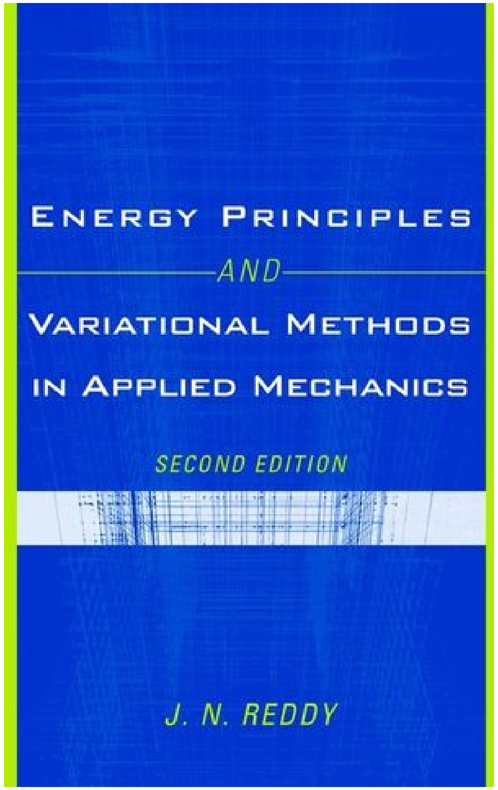
MEEN 618: ENERGY AND VARIATIONAL METHODS

WORK, ENERGY , AND VARIATIONAL CALCULUS

Read: Chapter 4



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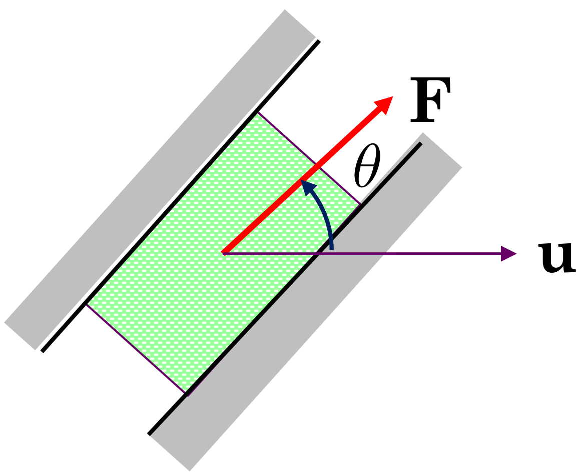
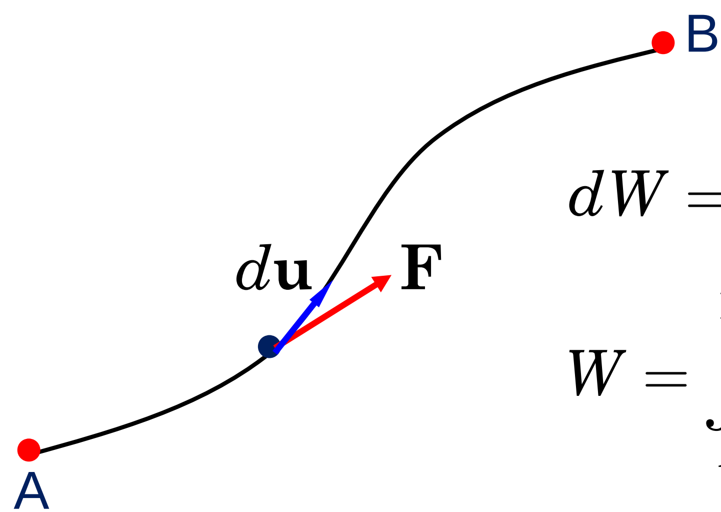
Work done Magnitude of the force multiplied by the magnitude of the displacement in the direction of the force: $W = \mathbf{F} \cdot \mathbf{u}$

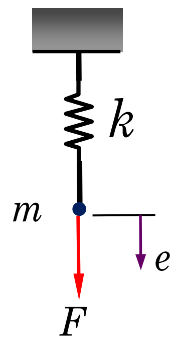


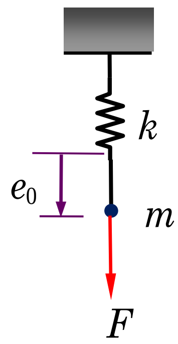


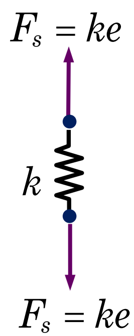
$$dW = \mathbf{F} \cdot d\mathbf{u}$$

$$W = \int_{A}^{B} \mathbf{F} \cdot d\mathbf{u}$$

Energy is the capacity to do work



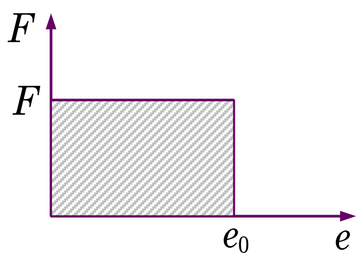




$$dE = F_s \cdot de, \quad F_s = ke$$

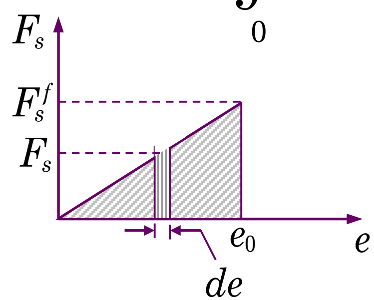
Work done

$$W = Fe_0$$



$$E = \int_0^{e_0} F_s \cdot de = \left[\tfrac{1}{2} k e^2\right]_0^{e_0} = \tfrac{1}{2} k e_0^2$$

(strain energy)

