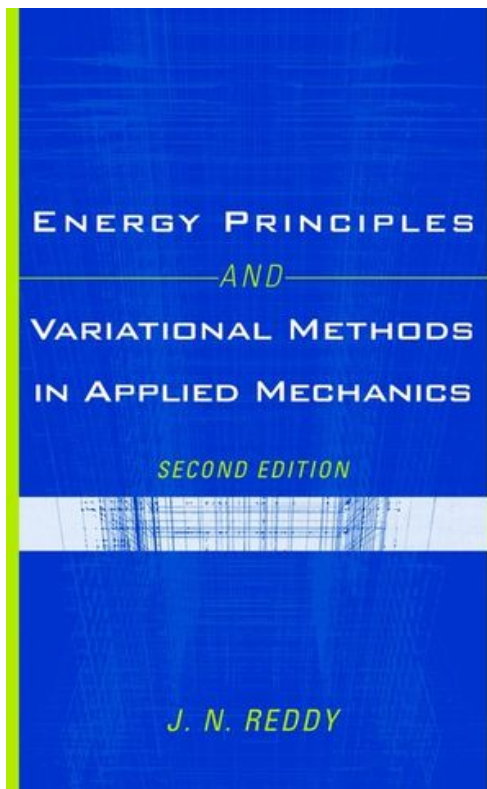


MEEN 618: ENERGY AND VARIATIONAL METHODS

KINEMATICS and KINETICS OF SOLIDS

Read: **Chapter 3**



CONTENTS

- Continuum assumption
- Kinematics of deformation
- Kinetics: Stress vector
- Cauchy's formula
- Balance of linear and angular momenta
- Theories of straight beams
 - Euler-Bernoulli beam theory
 - Timoshenko beam theory

In the study of deformation and motion of solid bodies, we make the simplifying assumption for convenience that the matter is distributed continuously, without any macroscopic gaps or empty spaces; that is, we disregard the molecular structure of matter.

Continuum assumption

$$\rho(\mathbf{x}, t) \equiv \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V}$$

The study of motion and deformation of a continuum can be broadly classified into four basic categories:

- (1) Kinematics** (strain-displacement equations)
- (2) Kinetics** (balance of linear and angular momentum)
- (3) Thermodynamics** (first and second laws of thermodynamics)
- (4) Constitutive equations** (stress-strain relations)

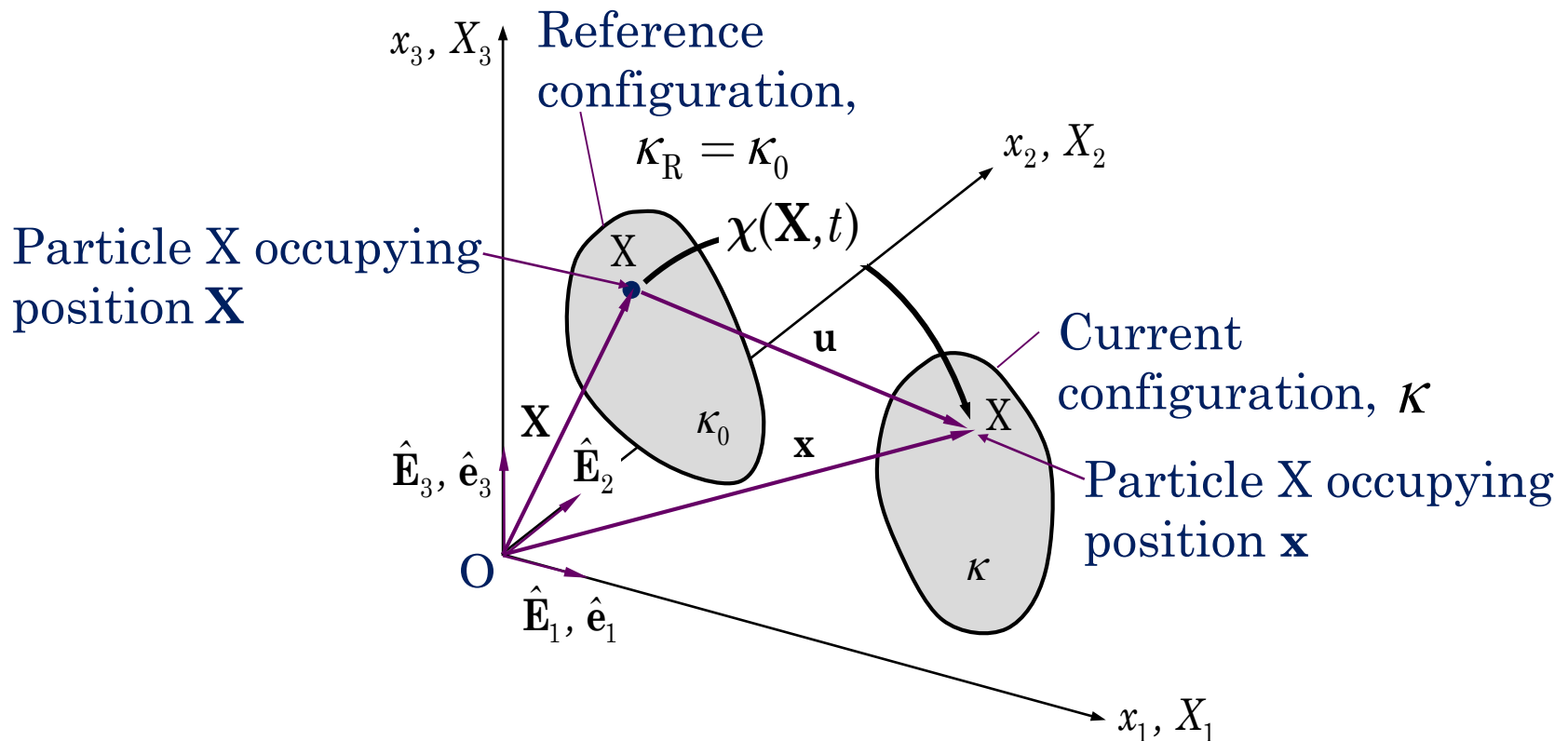
KINEMATICS OF SOLIDS

Material Description: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, 0)$

(X_1, X_2, X_3) are the *material coordinates*.

(x_1, x_2, x_3) are the *spatial (current) coordinates*.

Displacement vector: $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$



AN EXAMPLE

Problem statement:

Consider the uniform deformation of a square block of side 2 units and initially centered at $\{\mathbf{X}\}=(0,0)$. If the deformation is defined by the mapping

$$\chi(\mathbf{X}) = (3.5 + X_1 + 0.5X_2)\hat{\mathbf{e}}_1 + (4 + X_2)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3$$

(a) sketch the deformation, and (c) compute the displacements.

Solution:

(a) From the given mapping, we have in matrix form, we have

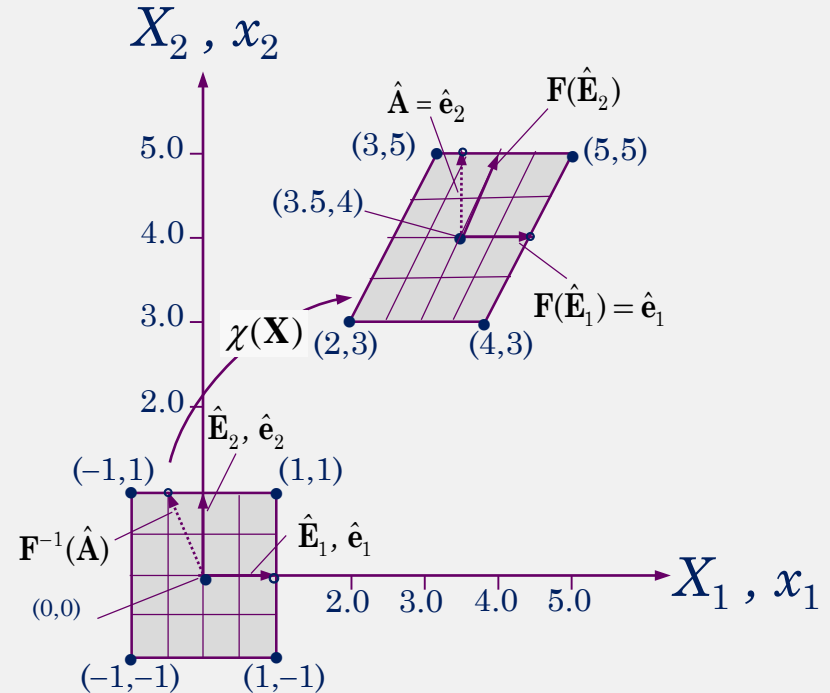
$$x_1 = 3.5 + X_1 + 0.5X_2, \quad x_2 = 4 + X_2, \quad x_3 = X_3$$

