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ENERGY PRINCIPLES AND VARIATIONAL METHODS IN APPLIED MECHANICS

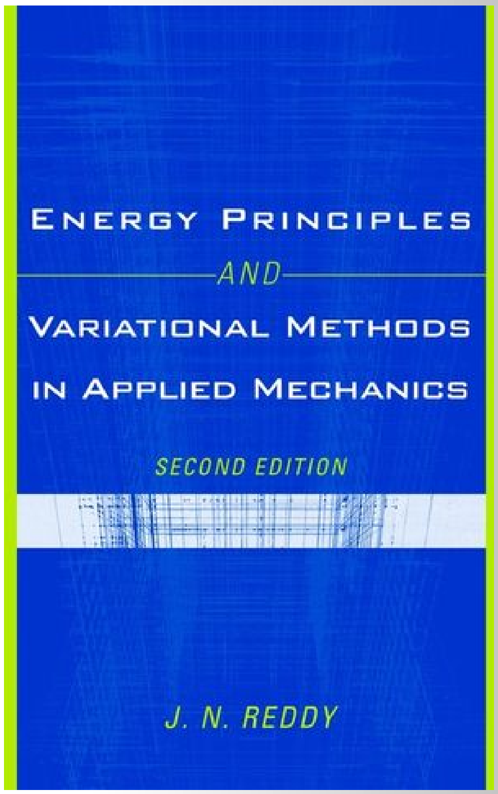
THIRD EDITION

WILEY

MEEN 618:
ENERGY AND VARIATIONAL METHODS

A REVIEW OF VECTORS AND TENSORS

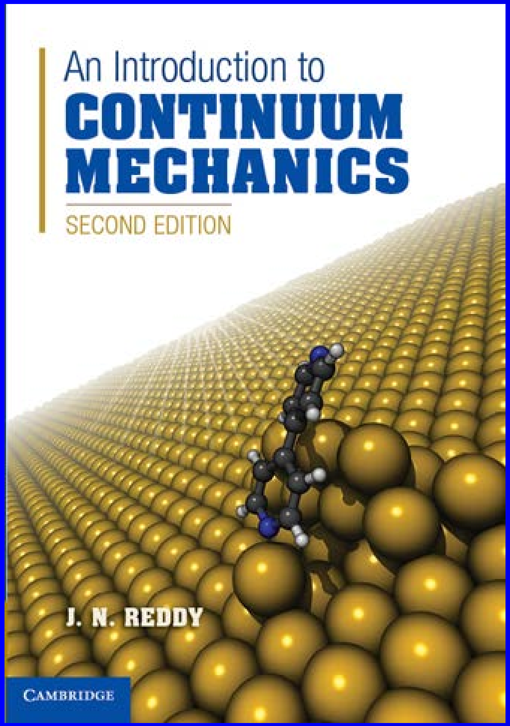
Read: Chapter 2



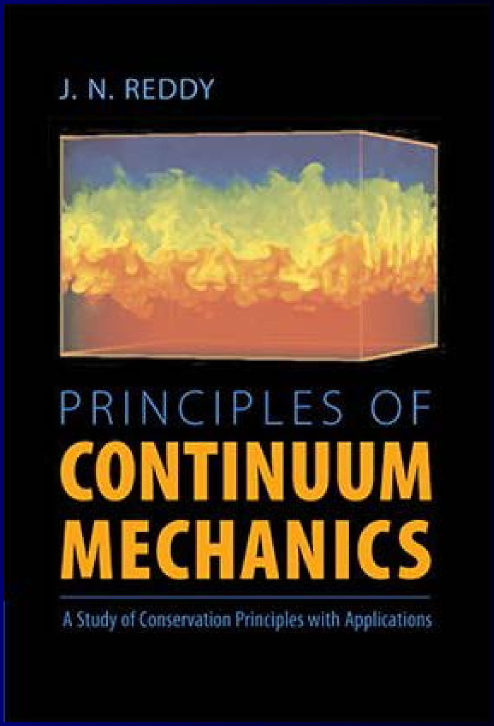
CONTENTS

- Physical vector
- Axioms of mathematical vector
- Scalar and vector products
- Triple products of vectors
- Components of vectors
- Summation convention
- Second-order tensors
- Transformation of components
- The del operator and calculus of vectors and tensors
- Cylindrical and spherical coordinates systems