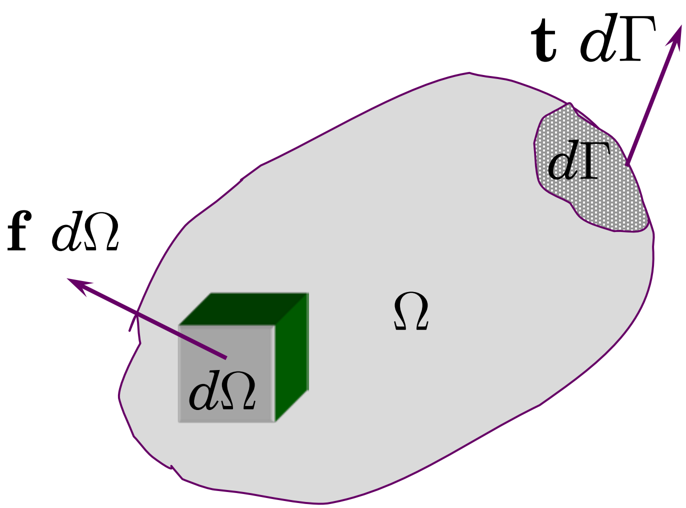


Note that

$$\left.\frac{dE}{de}\right|_{e=e_0} = W$$

EXTERNAL AND INTERNAL WORK IN A DEFORMABLE BODY

Work done by external forces

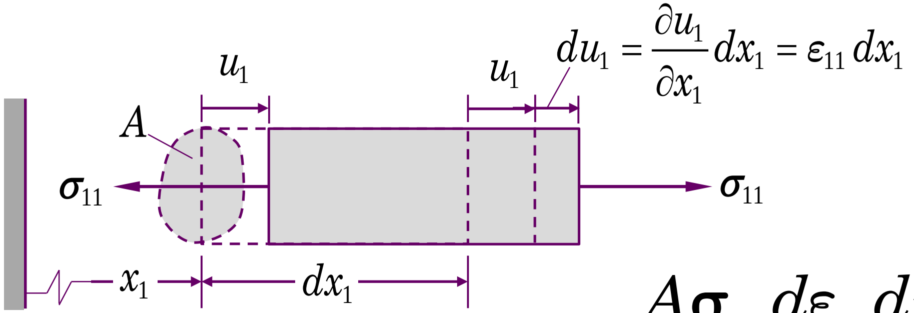


$$W_E = -\left[\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,d\Omega + \oint_\Gamma \mathbf{t}(s)\cdot\mathbf{u}(s)\,d\Gamma\right]$$

In calculating the external work done, the applied (external) forces (or moments) are assumed to be independent of the displacements (or rotations) they cause in a body.

STRAIN ENERGY DENSITY AND STRAIN ENERGY

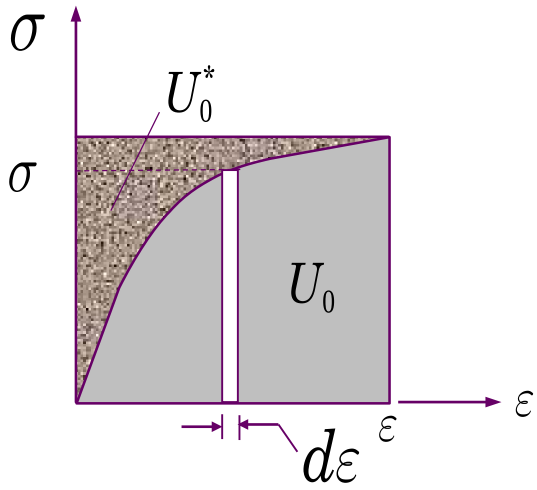
Work done by internal forces (1D)



$$A\boldsymbol{\sigma}_{11} d\varepsilon_{11} dx_1 = \boldsymbol{\sigma}_{11} d\varepsilon_{11} (A dx_1)$$

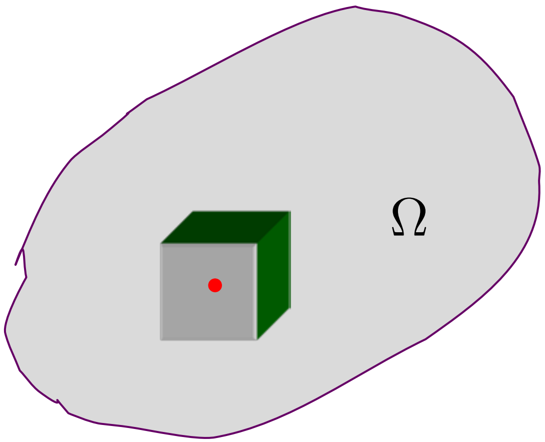
$$\equiv dU_0 (A dx_1)$$

$$U_0(\varepsilon_{11}) = \int_0^{\varepsilon_{11}} dU_0,\; U = \int_\Omega U_0 \, d\Omega$$



STRAIN ENERGY DENSITY AND STRAIN ENERGY

Strain energy of a 3D solid



$$U_0(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}, \quad U = \int_\Omega U_0\, d\Omega$$

$$U = \int_\Omega \left(\int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij} \right) d\Omega$$

For a linear elastic body, we have

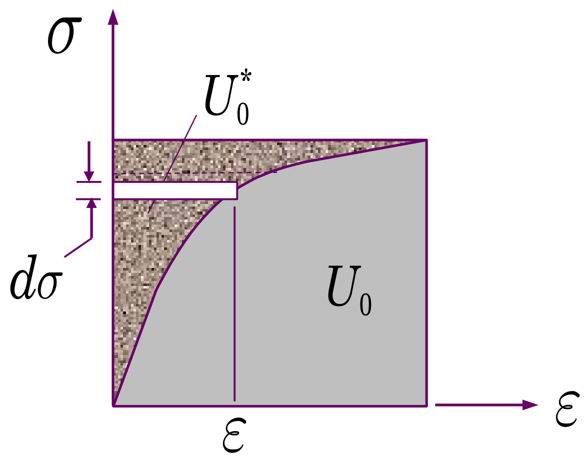
$$U = \tfrac{1}{2} \int_\Omega \boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x})\, d\Omega$$

If the only energy stored in the body is the strain energy, we write

$$U = W_I$$

COMPLEMENTARY STRAIN ENERGY DENSITY AND COMPLEMENTRY STRAIN ENERGY

Complementary strain energy for 1D



$$A\varepsilon_{11}d\sigma_{11}dx_1 = \varepsilon_{11}d\sigma_{11}(Adx_1) \equiv dU_0^*(Adx_1)$$

$$U_0^*(\sigma_{11}) = \int_0^{\sigma_{11}} dU_0^*,\ U^* = \int_\Omega U_0^*\ d\Omega$$

Complementary strain energy for 3D

$$U_0^*(\varepsilon_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{ij}\ d\sigma_{ij},\ U^* = \int_\Omega U_0^*\ d\Omega$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES

A truss is a collection of uniaxial members, each of which can only carry axial force (compressive or tensile). The members are connected through pins that allow relative rotation.