AN EXAMPLE (continued)

(a) $x_1 = 3.5 + X_1 + 0.5X_2,$
 $x_2 = 4 + X_2, \quad x_3 = X_3.$
 $X_1 = -1.5 + x_1 - 0.5x_2,$
 $X_2 = -4 + x_2, \quad X_3 = x_3.$

Mapping

(X_1, X_2)	(x_1, x_2)
$(-1, -1) \rightarrow$	$(2, 3)$
$(1, -1) \rightarrow$	$(4, 3)$
$(1, 1) \rightarrow$	$(5, 5)$
$(-1, 1) \rightarrow$	$(3, 5)$



(b) $\mathbf{u} = \mathbf{x} - \mathbf{X} = (3.5 + 0.5X_2)\hat{\mathbf{e}}_1 + 4\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$

GREEN-LAGRANGE STRAIN TENSOR

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X}, \quad (ds)^2 = d\mathbf{x} \cdot d\mathbf{x}$$

Define the Green-Lagrange strain tensor \mathbf{E} as

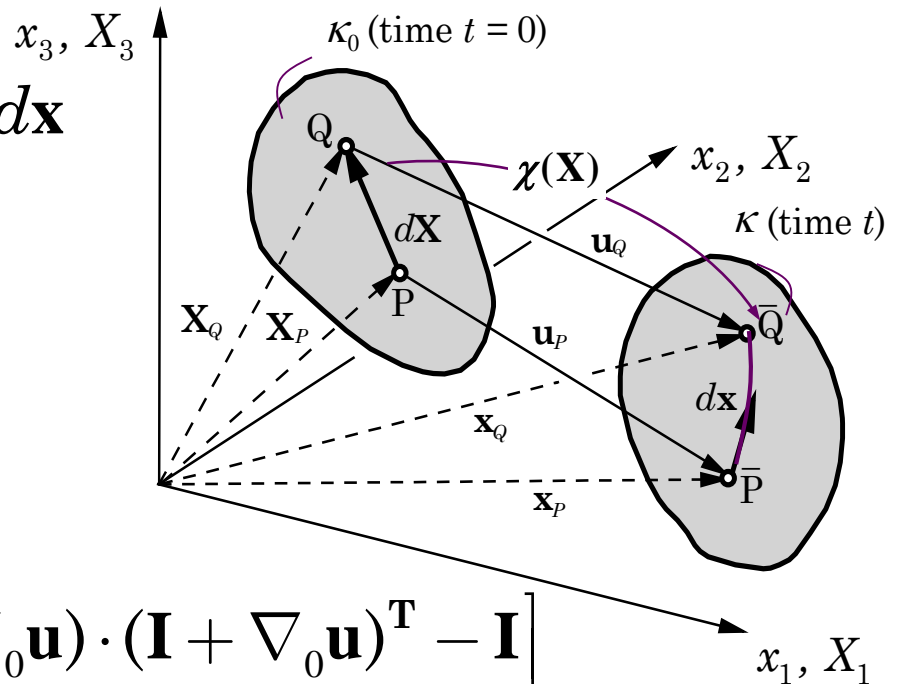
$$(ds)^2 - (dS)^2 \equiv 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X}$$

where \mathbf{E} is

$$\begin{aligned} \mathbf{E} &= \frac{1}{2} \left(\frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{x}}{d\mathbf{X}} - \mathbf{I} \right) = \frac{1}{2} \left[(\mathbf{I} + \nabla_0 \mathbf{u}) \cdot (\mathbf{I} + \nabla_0 \mathbf{u})^T - \mathbf{I} \right] \\ &= \frac{1}{2} \left[\nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^T + (\nabla_0 \mathbf{u}) \cdot (\nabla_0 \mathbf{u})^T \right] \end{aligned}$$

The rectangular Cartesian component form of \mathbf{E} is

$$E_{IJ} = \frac{1}{2} \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I} \frac{\partial u_K}{\partial X_J} \right)$$



GREEN-LAGRANGE STRAIN TENSOR

The rectangular Cartesian components in explicit form are given by

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial X_1} \right)^2 + \left(\frac{\partial u_2}{\partial X_1} \right)^2 + \left(\frac{\partial u_3}{\partial X_1} \right)^2 \right],$$

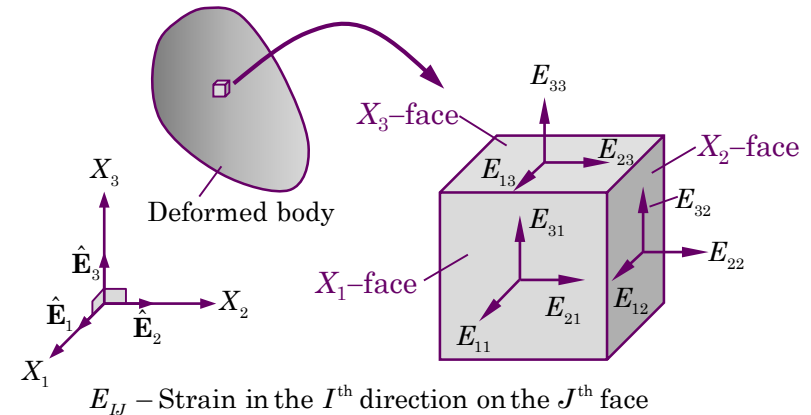
$$E_{22} = \frac{\partial u_2}{\partial X_2} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial X_2} \right)^2 + \left(\frac{\partial u_2}{\partial X_2} \right)^2 + \left(\frac{\partial u_3}{\partial X_2} \right)^2 \right]$$

$$E_{33} = \frac{\partial u_3}{\partial X_3} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial X_3} \right)^2 + \left(\frac{\partial u_2}{\partial X_3} \right)^2 + \left(\frac{\partial u_3}{\partial X_3} \right)^2 \right]$$

$$E_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} \right)$$

$$E_{13} = \frac{1}{2} \left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_3} \right)$$

$$E_{23} = \frac{1}{2} \left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \frac{\partial u_3}{\partial X_3} \right)$$



INFITESIMAL STRAIN TENSOR

If \mathbf{E} is of the order $O(\epsilon)$ in $\nabla_0 \mathbf{u}$, then we mean

$$\frac{\partial u_I}{\partial X_J} = O(\epsilon) \text{ as } \epsilon \rightarrow 0$$

If terms of the order $O(\epsilon^2)$ can be omitted, then

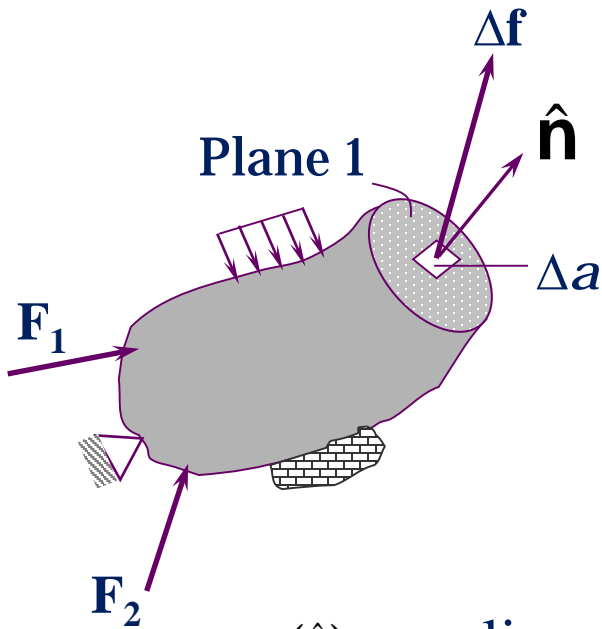
$$E_{IJ} = \frac{1}{2} \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I} \frac{\partial u_K}{\partial X_J} \right)$$

can be approximated as

$$E_{IJ} \approx \frac{1}{2} \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} \right) = O(\epsilon) \text{ as } \epsilon^2 \rightarrow 0$$