A REVIEW OF VECTORS AND TENSORS



Much of the material included herein is taken from the instructor's two books exhibited here (both published by the Cambridge University Press)



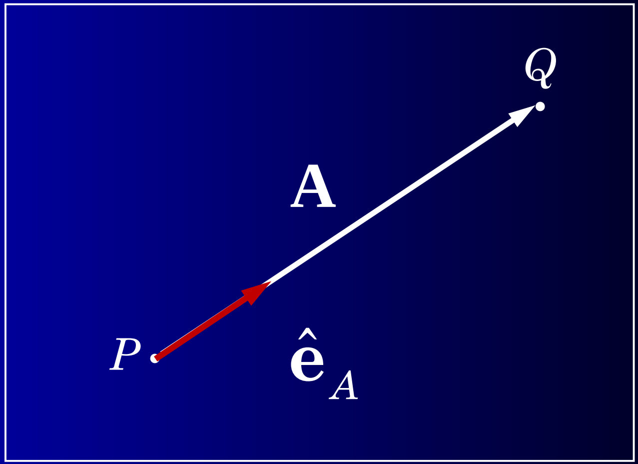
PHYSICAL VECTORS

Physical vector: A directed line segment with an arrow head.

Examples: *force, displacement, velocity, weight*

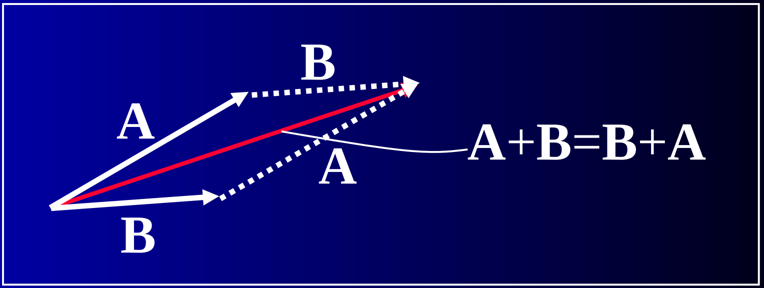
Unit vector along a given vector **A**:

The *unit vector*, $\hat{\mathbf{e}}_A \equiv \dfrac{\mathbf{A}}{A} \;\; (A \neq 0)$ is that vector which has the same direction as **A** but has a magnitude that is unity.



MATHEMATICAL VECTORS

Rules or Axioms



Vector addition:

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B}+\mathbf{A}$ (commutative)

(ii) $(\mathbf{A} + \mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ (associative)

(iii) $\mathbf{A}+\mathbf{0}=\mathbf{A}$ (zero vector)

(iv) $\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$ (negative vector)

Scalar multiplication of a vector:

(i) $\alpha(\beta\mathbf{A})= \alpha\beta(\mathbf{A})$ (associative)

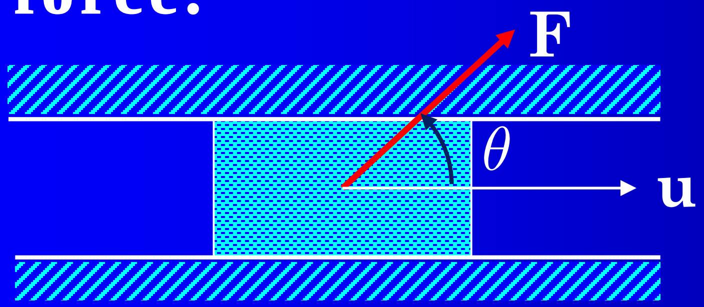
(ii) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A}+ \beta\mathbf{A}$ (distributive w.r.t. scalar addition)

(iii) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$ (distributive w.r.t. vector addition)

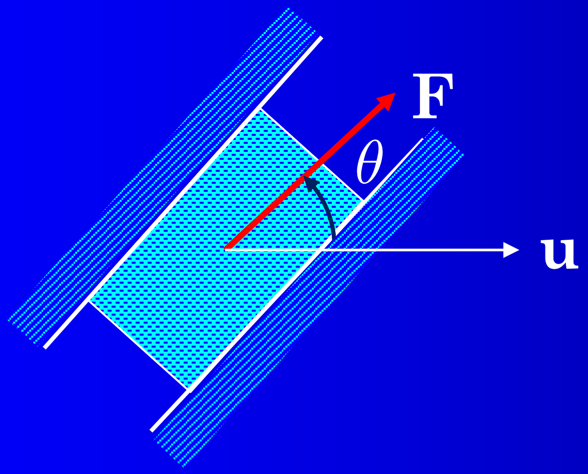
(iv) $1.\mathbf{A}=\mathbf{A}.1$

VECTORS (continued)