The strain energy and complementary strain energy of a truss with N members each having ist own length, area of cross section, and modulus are

$$U = \sum_{i=1}^{N} \int_{\Omega_i} U_0^{(i)} \, d\Omega = \sum_{i=1}^{N} A_i L_i U_0^{(i)}, \quad U_0^{(i)} = \int_0^{\varepsilon_i} \sigma_i(\varepsilon_i) \, d\varepsilon_i$$

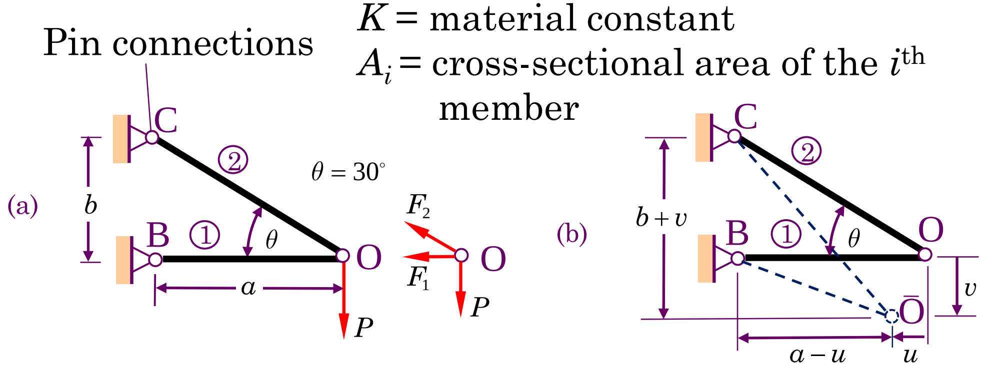
$$U^* = \sum_{i=1}^{N} \int_{\Omega_i} U_0^{*(i)} \, d\Omega = \sum_{i=1}^{N} A_i L_i U_0^{*(i)}, \; U_0^{*(i)} = \int_0^{\sigma_i} \varepsilon_i(\sigma_i) \, d\sigma_i$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

AN EXAMPLE

$$\sigma = \begin{cases} K\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -K\sqrt{-\varepsilon}, & \varepsilon \leq 0 \end{cases}$$

K = material constant
A_i = cross-sectional area of the i^{th} member



Find the (1) strain energy and (2) complementary strain energy of the truss

(1) From Fig. (b), we have the following strains:

$$\varepsilon^{(1)} = \frac{\mathbf{B\bar{O}} - \mathbf{BO}}{\mathbf{BO}} = \left[\frac{(a-u)^2 + v^2}{a^2}\right]^{\frac{1}{2}} - 1 = \left[\frac{a^2 + u^2 + v^2 - 2au}{a^2}\right]^{\frac{1}{2}} - 1 \approx \left(1 - 2\frac{u}{a}\right)^{\frac{1}{2}} - 1 \approx -\frac{u}{a}$$

$$\varepsilon^{(2)} = \frac{\mathbf{C\bar{O}} - \mathbf{CO}}{\mathbf{CO}} = \left[\frac{(a-u)^2 + (b+v)^2}{a^2 + b^2}\right]^{\frac{1}{2}} - 1 = \left[\frac{a^2 + b^2 + u^2 + v^2 + 2(bv - au)}{a^2 + b^2}\right]^{\frac{1}{2}} - 1$$

$$= \left(1 + 2\frac{bv - au}{a^2 + b^2}\right)^{\frac{1}{2}} - 1 \approx \frac{bv - au}{a^2 + b^2} = \frac{\sqrt{3}v - 3u}{4a}$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

(1) continued [note that the stress in member 1 is compressive and it is tensile in member 2; see part (b) to confirm this]

The strain energy densities of each member is

$$U_0^{(1)} = \int_0^{\varepsilon^{(1)}} \sigma^{(1)} d\varepsilon^{(1)} = \int_0^{\varepsilon^{(1)}} \left(-K\sqrt{-\varepsilon^{(1)}}\right) d\varepsilon^{(1)} = \frac{2K}{3}\left(-\varepsilon^{(1)}\right)^{\frac{3}{2}} = \frac{2K}{3}\left(\frac{u^3}{a^3}\right)^{\frac{1}{2}},$$

$$U_0^{(2)} = \int_0^{\varepsilon^{(2)}} \sigma^{(2)}\, d\varepsilon^{(2)} = \int_0^{\varepsilon^{(2)}} \left(K\sqrt{\varepsilon^{(2)}}\right) d\varepsilon^{(2)} = \frac{2K}{3}\left(\varepsilon^{(2)}\right)^{\frac{3}{2}} = \frac{2K}{3}\left(\frac{\sqrt{3}v - 3u}{4a}\right)^{\frac{3}{2}}$$

The total strain energy of the truss is

$$U = \int_{V_1} U_0^{(1)} d\Omega + \int_{V_2} U_0^{(2)} d\Omega = A_1 L_1 U_0^{(1)} + A_2 L_2 U_0^{(2)}$$

$$= \frac{2KA_1 a}{3}\left(\frac{u^3}{a^3}\right)^{\frac{1}{2}} + \frac{4KA_2 a}{3\sqrt{3}}\left(\frac{\sqrt{3}v - 3u}{4a}\right)^{\frac{3}{2}}$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

(2) From Fig. (a), we have

$$F_2 \sin\theta = P,\;\; F_1 + F_2\cos\theta = 0,\;\; \theta = 30^\circ,\;\; F_1 = -\sqrt{3}P,\;\; F_2 = 2P$$

The complementary strain energy densities of each member are

$$U_0^{*(1)} = \int_0^{\sigma^{(1)}} \varepsilon^{(1)} d\sigma^{(1)} = \int_0^{\sigma^{(1)}} \left[-\left(\frac{\sigma^{(1)}}{K}\right)^2\right] d\sigma^{(1)} = -\frac{1}{3K^2}\left(\sigma^{(1)}\right)^3 = \frac{\sqrt{3}P^3}{A_1^3K^2},$$

$$U_0^{*(2)} = \int_0^{\sigma^{(2)}} \varepsilon^{(2)}\, d\sigma^{(2)} = \int_0^{\sigma^{(2)}} \left[\frac{\left(\sigma^{(2)}\right)^2}{K^2}\right] d\sigma^{(2)} = \frac{1}{3K^2}\left(\sigma^{(2)}\right)^3 = \frac{1}{3}\left(\frac{8P^3}{A_2^3K^2}\right)$$

The total complementary strain energy of the truss is

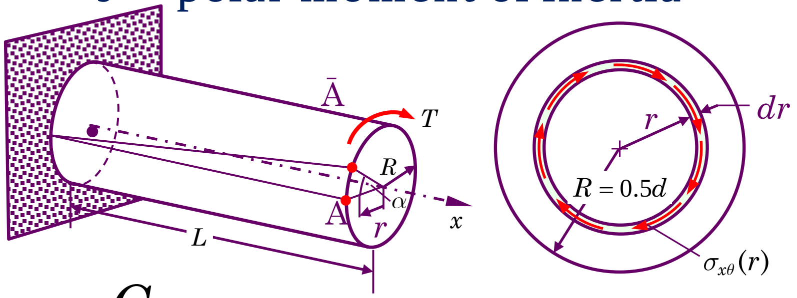
$$U^* = U_0^{*(1)}A_1L_1 + U_0^{*(2)}A_2L_2 = \frac{1}{3}\left[\left(\frac{3\sqrt{3}P^3a}{A_1^2K^2}\right) + \left(\frac{16P^3a}{\sqrt{3}A_2^2K^2}\right)\right]$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TORSION OF CIRCULAR SHAFTS