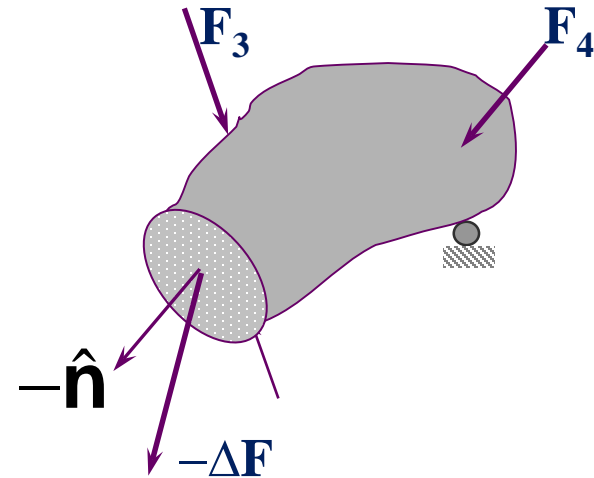
$$\mathbf{E} \approx \boldsymbol{\varepsilon} = \frac{1}{2} [\nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^T], \text{ the infinitesimal strain tensor}$$

KINETICS: Stress Vector



$$\mathbf{t}^{(\hat{n})} = \lim_{\Delta a \rightarrow 0} \frac{\Delta \mathbf{f}}{\Delta a}$$

$$\mathbf{t}^{(\hat{n})} = -\mathbf{t}^{(-\hat{n})}$$



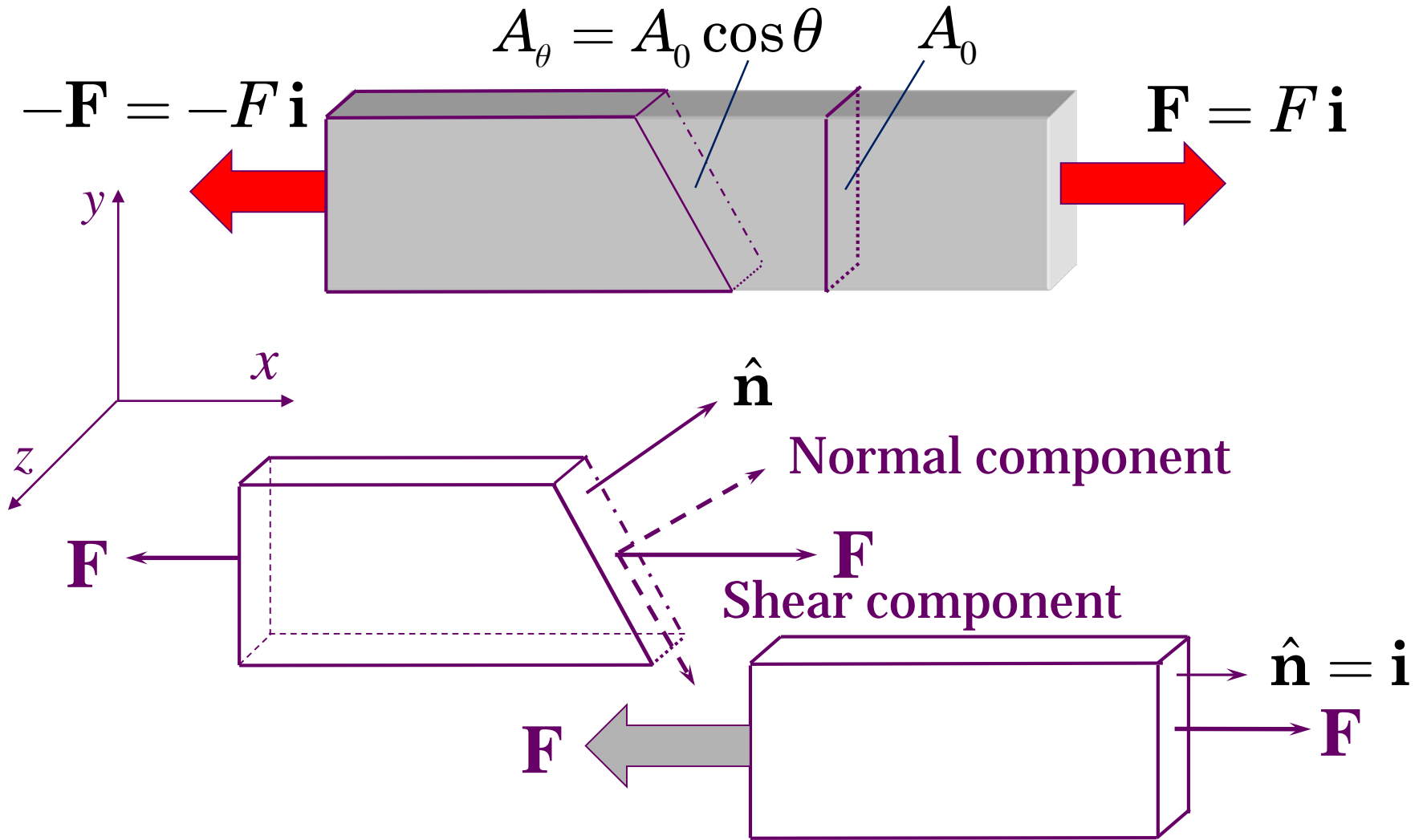
$$\mathbf{t}^{(-\hat{n})} = \lim_{\Delta a \rightarrow 0} \frac{-\Delta \mathbf{f}}{\Delta a}$$