Work done **Magnitude of the force multiplied by the magnitude of the displacement in the direction of the force:**



$$\mathrm{WD} = |\mathbf{F}|\cos\theta \times |\mathbf{u}|$$

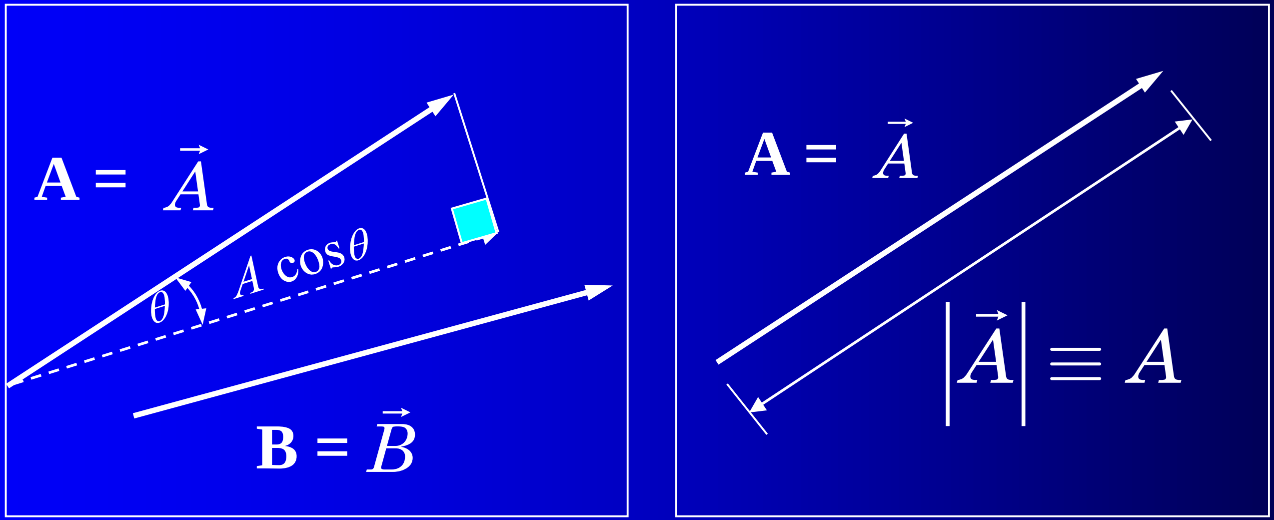


$$\mathrm{WD} = |\mathbf{F}| \times |\mathbf{u}|\cos\theta$$

VECTORS (continued)

Inner product (or scalar product) of two vectors is defined as

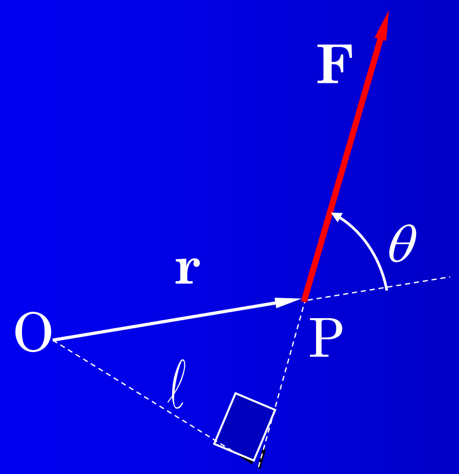
$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\;|\mathbf{B}|\cos\theta = AB\cos\theta$$

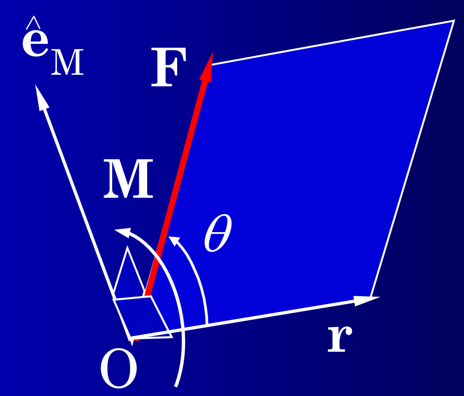


VECTORS (continued)