The shear stress in a shaft subjected to torque

$$\sigma_{x\theta}(r) = \frac{Tr}{J}$$

d = diameter of the shaft
J = polar moment of inertia



The strain energy density and strain energy are

$$\gamma_{x\theta} = 2\varepsilon_{x\theta} = \frac{r\alpha}{L}, \quad \alpha = \frac{TL}{GJ}$$

$$U_0 = 2\int_0^{\varepsilon_{x\theta}} \sigma_{x\theta}\, d\varepsilon_{x\theta} = 4\int_0^{\varepsilon_{x\theta}} G\varepsilon_{x\theta}\, d\varepsilon_{x\theta} = 2G\varepsilon_{x\theta}^2 \equiv \frac{G}{2}\gamma_{x\theta}^2$$

$$U = \int_\Omega \frac{G}{2}\gamma_{x\theta}^2\, d\Omega = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_0^{d/2} G\left(\frac{r\alpha}{L}\right)^2 dr\, rd\theta\, dx = \frac{1}{2}\int_0^L GJ\,\frac{\alpha^2}{L^2} dx$$

The complementary strain energy density is

$$U_0^* = 2\int_0^{\sigma_{x\theta}} \varepsilon_{x\theta}\, d\sigma_{x\theta} = \frac{1}{2G}\sigma_{x\theta}^2$$

$$U^* = \int_v \frac{1}{2G}\sigma_{x\theta}^2\, d\Omega = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_0^{d/2} \frac{1}{G}\left(\frac{Tr}{J}\right)^2 dr\, rd\theta\, dx = \frac{1}{2}\int_0^L \frac{T^2}{GJ} dx$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF STRAIGHT E-B BEAMS

The strain energy density and strain energy of the Euler-Bernoulli (E-B) beam are (linear elastic material)

$$U_0 = \int_0^{\varepsilon_{xx}} \sigma_{xx}\, d\varepsilon_{xx} = \int_0^{\varepsilon_{xx}} E(x)\varepsilon_{xx}\, d\varepsilon_{xx} = \frac{E}{2}\varepsilon_{xx}^2 = \frac{E}{2}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right)^2$$

$$U = \int_v U_0\, d\Omega = \int_0^L \int_A \frac{E}{2}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right)^2 dA dx = \frac{1}{2}\int_0^L \left[A_{xx}\left(\frac{du}{dx}\right)^2 + D_{xx}\left(\frac{d^2w}{dx^2}\right)^2\right] dx$$

$$A_{xx} = \int_A E(x) dA = EA, \quad D_{xx} = \int_A z^2 E(x) dA = EI$$

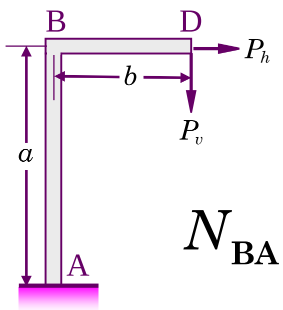
The complementary strain energy density and complementary strain energy of the Euler-Bernoulli beam are

$$U_0^* = \int_0^{\sigma_{xx}} \varepsilon_{xx}\, d\sigma_{xx} + 2\int_0^{\sigma_{xz}} \varepsilon_{xz}\, d\sigma_{xz} = \frac{\sigma_{xx}^2}{2E} + \frac{\sigma_{xz}^2}{2G} = \frac{1}{2E}\left(\frac{N}{A}\right)^2 + \frac{1}{2G}\left(\frac{VQ}{Ib}\right)^2$$

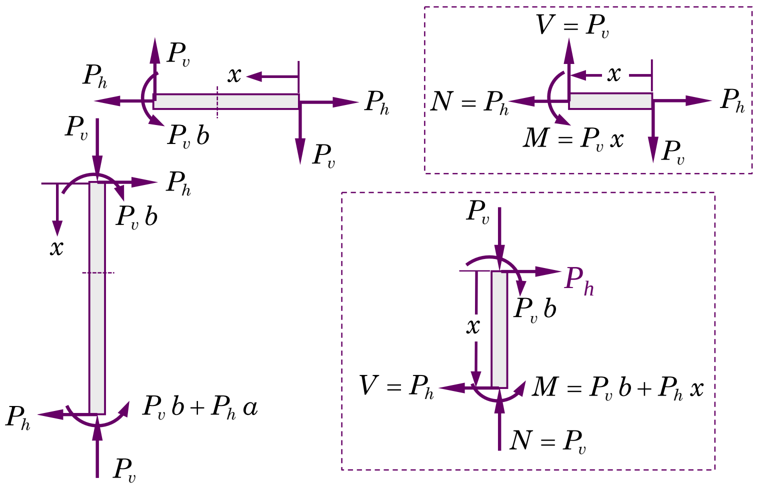
$$U^* = \int_0^L \int_A U_0^* dA dx = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{f_s V^2}{GA}\right) dx, \; f_s = \frac{A}{I^2 b^2}\int_A Q^2(z) dA$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF E-B BEAMS - AN EXAMPLE

AN EXAMPLE: **Determine the complementary strain energy of the (determinate) frame structure.**



All members have the same *E*, *A*, and *I*



$$N_{\mathrm{BA}} = P_v,\ \ V_{\mathrm{BA}} = P_h,$$

$$M_{\mathrm{BA}} = P_v \cdot b + P_h \cdot x$$

$$N_{\mathrm{DB}} = P_h,\ \ V_{\mathrm{DB}} = P_v,\ \ M_{\mathrm{DB}} = P_v$$

$$U^*_{\mathrm{DB}} = \int_0^b \left[\frac{P_h^2}{2EA} + \frac{1}{2EI}(P_v x)^2 + \frac{f_s P_v^2}{2GA}\right] dx = \frac{P_h^2 b}{2EA} + \frac{P_v^2 b^3}{6EI} + \frac{f_s P_v^2 b}{2GA},$$

$$U^*_{\mathrm{BA}} = \int_0^a \left[\frac{P_v^2}{2EA} + \frac{1}{2EI}(P_v b + P_h x)^2 + \frac{f_s P_h^2}{2GA}\right] dx$$

$$= \frac{P_v^2 a}{2EA} + \frac{1}{2EI}\left(P_v^2 a b^2 + P_v P_h a^2 b + \frac{1}{3} P_h^2 a^3\right) + \frac{f_s P_h^2 a}{2GA}$$

TOTAL POTENTIAL ENERGY AND COMPLEMENTRY ENERGY

The potential energy of a 3D solid $(W_I = U,\ W_E = V_E)$

$$\Pi(\mathbf{u}) = U + V_E = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon}\, d\Omega - \left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\Omega + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, d\Gamma\right]$$

The complementary energy of a 3D solid

$$\Pi^*(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma})\, d\Omega - \left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\Omega + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, d\Gamma\right]$$