$$\mathbf{t}^{(-\hat{n})} = -\mathbf{t}^{(\hat{n})}$$

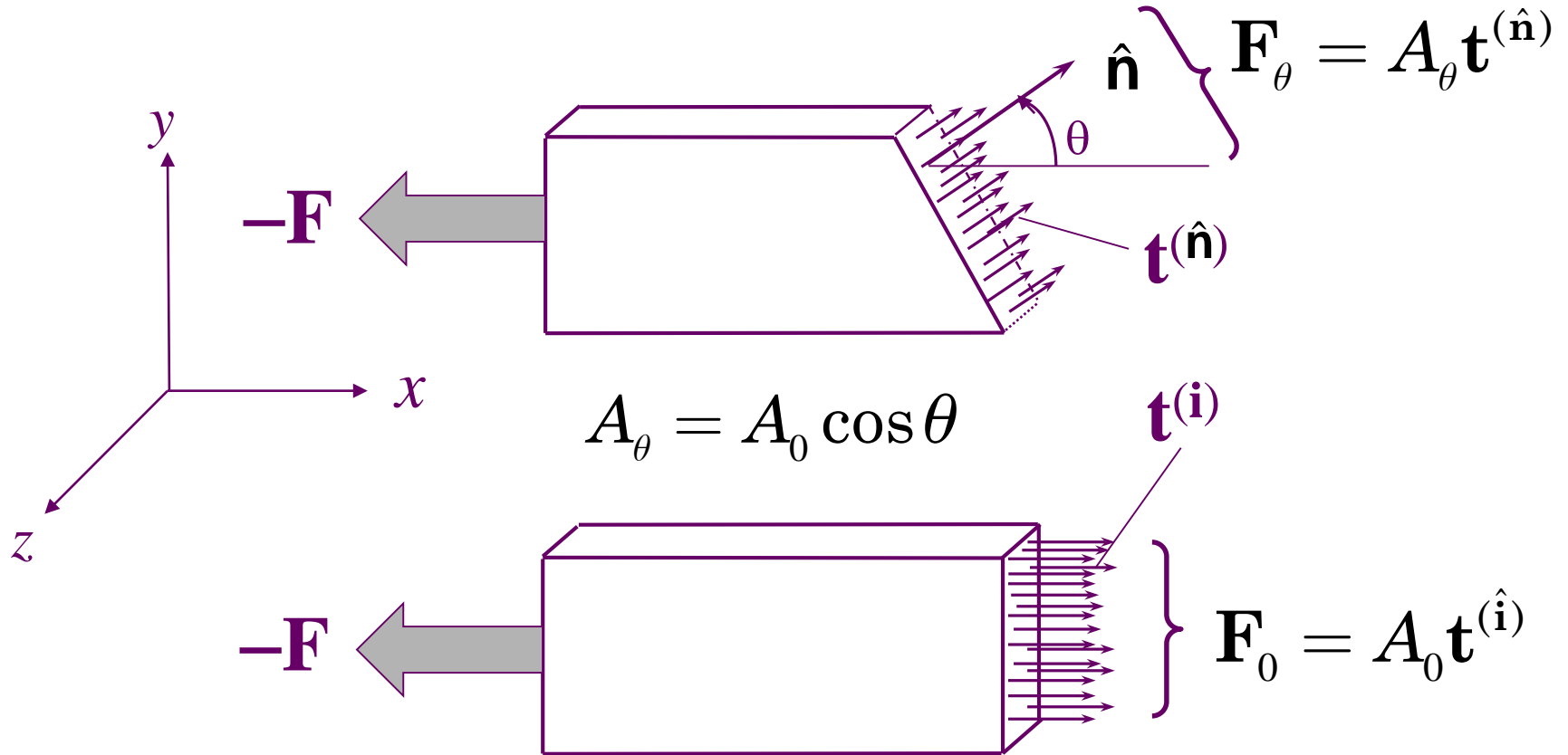
or

Cauchy's Lemma

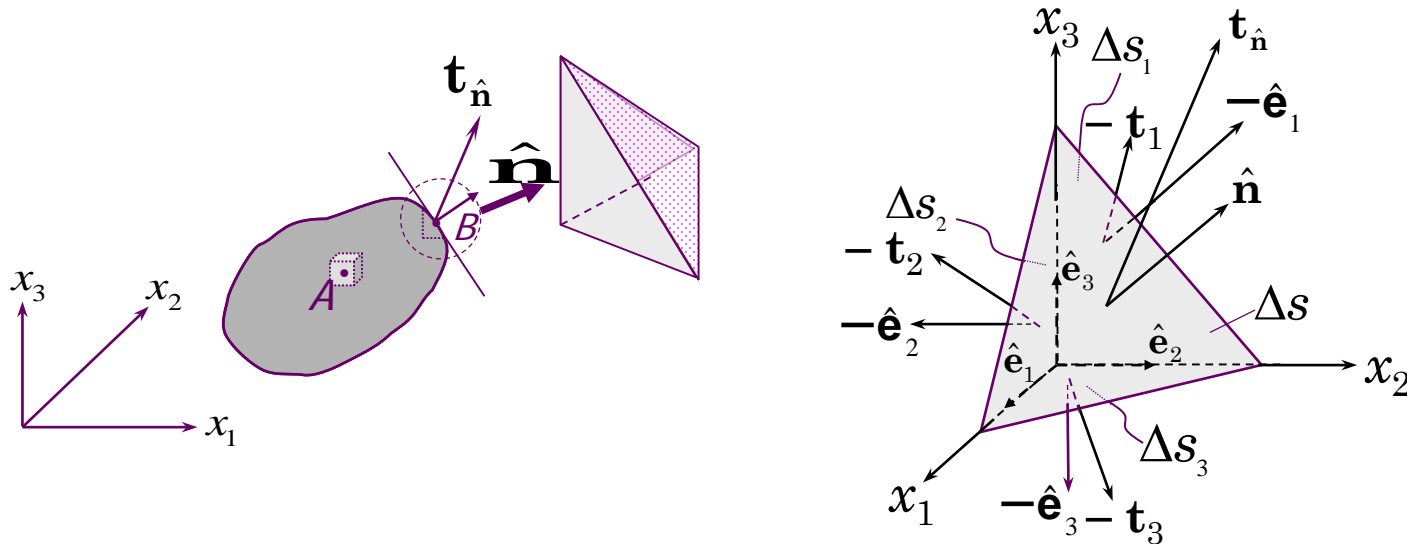
Traction Vector: Examples



Traction Vector: Examples



CAUCHY STRESS TENSOR-1



$$\mathbf{t} \Delta s - \mathbf{t}_1 \Delta s_1 - \mathbf{t}_2 \Delta s_2 - \mathbf{t}_3 \Delta s_3 + \rho \Delta v \mathbf{f} = \rho \Delta v \mathbf{a}$$

$$\mathbf{t} = \mathbf{t}_1 (\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}}) + \mathbf{t}_2 (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}}) + \mathbf{t}_3 (\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{n}}) + \rho \frac{\Delta h}{3} (\mathbf{a} - \mathbf{f})$$

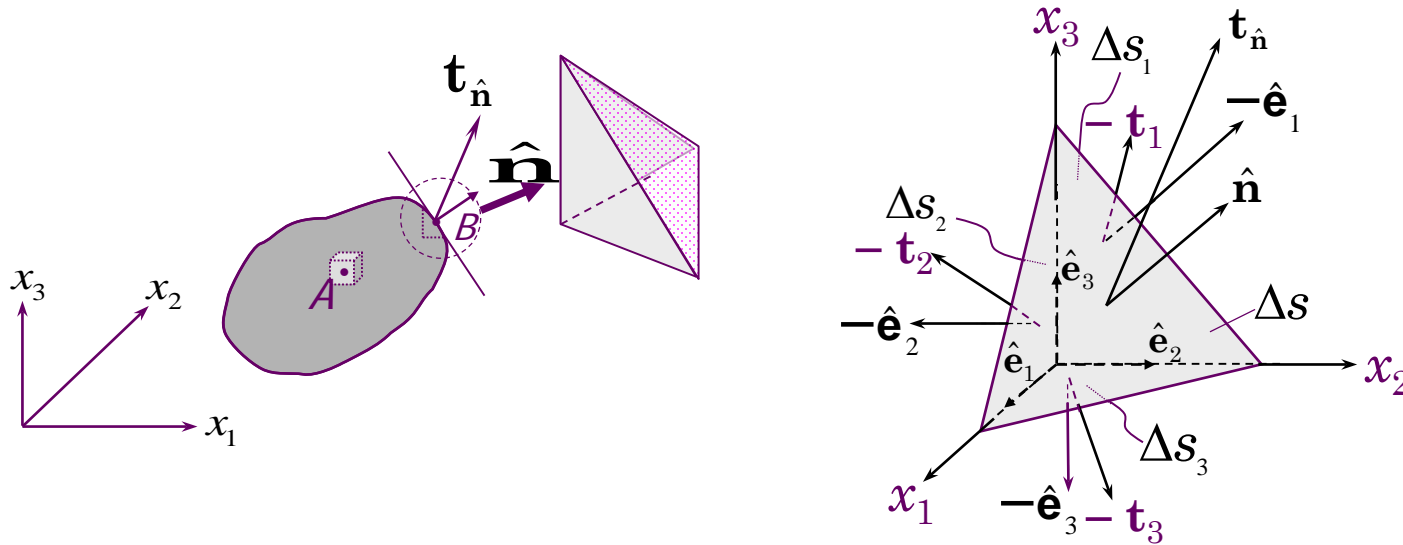
As $\Delta h \rightarrow 0$, we obtain

$$\mathbf{t} = \mathbf{t}_1 (\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}}) + \mathbf{t}_2 (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}}) + \mathbf{t}_3 (\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{n}}) = \mathbf{t}_j \hat{\mathbf{e}}_j \cdot \hat{\mathbf{n}} \equiv \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \quad \text{and} \quad \boldsymbol{\sigma} = \mathbf{t}_j \hat{\mathbf{e}}_j \quad \left[\quad t_i^{(\hat{\mathbf{n}})} = \sigma_{ij} n_j; \quad \mathbf{t}_i = \sigma_{ji} \hat{\mathbf{e}}_j, \quad \boldsymbol{\sigma} = \sigma_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \quad \right]$$

plane direction **Kinetics - 14**

CAUCHY'S FORMULA



$$\mathbf{t} \Delta s - \mathbf{t}_1 \Delta s_1 - \mathbf{t}_2 \Delta s_2 - \mathbf{t}_3 \Delta s_3 + \rho \Delta v \mathbf{f} = \rho \Delta v \mathbf{a}$$