Moment of a force Magnitude of the force multiplied by the magnitude of the perpendicular distance to the action of the force:

$$|\mathbf{M}| = \ell F, \;\; \mathbf{M} = \left(r \sin\theta \times F \right) \hat{\mathbf{e}}_M = \mathbf{r} \times \mathbf{F}$$



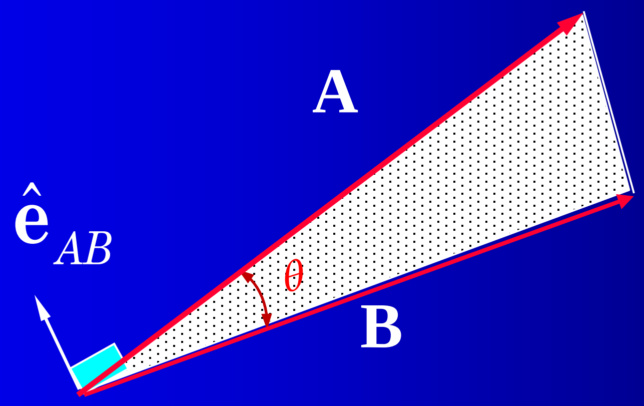


$$\ell = |\mathbf{r}| \sin\theta = r \sin\theta$$

VECTORS (continued)

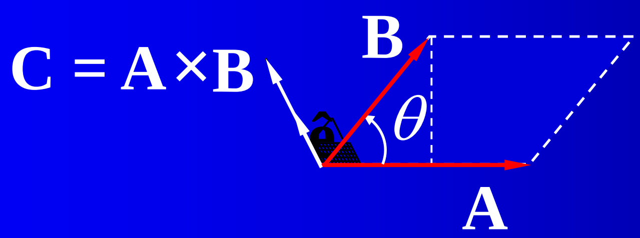
Vector product of two vectors is defined as

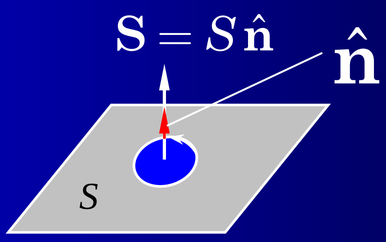
$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{\mathbf{e}}_{AB} = AB\sin\theta\,\hat{\mathbf{e}}_{AB}$$

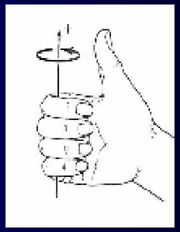


PLANE AREA AS A VECTOR

The magnitude of the vector **C** = **A** × **B** is equal to the area of the parallelogram formed by the vectors **A** and **B**.



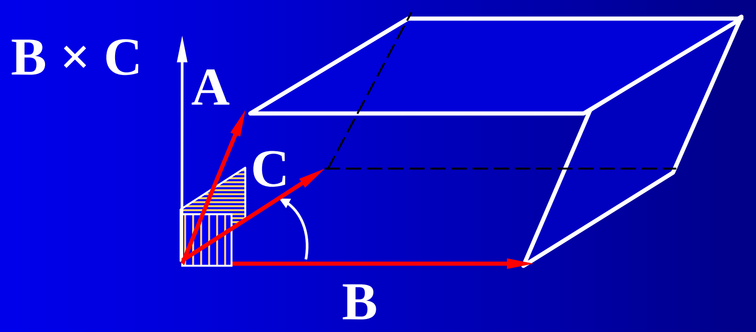




In fact, the vector **C** may be considered to represent <u>both</u> the *magnitude* and *the direction* of the product of **A** and **B**. Thus, a plane area in space may be looked upon as possessing a direction in addition to a magnitude, the directional character arising out of the need to specify an orientation of the plane area in space. Representation of an area as a vector has many uses in mechanics, as will be seen in the sequel.

SCALAR TRIPLE PRODUCT

The product **A** . (**B** × **C**) is a scalar and it is termed *the scalar triple product*. It can be seen from the figure that the product **A** . (**B** × **C**) , except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A**, **B**, and **C**.