For an E-B beam, they are

$$\Pi(u,w) = \frac{1}{2}\int_0^L \left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2w}{dx^2}\right)^2\right]dx - \int_0^L (fu + qw)\,dx + \ldots$$

$$\Pi^*(\sigma_{xx},\sigma_{xz}) = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{f_s V^2}{GA}\right)dx - \int_0^L (fu + qw)\,dx + \ldots$$

THE PRINCIPLE OF MINIMUM TOTAL POTENTIAL ENERGY

Minimum nature of PE:

$$\Pi(\bar{u},\bar{w}) = \frac{1}{2}\int_0^L \left[EA\left(\frac{d\bar{u}}{dx}\right)^2 + EI\left(\frac{d^2\bar{w}}{dx^2}\right)^2 \right] dx - \int_0^L (f\bar{u} + q\bar{w})\, dx$$

$$= \frac{1}{2}\int_0^L \left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2 w}{dx^2}\right)^2 \right] dx - \int_0^L (fu + qw)\, dx$$

$$+ \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2 w_0}{dx^2}\right)^2 \right] dx - \int_0^L (\alpha f u_0 + \beta q w_0)\, dx$$

$$+ \int_0^L \left[\alpha EA\left(\frac{du}{dx}\frac{du_0}{dx}\right) + \beta EI\left(\frac{d^2 w}{dx^2}\frac{d^2 w_0}{dx^2}\right) \right] dx$$

$$= \Pi(u,w) + \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2 w_0}{dx^2}\right)^2 \right] dx + \alpha(0) + \beta(0)$$

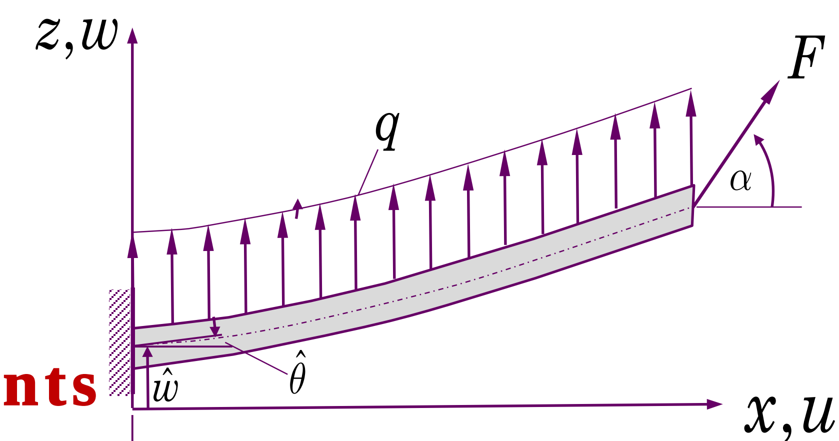
$$\Pi(\bar{u},\bar{w}) \geq \Pi(u,w)$$

because u and w satisfy the equilibrium equations

VIRTUAL WORK

Virtual displacements are those which satisfy the homogeneous form of the specified kinematic boundary conditions, but otherwise arbitrary.

$$u(0) = \hat{u} = 0,\;\; w(0) = \hat{w},\;\; -\frac{dw}{dx}\bigg|_{x=0} = \hat{\theta}$$



Set of admissible displacements

$$u_1(x) = \hat{u} + a_1 x, \quad w_1(x) = \hat{w} + \hat{\theta} x + b_1 x^2$$

Set of admissible virtual displacements

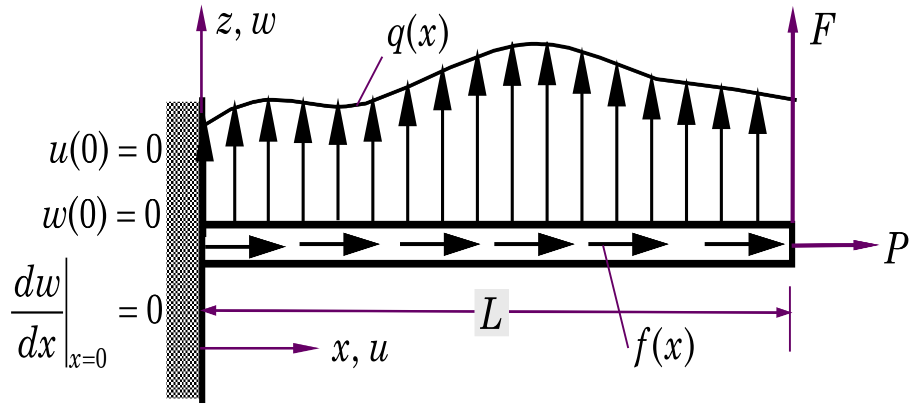
$$\delta u_1 = a_1 x,\;\; \delta w_1 = b_1 x^2;\quad \delta u_2 = a_1 x + a_2 x^2,\;\; \delta w_2 = b_1 x^2 + b_2 x^3$$

Virtual work done by actual forces (q, F) in moving through their respective displacements is

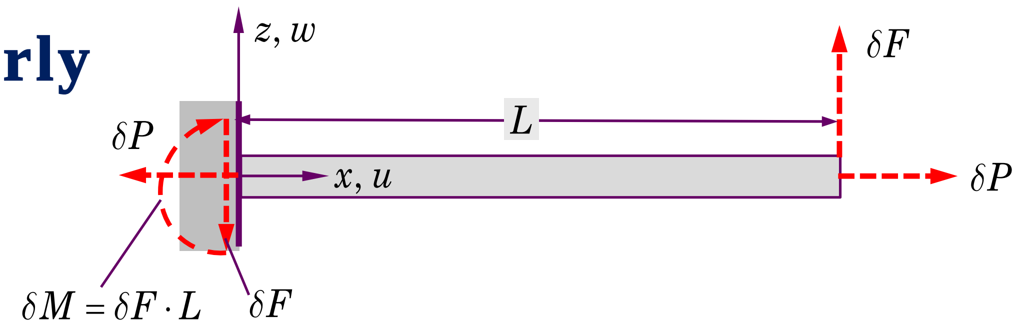
$$\delta W = \int_0^L q(s)\,\delta w(s)\,ds + F\cos\alpha \cdot \delta u(L) + F\sin\alpha \cdot \delta w(L)$$

COMPLEMENTARY VIRTUAL WORK

Virtual forces
are those which satisfy the self-equilibrium conditions, but otherwise arbitrary.



The set $(\delta P, \delta F)$ is clearly is in self-equilibrium.