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1) \mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2) \mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3) \mathbf{t}_3 + \rho \frac{\Delta h}{3} (\mathbf{a} - \mathbf{f})$$

As $\Delta h \rightarrow 0$, we obtain

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1) \mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2) \mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3) \mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i) \mathbf{t}_i \equiv \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$$

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\sigma} = \hat{\mathbf{e}}_i \mathbf{t}_i \quad \left[t_i^{(\hat{\mathbf{n}})} = n_j \sigma_{ji}, \quad \mathbf{t}_i = \sigma_{ij} \hat{\mathbf{e}}_j \right]$$

plane
direction

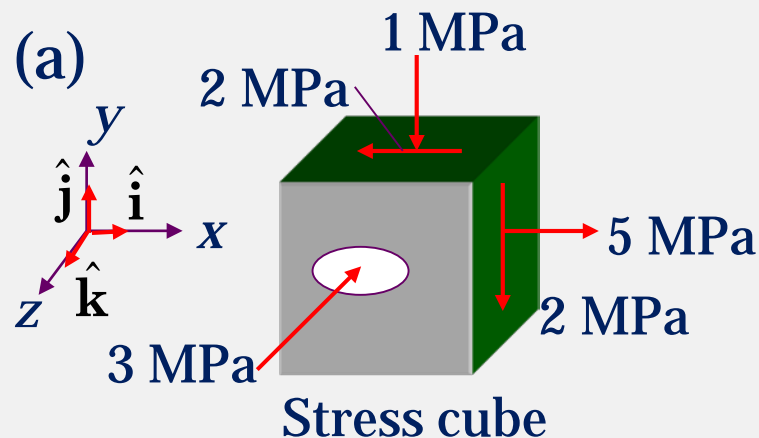
AN EXAMPLE

Problem: Given the following stress tensor components in Cartesian coordinates

$$[\sigma] = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ (MPa)}$$

- (a) Show the stress components on the stress cube.
 (b) Determine the traction vectors $\mathbf{t}^{(\hat{i})}$, $\mathbf{t}^{(\hat{j})}$, and $\mathbf{t}^{(\hat{k})}$
 (c) Sketch the traction vectors on the stress cube.

Solution: We have



(b)

$$\mathbf{t}^{(\hat{i})} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

$$\mathbf{t}^{(\hat{j})} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}}$$

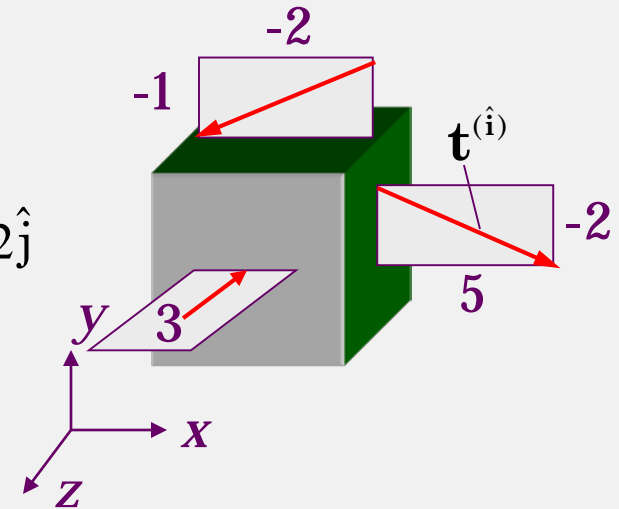
$$\mathbf{t}^{(\hat{k})} = -3\hat{\mathbf{k}}$$

Example (continued)**(b) Solution by use of Cauchy's formula**

$$[\sigma] = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ (MPa)}, \quad \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

(c)**(i) When $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ ($n_x = 1, n_y = 0, n_z = 0$)**

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{(\hat{\mathbf{i}})} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -2 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{t}^{(\hat{\mathbf{i}})} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

**(ii) When $\hat{\mathbf{n}} = \hat{\mathbf{j}}$ ($n_x = 0, n_y = 1, n_z = 0$)**

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{(\hat{\mathbf{j}})} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -1 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{t}^{(\hat{\mathbf{j}})} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}}$$

(iii) When $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ ($n_x = 0, n_y = 0, n_z = 1$) $\Rightarrow \mathbf{t}^{(\hat{\mathbf{k}})} = -3\hat{\mathbf{k}}$

AN EXAMPLE

Problem statement:

With reference to a rectangular Cartesian system (x_1, x_2, x_3) , the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ psi.}$$

Determine stress vector and its normal and tangential components at the point on the plane

$$\phi(x_1, x_2, x_3) \equiv x_1 + 2x_2 + 2x_3 = \text{constant}$$

which is passing through the point.

AN EXAMPLE

Solution:

First, we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal to the plane is

$$\hat{\mathbf{n}} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)$$

The components of the stress vector are

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi}$$

Solution (continued):

The traction vector normal to the plane is given by

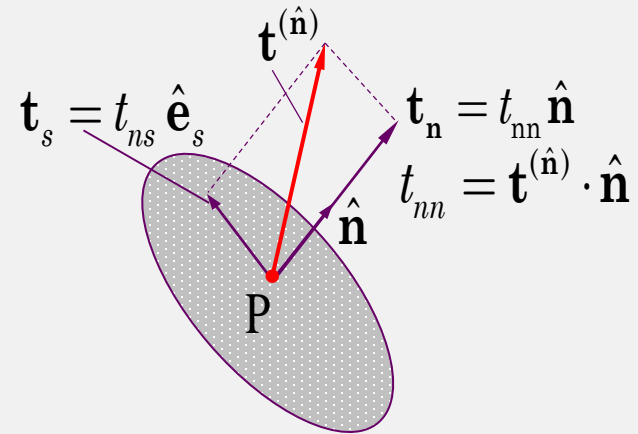
$$\begin{aligned}\mathbf{t}_{nn} &= (\mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \frac{2600}{9}\hat{\mathbf{n}} \\ &= \frac{2600}{27}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \text{ psi}\end{aligned}$$

and the traction vector projected onto the plane (i.e., shear traction) is given by

$$\mathbf{t}_{ns} = \mathbf{t}(\hat{\mathbf{n}}) - \mathbf{t}_{nn} = \frac{100}{27}(118\hat{\mathbf{e}}_1 - 16\hat{\mathbf{e}}_2 - 43\hat{\mathbf{e}}_3) \text{ psi}.$$

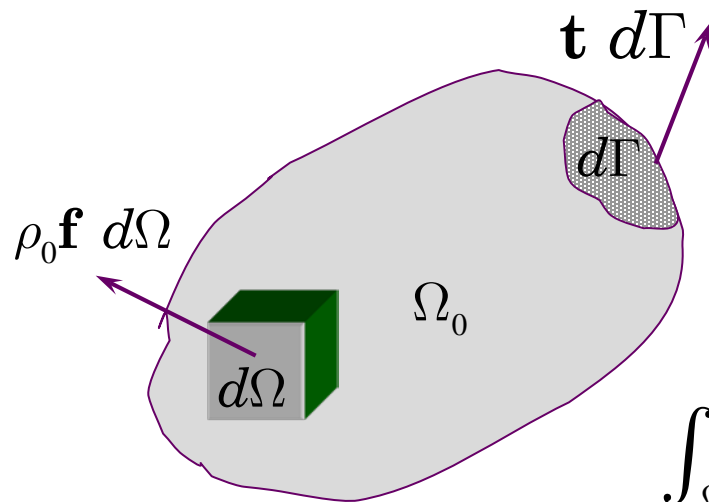
The magnitudes are

$$|\mathbf{t}_{nn}| = t_{nn} = \frac{2600}{9} = 288.89 \text{ psi}, \quad |\mathbf{t}_{ns}| = t_{ns} = 468.91 \text{ psi}.$$



Conservation of Linear Momentum

The time rate of change of total linear momentum of a given continuum equals the vector sum of all external forces acting on the continuum. This also known as **Newton's Second Law**.



$$\oint_{\Gamma_0} \mathbf{t} d\Gamma + \int_{\Omega_0} \rho_0 \mathbf{f} d\Omega = \frac{\partial}{\partial t} \int_{\Omega_0} \rho_0 \mathbf{v} d\Omega$$

$$\oint_{\Gamma_0} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} d\Gamma + \int_{\Omega_0} \rho_0 \mathbf{f} d\Omega = \int_{\Omega_0} \rho_0 \frac{\partial \mathbf{v}}{\partial t} d\Omega$$

$$\int_{\Omega_0} \left(\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{f} - \rho_0 \frac{\partial \mathbf{v}}{\partial t} \right) d\Omega = 0$$

Newton's First Law. Newton's First Law states that an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force.