A . (**B** × **C**)

EXERCISES ON VECTORS

1. If two vectors are such that $\mathbf{A} \cdot \mathbf{B} = 0$ what can we conclude?

2. If two vectors are such that $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ what can we conclude?

3. Prove that $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$

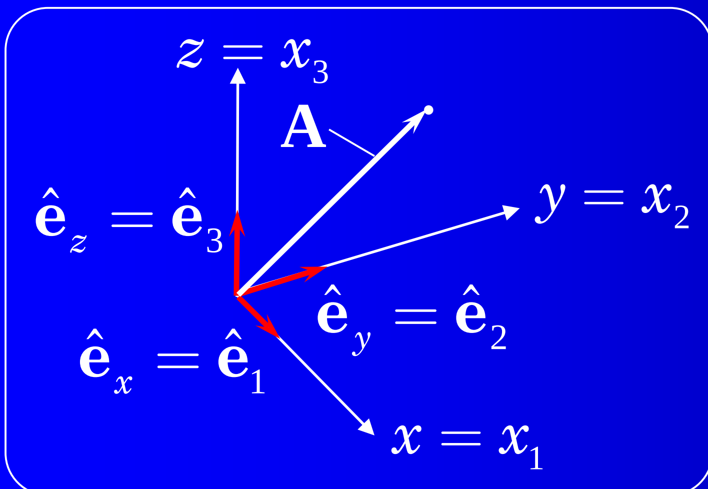
4. If three vectors are such that $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = 0$ what can we conclude?

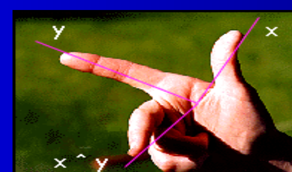
5. The velocity vector in a flow field is $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (m/ s). Determine (a) the velocity vector $\mathbf{v}_n$ normal to the plane $\mathbf{n} = 3\hat{\mathbf{i}}$-$4\hat{\mathbf{k}}$ passing through the point, (b) the angle between $\mathbf{v}$ and $\mathbf{v}_n$, (c) tangential velocity vector on the plane, and (d) The mass flow rate across the plane through an area $A = 0.15\ \mathrm{m}^2$ if the fluid density is $\rho = 10^3\ \mathrm{kg/m^3}$ and the flow is uniform.

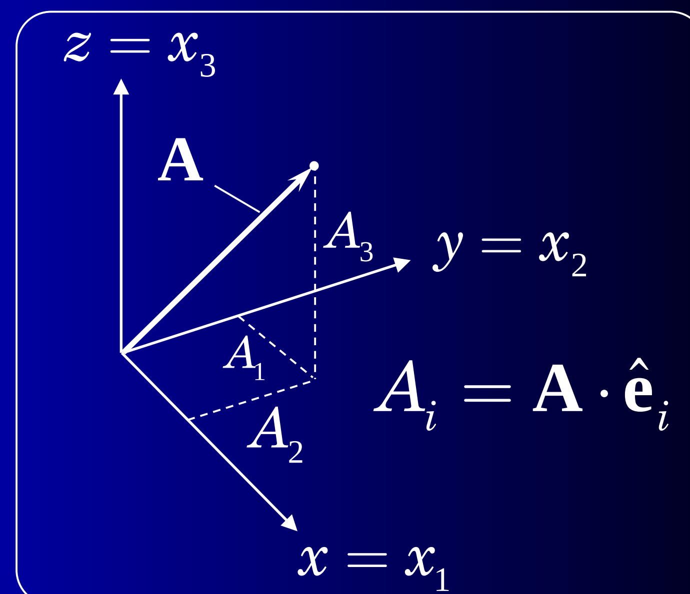
COMPONENTS OF VECTORS

Components of a vector

$$\begin{aligned}\mathbf{A} &= A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z \\ &= A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3 \\ \hat{\mathbf{n}} &= n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y + n_z\hat{\mathbf{e}}_z \\ &= n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2 + n_3\hat{\mathbf{e}}_3\end{aligned}$$







$$\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 = 0,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3 = 0,$$

$$\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 = 1,\quad \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3 = 0,\quad \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3 = 1,$$

$$\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 = 0,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3,$$

$$\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2$$

SUMMATION CONVENTION

Omit the summation sign and understand that a repeated index is to be summed over its range:

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

$$= \sum_{i=1}^{3} A_i\hat{\mathbf{e}}_i = A_i\hat{\mathbf{e}}_i \text{ (summation convention)}$$