The virtual work done by virtual forces in moving through actual Displacements is

$$\delta W^* = -\left[\delta P \cdot u(L) + \delta F \cdot w(L) - \delta P \cdot u(0) - \delta F \cdot w(0) + \delta M \cdot \theta(0)\right]$$

$$= -\left[\delta P \cdot u(L) + \delta F \cdot w(L)\right]$$

VIRTUAL WORK DONE

Internal virtual work done in a 3D body

$$\delta W_I = \int_\Omega \delta U_0 \, d\Omega = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) : \delta\boldsymbol{\varepsilon} \, d\Omega = \int_\Omega \sigma_{ij}(\varepsilon_{kl}) \delta \varepsilon_{ij} \, d\Omega$$

Internal complementary virtual work done

$$\delta W_I^* = \int_\Omega \delta U_0^* \, d\Omega = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \delta\boldsymbol{\sigma} \, d\Omega = \int_\Omega \varepsilon_{ij}(\sigma_{kl}) \delta \sigma_{ij} \, d\Omega$$

External virtual work done in a 3D body

$$\delta W_E = \int_\Omega \mathbf{f} \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_\sigma} \mathbf{t} \cdot \delta \mathbf{u} \, d\Gamma$$

External complementary virtual work done

$$\delta W_E^* = \int_\Omega \delta \mathbf{f} \cdot \mathbf{u} \, d\Omega + \int_{\Gamma_u} \delta \mathbf{t} \cdot \mathbf{u} \, d\Gamma$$

Total virtual virtual work done

$$\delta W = \delta W_I + \delta W_E; \quad \delta W^* = \delta W_I^* + \delta W_E^*$$

VIRTUAL WORK DONE FOR E-B BEAMS

Internal virtual work done

$$\delta W_I = \int_0^L \left(N \frac{d\delta u}{dx} - M \frac{d^2 \delta w}{dx^2} \right) dx$$

Internal complementary virtual work done

$$\delta W_I^* = \int_0^L \left(\varepsilon_{xx}^{(0)} \delta N + \varepsilon_{xx}^{(1)} \delta M + 2\varepsilon_{xz}^{(1)} \delta V \right) dx$$

External virtual work done in a 3D body

$$\delta W_E = -\left[\int_a^b f(x)\delta u(x)\, dx + \int_c^d \mathrm{q}(x)\delta w(x)\, dx + \text{VW point forces} \right]$$

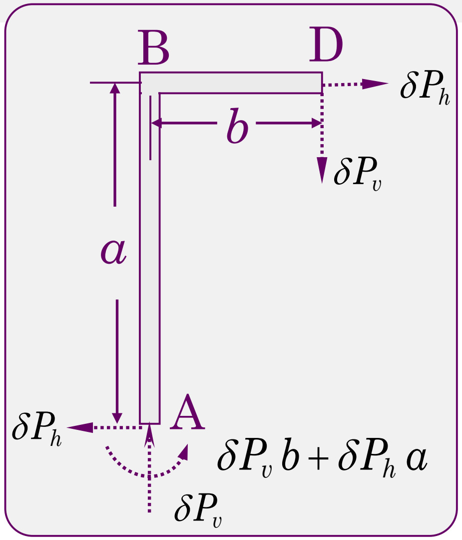
External complementary virtual work done

$$\delta W_E^* = -\left[\int_a^b \delta f(x) u(x)\, dx + \int_c^d \delta q(x) w(x)\, dx + \text{VW} \right]$$

COMPLEMENTARY VIRTUAL WORK DONE: AN EXAMPLE

$$\delta W_I^* = \delta U^* = \int_0^L \left[\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{V}{K_s GA}\delta V \right] dx,$$

$$\delta W_E^* = \delta V_E^* = -\left(v\delta P_v + u\delta P_h \right)$$

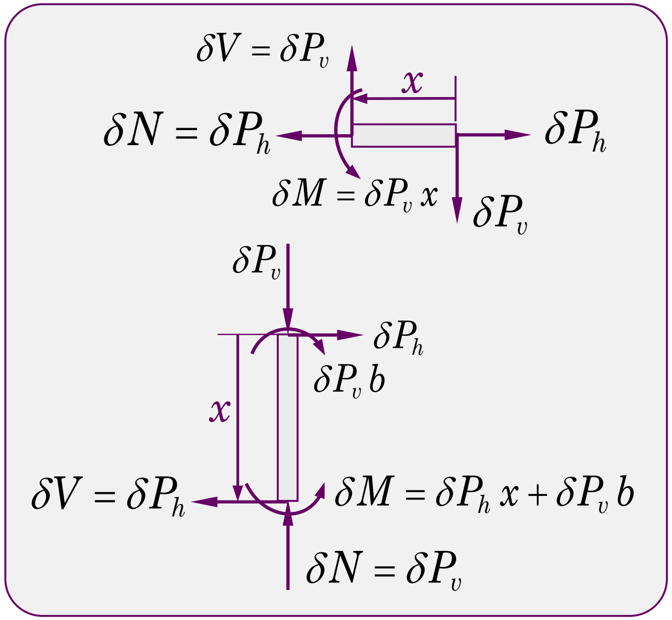


$$\delta N_{\mathbf{DB}} = \delta P_h,\ \delta N_{\mathbf{BA}} = \delta P_v, \delta V_{\mathbf{DB}} = \delta P_v,\ \delta V_{\mathbf{BA}} = \delta P_h$$

$$\delta M_{\mathbf{DB}} = \delta P_v \cdot x,\ \delta M_{\mathbf{BA}} = \delta P_v \cdot b + \delta P_h \cdot x$$

$$\delta U_{\mathbf{DB}}^* = \int_0^b \left[\frac{N_{\mathbf{DB}}}{EA}\delta N_{\mathbf{DB}} + \frac{M_{\mathbf{DB}}}{EI}\delta M_{\mathbf{DB}} + \frac{V_{DB}}{K_s GA}\delta V_{\mathbf{DB}} \right] dx$$

$$= \frac{P_h b}{EA}\delta P_h + \left(\frac{P_v b^3}{3EI} + \frac{P_v b}{K_s GA} \right)\delta P_v,$$



$$\delta U_{\mathbf{BA}}^* = \int_0^b \left[\frac{N_{\mathbf{BA}}}{EA}\delta N_{\mathbf{BA}} + \frac{M_{\mathbf{BA}}}{EI}\delta M_{\mathbf{BA}} + \frac{V_{BA}}{K_s GA}\delta V_{\mathbf{BA}} \right] dx$$

$$= \left[\frac{P_v a}{EA} + \frac{1}{EI}\left(P_v b^2 a + P_h b \frac{a^2}{2} \right) \right] \delta P_v$$

$$+ \left[\frac{1}{EI}\left(P_v b \frac{a^2}{2} + P_h \frac{a^3}{3} \right) + \frac{P_h a}{K_s GA} \right] \delta P_h$$

FIRST VARIATION and VARIATIONAL SYMBOL

The delta operator δ used in conjunction with virtual quantities has special importance in variational calculus. The operator is called the **variational operator** because it is used to denote a variation (or change) in a given quantity.