Conservation of Linear Momentum

(continued)

Vector form of the equation of motion

$$\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Cartesian Component Form

$$\left(\hat{\mathbf{e}}_k \frac{\partial}{\partial x_k} \right) \cdot \left(\sigma_{ji} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i \right) + \rho_0 f_i \hat{\mathbf{e}}_i = \rho_0 \frac{\partial^2 (u_i \hat{\mathbf{e}}_i)}{\partial t^2}$$

$$\frac{\partial \sigma_{ji}}{\partial x_j} \hat{\mathbf{e}}_i + \rho_0 f_i \hat{\mathbf{e}}_i = \rho_0 \frac{\partial v_i}{\partial t} \hat{\mathbf{e}}_i \Rightarrow \frac{\partial \sigma_{ji}}{\partial x_j} + \rho_0 f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Conservation of Linear Momentum (continued)

Cartesian Component Form (expanded form)

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho_0 f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho_0 f_1 = \rho_0 \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho_0 f_2 = \rho_0 \frac{\partial^2 u_2}{\partial t^2}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho_0 f_3 = \rho_0 \frac{\partial^2 u_3}{\partial t^2}$$

Balance of Angular Momentum

The principle of balance of angular momentum can be stated as: *the time rate of change of the total moment of momentum for a continuum is equal to vector sum of the moments of external forces acting on the continuum.* We assume that there are no body (volume dependent) couples \mathbf{M} :

$$\lim_{\Delta V \rightarrow 0} \Delta \mathbf{M} / \Delta V = \mathbf{0}$$

Then the balance of angular momentum requires

$$\oint_{\Gamma} \mathbf{x} \times \mathbf{t} \, d\Gamma + \int_{\Omega} \mathbf{x} \times \mathbf{f} \, d\Omega = \frac{D}{Dt} \int_{\Omega} \mathbf{x} \times \rho \mathbf{v} \, d\Omega$$

which results in the symmetry of the Cauchy stress tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad \text{or} \quad \sigma_{ij} = \sigma_{ji}$$

THEORIES OF STRAIGHT BEAMS

Euler-Bernoulli beam theory

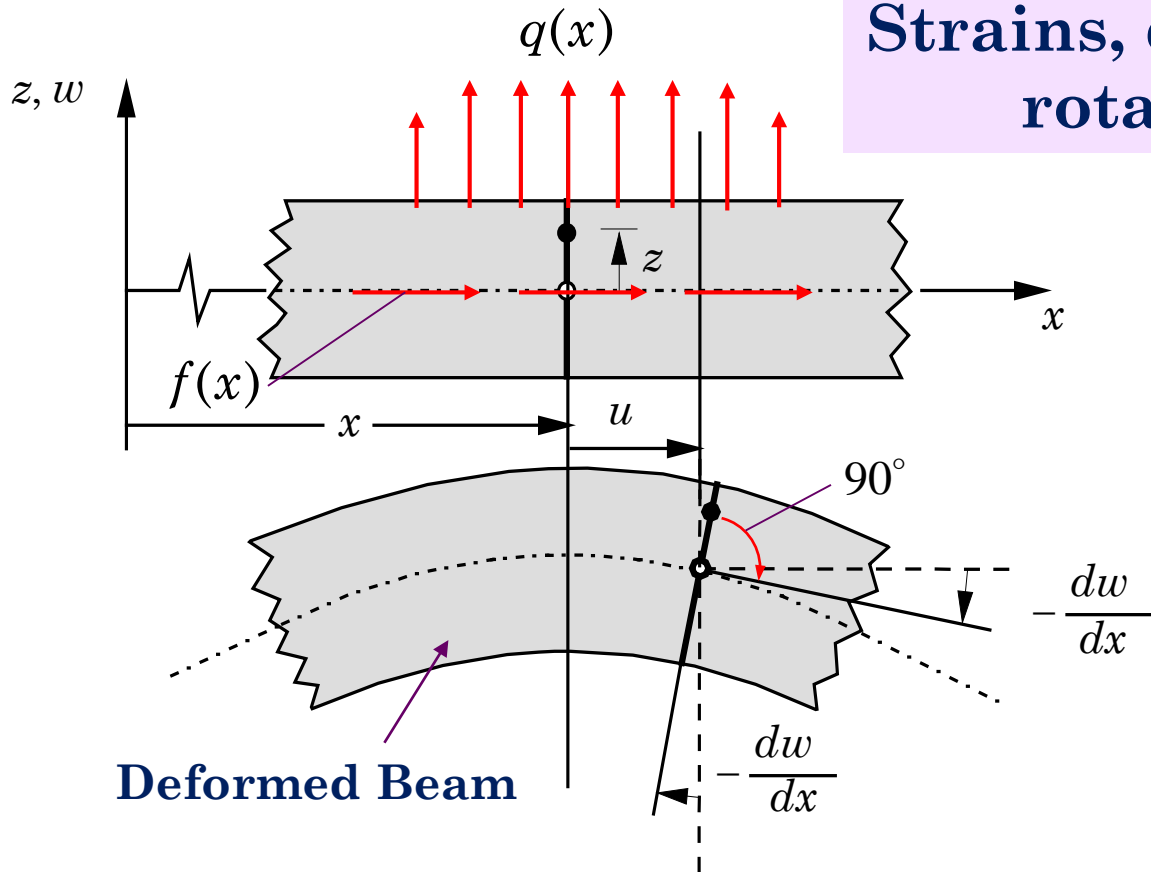
➤ **Governing Equations**

Timoshenko beam theory

➤ **Governing Equations**

KINEMATICS OF THE LINEARIZED EULER-BERNOULLI BEAM THEORY

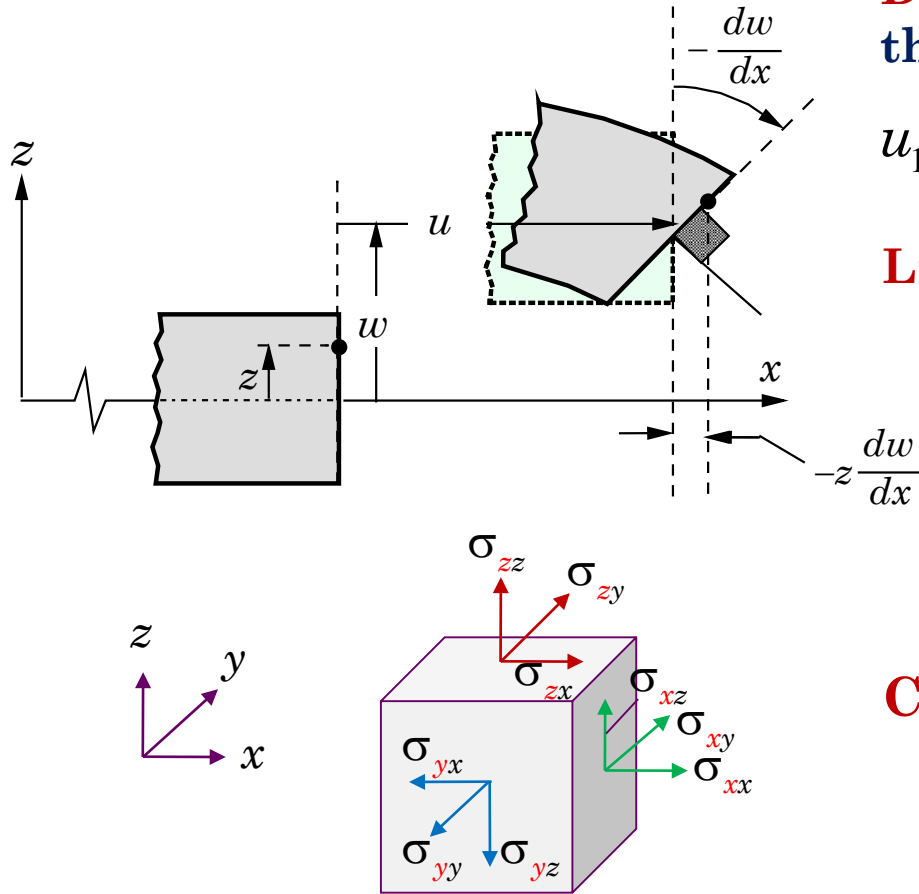
Strains, displacements, and rotations are small



