theory to determine the expressions for the deflection, slope, bending moment, and shear force for a beam simply-supported at both ends and subjected to uniformly distributed load of intensity q_0.

Solution: For this problem, EI and $q = q_0$ are constant. Hence, Eqs. (A4.4)–(A4.7) become

$$GAK_s\left(\phi_x + \frac{dw}{dx}\right) = -q_0 x + c_1 = V(x), \tag{1}$$

$$EI\frac{d\phi_x}{dx} = -\frac{q_0 x^2}{2} + c_1 x + c_2 = M(x), \tag{2}$$

$$EI\phi_x(x) = -\frac{q_0 x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3, \tag{3}$$

$$EIw(x) = \Omega L^2\left(-\frac{q_0 x^2}{2} + c_1 x\right) - \left(-\frac{q_0 x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4\right). \tag{4}$$

The boundary conditions of the beam are

$$w(0) = 0, \quad M(0) = 0, \quad w(L) = 0, \quad M(L) = 0. \tag{5}$$

Using the first two boundary conditions from Eq. (5) in (2) and (4) gives $c_2 = c_4 = 0$. The last two boundary conditions give

$$\Omega L^2 \left(-\frac{q_0 L^2}{2} + c_1 L \right) - \left(-\frac{q_0 L^4}{24} + c_1 \frac{L^3}{6} + c_3 L \right) = 0, \quad c_1 = \frac{q_0 L}{2}. \tag{7}$$

Solving for c_2, we obtain

$$c_3 = -\frac{q_0 L^3}{24}. \tag{8}$$

Thus, we have

$$\begin{aligned}
w(x) &= \frac{q_0 L^4}{24EI} \left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4} \right) + \frac{q_0 L^2}{24GAK_s} \left(\frac{x}{L} - \frac{x^2}{L^2} \right), \\
\phi_x(x) &= -\frac{q_0 L^3}{24EI} \left(1 - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3} \right), \\
M(x) &= \frac{q_0 L^2}{2} \left(\frac{x}{L} - \frac{x^2}{L^2} \right), \\
V(x) &= \frac{q_0 L}{2} \left(1 - 2\frac{x}{L} \right).
\end{aligned} \tag{10}$$

This is the same result obtained using the relationships between EBT and TBT in **Example 7.2.1**.

Example A4.2:

Use the Timoshenko beam theory to determine the expressions for the deflection, slope, bending moment, and shear force for a beam clamped at both ends and subjected to uniformly distributed load of intensity q_0.

Solution: For this problem, EI is a constant and $q = q_0$. We have

$$GAK_s \left(\phi_x + \frac{dw}{dx} \right) = -q_0 x + c_1 = V(x), \tag{1}$$

$$EI \frac{d\phi_x}{dx} = -\frac{q_0 x^2}{2} + c_1 x + c_2 = M(x), \tag{2}$$

$$EI \phi_x(x) = -\frac{q_0 x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3, \tag{3}$$

$$EIw(x) = \Omega L^2 \left(-\frac{q_0 x^2}{2} + c_1 x \right) - \left(-\frac{q_0 x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4 \right). \tag{4}$$

The boundary conditions are

$$w(0) = 0, \quad \phi_x(0) = 0, \quad w(L) = 0 \quad \phi_x(L) = 0. \tag{5}$$

Using the first two boundary conditions in Eqs. (3) and (4), we obtain $c_3 = c_4 = 0$. Using the boundary condition $\phi_x(L) = 0$ in Eq. (3), we obtain

$$-\frac{q_0 L^3}{6} + c_1 \frac{L^2}{2} + c_2 L = 0. \tag{6}$$

The use of $w(L) = 0$ in Eq. (4) gives

$$\Omega L^2 \left(-\frac{q_0 L^2}{2} + c_1 L \right) - \left(-\frac{q_0 L^4}{24} + c_1 \frac{L^3}{6} + c_2 \frac{L^2}{2} \right) = 0. \tag{7}$$

The solution to Eqs. (6) and (7) is

$$c_1 = \frac{q_0 L}{2}, \quad c_2 = -\frac{q_0 L^2}{12}. \tag{8}$$

Thus, we have

$$w(x) = \frac{q_0 L^4}{24EI} \frac{x^2}{L^2} \left(1 - \frac{x}{L} \right)^2 - \frac{1}{K_s GA} \left[\frac{q_0 L^2}{24} \left(1 - 12\frac{x}{L} + 12\frac{x^2}{L^2} \right) \right] + \Omega \frac{q_0 L^4}{24EI}, \tag{9}$$

$$\phi_x(x) = -\frac{q_0 L^3}{24EI} \left(2\frac{x}{L} - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3} \right), \tag{10}$$

$$M(x) = -\frac{q_0 L^2}{24} \left(2 - 12\frac{x}{L} + 12\frac{x^2}{L^2} \right), \tag{11}$$

$$V(x) = \frac{q_0 L}{2} \left(1 - 2\frac{x}{L} \right). \tag{12}$$