$$\bar{\mathbf{u}} = \mathbf{u} + \alpha \mathbf{v}, \;\; \alpha \mathbf{v} = \delta \mathbf{u}; \;\; \delta(\nabla \mathbf{u}) = \alpha(\nabla \mathbf{v}) = \nabla(\alpha \mathbf{v}) = \nabla(\delta \mathbf{u})$$

$$\begin{aligned}
\Delta F &= F(x, u + \alpha v, u' + \alpha v') - F(x, u, u') \\
&= F(x, u, u') + \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' \\
&\quad + \frac{(\alpha v)^2}{2!}\frac{\partial^2 F}{\partial u^2} + \frac{2(\alpha v)(\alpha v')}{2!}\frac{\partial^2 F}{\partial u \partial u'} + \cdots - F(x, u, u') \\
&= \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + \mathbf{O}(\alpha^2)
\end{aligned}$$

FIRST VARIATION OF A FUNCTION OF A DEPENDENT VARIABLE

Define

$$\delta F \equiv \alpha \left[\lim_{\alpha \to 0} \frac{\Delta F}{\alpha} \right] = \alpha \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial u'} v' \right)$$

$$= \frac{\partial F}{\partial u} \alpha v + \frac{\partial F}{\partial u'} \alpha v' = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

Alternatively,

$$\delta F = \alpha \left[\frac{dF(u + \alpha v, u' + \alpha v')}{d\alpha} \right]_{\alpha = 0}$$

$$= \alpha \left[\frac{\partial F}{\partial (u + \alpha v)} \frac{\partial (u + \alpha v)}{\partial \alpha} + \frac{\partial F}{\partial (u' + \alpha v')} \frac{\partial (u' + \alpha v')}{\partial \alpha} \right]_{\alpha = 0}$$

$$= \frac{\partial F}{\partial u} \alpha v + \frac{\partial F}{\partial u'} \alpha v' = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

ANALOGY BETWEEN TOTAL DIFFERENTIAL AND VARIATIONAL OPERTOR

Analogy

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

$$\delta F = \frac{\partial F}{\partial x}\cancel{\delta x} + \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \qquad (\delta x \to 0)$$

Properties

(1) $\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2$, (2) $\delta(F_1\, F_2) = \delta F_1\, F_2 + F_1 \delta F_2$

(3) $\delta\left(\frac{F_1}{F_2}\right) = \frac{\delta F_1\, F_2 - F_1\, \delta F_2}{F_2^2}$, (4) $\delta(F_1)^n = n(F_1)^{n-1}\delta F_1$

(1) $\delta\left(\frac{du}{dx}\right) = \alpha\frac{dv}{dx} = \frac{d}{dx}(\alpha v) = \frac{d}{dx}(\delta u)$

(2) $\delta\left(\int_0^a u\,dx\right) = \alpha\int_0^a v\,dx = \int_0^a \alpha v\,dx = \int_0^a \delta u\,dx$

FIRST VARIATION OF A FUNCTIONAL

A functional

A *functional* F is a mapping (or operator) from a vector space U into the real number field R. Thus, if $u \in U$ (i.e. u is an element of U), then $F(u)$ is a real number.

The First Variation of a Functional

$$I(u) = \int_a^b F(x,u,u')\,dx, \quad u' = \frac{du}{dx}$$

$$\delta I(u;\delta u) = \delta \int_a^b F(x,u,u')\,dx = \int_a^b \delta F\,dx = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \delta u\,dx + \left[\delta u \cdot \frac{\partial F}{\partial u'} \right]_{x=a}^{x=b}$$

FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS AND EULER EQUATIONS

Lemma: If G is an integrable function and $\eta(x)$ is arbitrary in $a < x < b$ and $\eta(a)$ is arbitrary, then the statement

$$\int_a^b G(x)\eta(x)dx + B(a)\eta(a) = 0$$

implies that $G(x) = 0 \;\; a < x < b$ and $B(a) = 0$

which are called the **Euler equations.** If $\delta I = 0$ and δu is arbitrary in (a,b) and at $x = a$ and $x = b$, then

$$\delta I(u;\delta u) = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u\, dx + \left[\delta u \cdot \frac{\partial F}{\partial u'}\right]_{x=a}^{x=b} = 0$$

$$\Rightarrow \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] = 0 \;\; a < x < b; \quad \left[\frac{\partial F}{\partial u'}\right]_{x=a}^{x=b} = 0$$

EULER EQUATIONS OF FUNCTIONAL IN 2D, INVOLVING TWO DEPENDENT VARIABLES

Given the functional

$$I(u,v)=\int_{\Omega}F(x,y,u,v,u_x,v_x,u_y,v_y)\,dxdy,\; u_x\equiv\frac{\partial u}{\partial x},\; etc.$$

find the Euler equations if $\delta I=0$.

We have

$$0=\int_{\Omega}\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u_x}\delta u_x+\frac{\partial F}{\partial u_y}\delta u_y+\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v_x}\delta v_x+\frac{\partial F}{\partial v_y}\delta v_y\right)dxdy$$

$$=\int_{\Omega}\left\{\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u+\left[\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v\right\}dxdy$$

$$+\oint_{\Gamma}\left[\left(\frac{\partial F}{\partial u_x}n_x+\frac{\partial F}{\partial u_y}n_y\right)\delta u+\left(\frac{\partial F}{\partial v_x}n_x+\frac{\partial F}{\partial v_y}n_y\right)\delta v\right]dxdy$$

EULER EQUATIONS OF FUNCTIONAL IN 2D, INVOLVING TWO DEPENDENT VARIABLES

If δu is arbitrary in Ω and on Γ, then the Euler Equations are as follows:

$$\left.\begin{aligned} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \end{aligned}\right\} \text{in } \Omega$$

$$\left.\begin{aligned} \frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y = 0 \\ \frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y = 0 \end{aligned}\right\} \text{on } \Gamma$$

See the textbook for examples (some will be discussed in the class)

Problems with Constraints-1

Problem 1: Find the minimum of the function $F(x, y)$ with no constraints.