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT) is based on the assumptions of (1) *straightness*, (2) *inextensibility*, and (3) *normality*

Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



Displacement field (constructed using the hypothesis)

$$u_1(x, z) = u - z \frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x)$$

Linear strains

$$\epsilon_{11} = \epsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} - z \frac{d^2w}{dx^2},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\frac{dw}{dx} + \frac{dw}{dx} = 0.$$

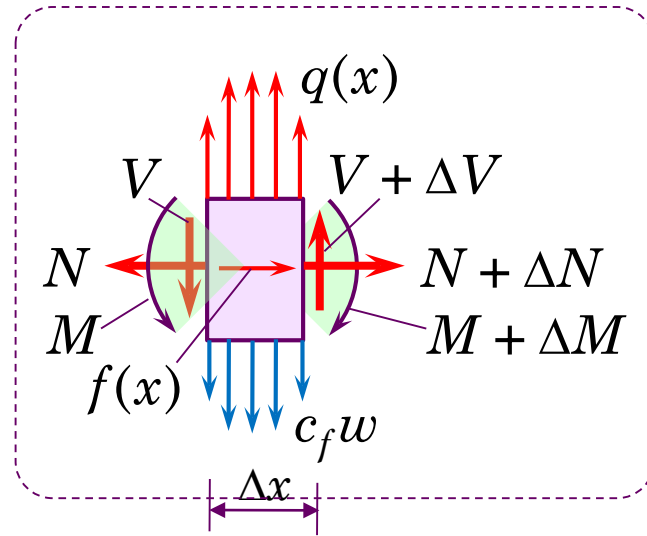
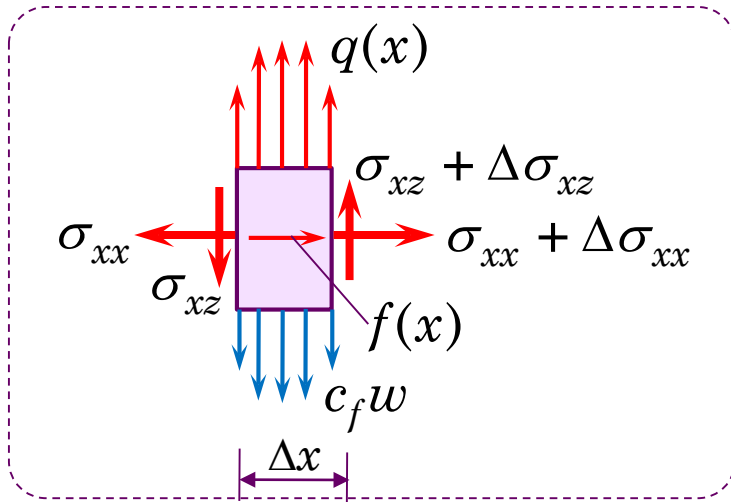
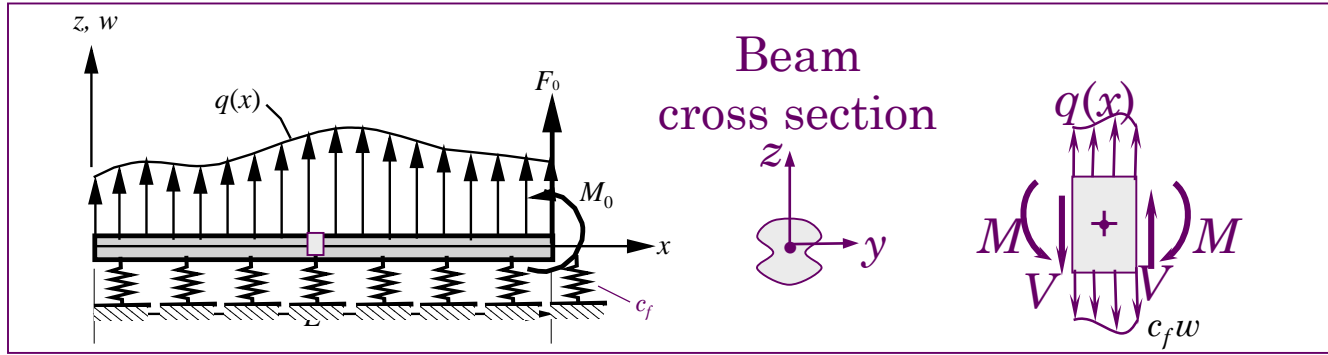
Constitutive relations

$$\sigma_{xx} = E \epsilon_{xx} = E \frac{du}{dx} - Ez \frac{d^2w}{dx^2},$$

$$\sigma_{xz} = G \gamma_{xz} = 0$$

Notation for stress components

Euler-Bernoulli Beam Theory



Definition of stress resultants

$$N = \int_A \sigma_{xx} dA, \quad M = \int_A \sigma_{xx} \cdot z dA, \quad V = \int_A \sigma_{xz} dA.$$

Euler-Bernoulli Beam Theory (Continued)

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + q - c_f w = 0$$

$$\int_A 1 \cdot dA = A, \quad \int_A z \cdot dA = 0, \quad \int_A z^2 \cdot dA = I$$

Stress resultants in terms of deflection

$$N = \int_A \sigma_{xx} dA = \int_A \left(E \frac{du}{dx} - Ez \frac{d^2w}{dx^2} \right) dA = EA \frac{du}{dx}$$

$$M = \int_A \sigma_{xx} \times z dA = \int_A \left(E \frac{du}{dx} - Ez \frac{d^2w}{dx^2} \right) z dA = -EI \frac{d^2w}{dx^2}$$

$$V = \frac{dM}{dx} = \frac{d}{dx} \left(-EI \frac{d^2w}{dx^2} \right)$$

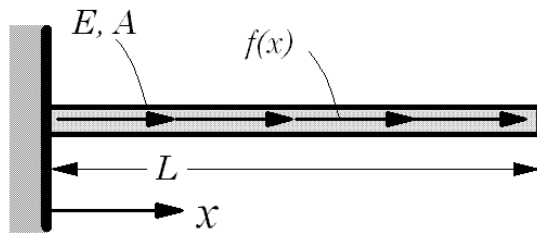
Euler-Bernoulli Beam Theory (Continued)

Governing equations in terms of the displacements

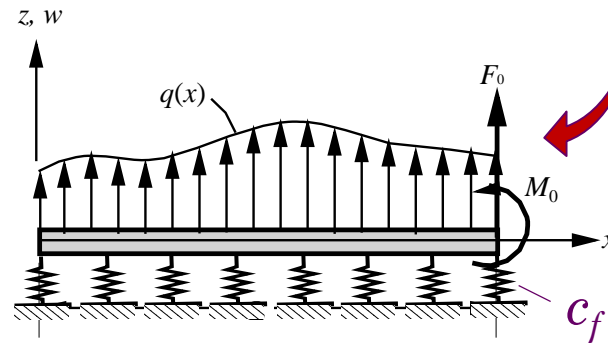
$$-\frac{d}{dx} \left(EA \frac{du}{dx} \right) - f = 0, \quad 0 < x < L$$

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 w}{dx^2} \right) + c_f w - q = 0, \quad 0 < x < L$$