Necessary condition

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

Since dx and dy are arbitrary and independent, we have

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0$$

Problems with Constraints-2

Problem 2: Find the minimum of the function $F(x, y)$ subjected to the constraint $G(x, y) = 0$

Necessary condition

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

But dx and dy are not independent of each other because of the constraint, we cannot set

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0$$

Problems with Constraints-3

Lagrange Multiplier Method: Introduce new function

$$F_L(x,y,\lambda) = F(x,y) + \lambda G(x,y)$$

where λ is the Lagrange multiplier. Set

$$dF_L = \frac{\partial F_L}{\partial x}dx + \frac{\partial F_L}{\partial y}dy + \frac{\partial F_L}{\partial \lambda}d\lambda = 0$$

Now dx, dy, and $d\lambda$ are independent of each other; so, we can set

$$\frac{\partial F_L}{\partial x} = \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x} = 0, \qquad \frac{\partial F_L}{\partial y} = \frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y} = 0,$$

$$\frac{\partial F_L}{\partial \lambda} = G(x,y) = 0.$$

Problems with Constraints-4

Penalty Function Method: Introduce new function

$$F_P(x, y, \lambda) = F(x, y) + \frac{\gamma}{2}\left[G(x, y)\right]^2$$

where γ is the penalty parameter. Set

$$dF_P = \frac{\partial F_P}{\partial x}dx + \frac{\partial F_P}{\partial y}dy = 0$$

Since dx and dy are independent of each other, we can set

$$\frac{\partial F_P}{\partial x} = \frac{\partial F}{\partial x} + \gamma G\frac{\partial G}{\partial x} = 0, \qquad \frac{\partial F_P}{\partial y} = \frac{\partial F}{\partial y} + \gamma G\frac{\partial G}{\partial y} = 0.$$

Comparison of the two methods

Lagrange Multiplier Method:

$$\frac{\partial F_L}{\partial x} = \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0, \qquad \frac{\partial F_L}{\partial y} = \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0.$$

Penalty Function Method:

$$\frac{\partial F_P}{\partial x} = \frac{\partial F}{\partial x} + \gamma G \frac{\partial G}{\partial x} = 0, \qquad \frac{\partial F_P}{\partial y} = \frac{\partial F}{\partial y} + \gamma G \frac{\partial G}{\partial y} = 0.$$

We find that in the penalty function method we can compute

$$\lambda = \gamma G(x_\gamma, y_\gamma)$$

AN EXAMPLE: ALGEBRAIC PROBLEM

$$F(x,y) = 2x^2 + y^2 - 8x + y + 1, \quad G(x,y) \equiv 2x - y = 0$$

Lagrange Multiplier Method

$$4x - 8 + 2\lambda = 0, \quad 2y + 1 - \lambda = 0, \quad 2x - y = 0$$

$$x = 0.5, \qquad y = 1.0, \qquad \lambda = 3.0$$

Penalty Function Method

$$4x - 8 + 2\gamma(2x - y) = 0, \qquad 2y + 1 - \gamma(2x - y) = 0$$

$$x_\gamma = \frac{8 + 3\gamma}{4 + 6\gamma}, \qquad y_\gamma = \frac{3\gamma - 1}{2 + 3\gamma}$$

Clearly, as $\gamma \to \infty$, we have

$$\lim_{\gamma\to\infty} x_\gamma = 0.5 = x, \qquad \lim_{\gamma\to\infty} y_\gamma = 1.0 = y$$

Penalty Function Method (Example - continued)

Table: A comparison of the penalty solution with the exact for various values of the penalty parameter γ.

γ	1.0	10.0	25.0	50.0	100.0	1000.0
x_γ	1.1	0.5938	0.5390	0.5197	0.5099	0.5010
y_γ	0.4	0.9063	0.9610	0.9803	0.9901	0.9990
$G(x_\gamma, y_\gamma)$	1.8	0.2813	0.1169	0.0592	0.0298	0.0030
λ_γ	1.8	2.8125	2.9221	2.9605	2.9801	2.9980

$$\lambda_\gamma = \gamma\, G(x_\gamma, y_\gamma)$$

AN EXAMPLE: CONTINUUM PROBLEM-1

Problem 3: Find the minimum of the functional $I(u,v)$ subjected to the constraint $G(u,v)=0$.

Minimize

$$I(u,v)=\int_a^b F(x,u,u',v,v')dx$$

subjected to

$$G(u,u',v,v')=0$$

Lagrange Multiplier Method

$$0=\delta I=\int_a^b\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'+\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v'}\delta v'\right)dx$$

$$0=\delta G=\frac{\partial G}{\partial u}\delta u+\frac{\partial G}{\partial u'}\delta u'+\frac{\partial G}{\partial v}\delta v+\frac{\partial G}{\partial v'}\delta v'$$

AN EXAMPLE: CONTINUUM PROBLEM-2

$$0 = \int_a^b \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' + \lambda\left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v'\right)\right] dx$$

$$= \int_a^b \left\{\left[\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)\right]\delta u + \left[\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)\right]\delta v\right\} dx$$

Select λ such that the coefficient of δv is zero. Then the coefficient of δu is zero because δu is independent.

AN EXAMPLE: CONTINUUM PROBLEM-3

$$I_L(u,v,\lambda) \equiv I(u,v) + \int_a^b \lambda G(u,u',v,v')\,dx$$
$$= \int_a^b (F + \lambda G)\,dx$$

$$\delta I_L = \int_a^b \left[\left(\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u}\right)\delta u + \left(\frac{\partial F}{\partial u'} + \lambda \frac{\partial G}{\partial u'}\right)\delta u' + \left(\frac{\partial F}{\partial v} + \lambda \frac{\partial G}{\partial v}\right)\delta v + \left(\frac{\partial F}{\partial v'} + \lambda \frac{\partial G}{\partial v'}\right)\delta v' + \delta\lambda G\right] dx$$

$$= \int_a^b \left\{\left[\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda \frac{\partial G}{\partial u'}\right)\right]\delta u + \left[\frac{\partial F}{\partial v} + \lambda \frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda \frac{\partial G}{\partial v'}\right)\right]\delta v + G\delta\lambda\right\} dx$$

AN EXAMPLE: CONTINUUM PROBLEM-3

Penalty Function Method

$$I_P(u,v) \equiv I(u,v) + \tfrac{1}{2}\int_a^b \gamma \left[G(u,u',v,v')\right]^2 dx$$

$$\delta I_P = \int_a^b \left[\left(\frac{\partial F}{\partial u} + \gamma G\frac{\partial G}{\partial u}\right)\delta u + \left(\frac{\partial F}{\partial u'} + \gamma G\frac{\partial G}{\partial u'}\right)\delta u' + \left(\frac{\partial F}{\partial v} + \gamma G\frac{\partial G}{\partial v}\right)\delta v + \left(\frac{\partial F}{\partial v'} + \gamma G\frac{\partial G}{\partial v'}\right)\delta v'\right] dx$$

$$= \int_a^b \left\{\left[\frac{\partial F}{\partial u} + \gamma G\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \gamma G\frac{\partial G}{\partial u'}\right)\right]\delta u + \left[\frac{\partial F}{\partial v} + \gamma G\frac{\partial G}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \gamma G\frac{\partial G}{\partial v'}\right)\right]\delta v\right\} dx$$

$$\lambda_\gamma = \gamma G(u_\gamma, u'_\gamma, v_\gamma, v'_\gamma)$$