Bars
u



Beams
w

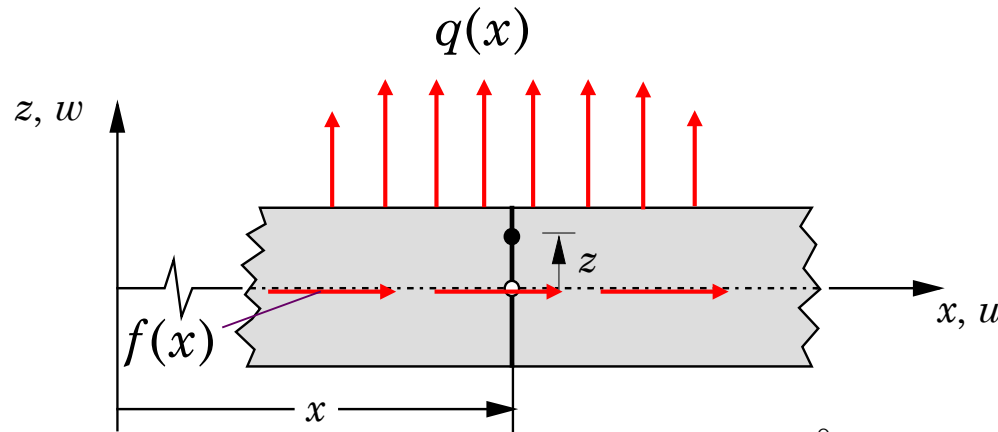


Axial deformation of a bar

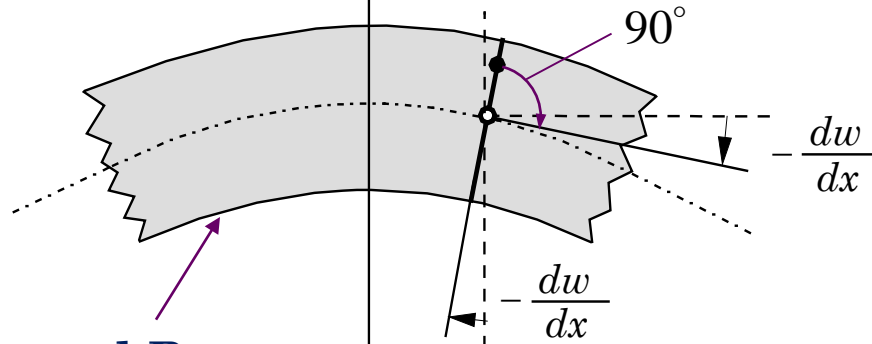
Bending of a beam

Axial displacement is uncoupled from transverse displacement

Kinematics of Timoshenko Beam Theory



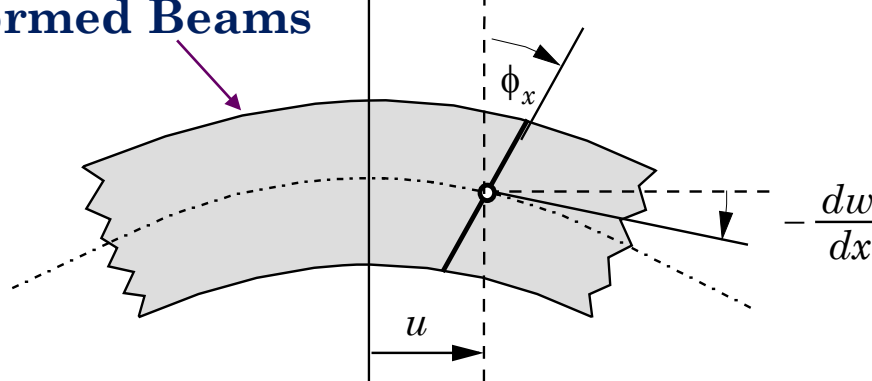
Undeformed Beam



Euler-Bernoulli Beam Theory (EBT)

Straightness, inextensibility, and normality

Deformed Beams



Timoshenko Beam Theory (TBT)

Straightness and inextensibility

Timoshenko Beam Theory

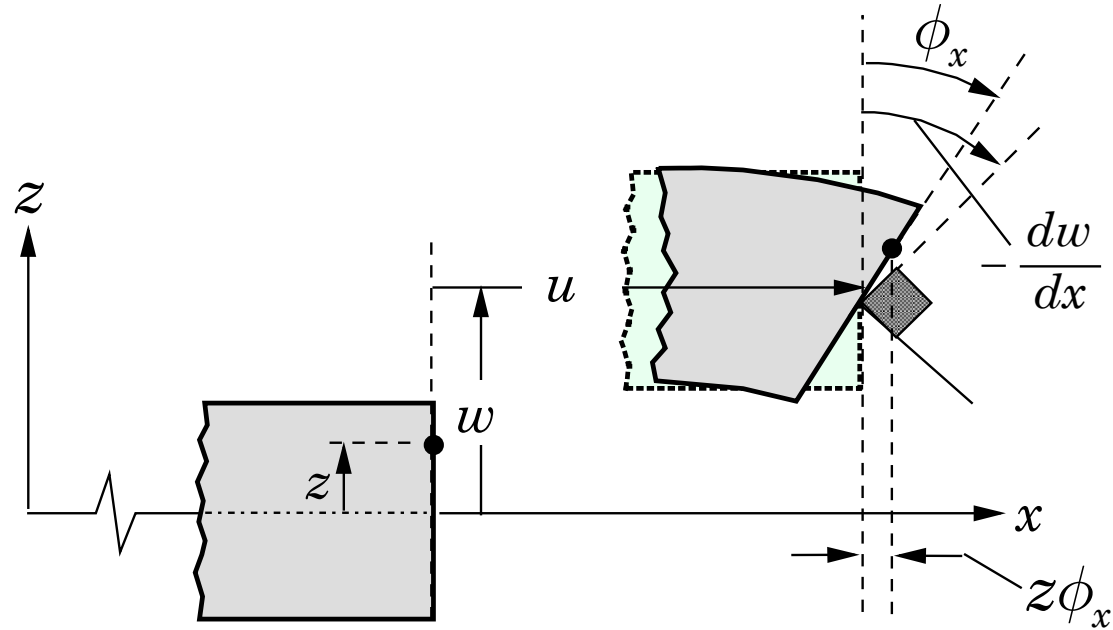
Kinematic Relations

$$u_1(x, z) = u(x) + z\phi_x(x),$$

$$u_2 = 0, \quad u_3(x, z) = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} + z \frac{d\phi_x}{dx},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \phi_x + \frac{dw}{dx}$$



Constitutive Equations

$$\sigma_{xx} = E \varepsilon_{xx} = E \left(\frac{du}{dx} + z \frac{d\phi_x}{dx} \right)$$

$$\sigma_{xz} = G \gamma_{xz} = G \left(\phi_x + \frac{dw}{dx} \right)$$

Timoshenko Beam Theory (Continued)

Equilibrium Equations $\frac{dN}{dx} + f = 0$, $-\frac{dV}{dx} - q + c_f w = 0$, $-\frac{dM}{dx} + V = 0$.

Beam Constitutive Equations

$$N = \int_A \sigma_{xx} dA = \int_A E \left(\frac{du}{dx} + z \frac{d\phi_x}{dx} \right) dA = EA \frac{du}{dx}$$

$$M = \int_A \sigma_{xx} z dA = \int_A E \left(\frac{du}{dx} + z \frac{d\phi}{dx} \right) z dA = EI \frac{d\phi}{dx}$$

$$V = K_s \int_A \sigma_{xz} dA = GK_s \left(\phi + \frac{dw}{dx} \right) \int_A dA = GAK_s \left(\phi + \frac{dw}{dx} \right)$$

Governing Equations in terms of the displacements

$$-\frac{d}{dx} \left[GAK_s \left(\phi + \frac{dw}{dx} \right) \right] + c_f w = q \quad (1)$$

$$-\frac{d}{dx} \left(EI \frac{d\phi}{dx} \right) + GAK_s \left(\phi + \frac{dw}{dx} \right) = 0 \quad (2)$$

SUMMARY

In these lectures we have covered the following topics with some examples:

- Continuum assumption
- Kinematics of deformation:
Introduced the Green-Lagrange strain tensor
- Kinetics: Defined Cauchy stress vector
- Cauchy's formula is derived and Cauchy stress tensor is introduced
- Balance of linear and angular momenta
- Developed the governing equations two beam theories:
 - Euler-Bernoulli beam theory
 - Timoshenko beam theory