Dummy index

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_i\right)\hat{\mathbf{e}}_i = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_j\right)\hat{\mathbf{e}}_j$$

Dummy indices

Scalar product

$$\mathbf{A}\cdot\mathbf{B} = \left(A_i\hat{\mathbf{e}}_i\right)\cdot\left(B_j\hat{\mathbf{e}}_j\right) = A_iB_j\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right) = A_iB_j\delta_{ij} = A_iB_i$$

$$\delta_{ij} \equiv \left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right) = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

SUMMATION CONVENTION (continued)

Vector product

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{e}}_{AB}$$
$$= \left(A_i\hat{\mathbf{e}}_i\right)\times\left(B_j\hat{\mathbf{e}}_j\right) = A_iB_j\left(\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\right)$$
$$= A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j = \varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\varepsilon_{ijk} \equiv \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k = \begin{cases}\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)\\ 0, \text{ if any two indices are the same}\\ 1, \textit{if } i\neq j\neq k, \text{ and they permute}\\ \quad\text{in a natural order}\\ -1, \textit{if } i\neq j\neq k, \text{and they permute}\\ \quad\textit{opposite} \text{ to a natural order}\end{cases}$$

SUMMATION CONVENTION (continued)

Contraction of indices:

The Kronecker delta δ_{ij} modifies (or contracts) the subscripts in the coefficients of an expression in which it appears:

$$A_i\delta_{ij} = A_j, \quad A_iB_j\delta_{ij} = A_iB_i = A_jB_j, \quad \delta_{ij}\delta_{ik} = \delta_{jk}$$

Correct expressions:

$$F_i = A_iB_jC_j, \; G_k = H_k(2 - 3A_iB_i) + P_jQ_jF_k$$

Free indices

Incorrect expressions:

$$A_i = B_jC_k, \; A_i = B_j \text{ and } F_k = A_iB_jC_k$$

SUMMATION CONVENTION (continued)

One must be careful when substituting a quantity with an index into an expression with indices or solving for one quantity with index in terms of the others with indices in an equation. For example, consider the equations

$$p_i = a_i b_j c_j \text{ and } c_k = d_i e_i q_k$$

It is correct to write

$$a_i = \frac{p_i}{b_j c_j}$$

but it is <u>incorrect</u> to write

$$b_j c_j = \frac{p_i}{a_i}$$

which has a totally different meaning

$$b_j c_j = \frac{p_i}{a_i} = \frac{p_1}{a_1} + \frac{p_2}{a_2} + \frac{p_3}{a_3}$$

SUMMATION CONVENTION (continued)

The permutation symbol and the Kronecker delta prove to be very useful in establishing vector identities. Since a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to prove it in one coordinate system. The following identity is useful:

ε-δ Identity:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

EXERCISES ON INDEX NOTATION

Exercise-1: Check which one of the following expressions are valid:

(a) $a_m b_s = c_m(d_r - f_r)$; (b) $a_m b_s = c_m(d_s - f_s)$

(c) $a_i = b_j c_i(d_i - f_i)$; (d) $x_m x_m = r^2$

(e) $a_i = 3$; (f) $\delta_{ij}\,\delta_{jk}\,\delta_{ki} = ?$

Exercise-2: Prove $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \varepsilon_{ijk}A_iB_jC_k = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Exercise-3: Simplify the expression $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$

Exercise-4: Simplify the expression $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$

Exercise-5: Rewrite the expression $\varepsilon_{mni}\,A_iB_jC_mD_n\hat{\mathbf{e}}_j$ in vector form

SECOND-ORDER TENSORS

A second-order tensor is one that has two basis vectors standing next to each other, and they satisfy the same rules as those of a vector (hence, mathematically, tensors are also called vectors). A second-order tensor and its *transpose* can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{S} = S_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ji}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i; \qquad \mathbf{S}^{\mathrm{T}} = S_{ji}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ij}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i$$

A second-order tensor is symmetric only if

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \Leftrightarrow S_{ij} = S_{ji}$$

Second-order identity tensor has the form

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

SECOND-ORDER TENSORS

We note that $\mathbf{S}\cdot\mathbf{T} \neq \mathbf{T}\cdot\mathbf{S}$ (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors) because

$$\mathbf{S}\cdot\mathbf{T} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(T_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{jl}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

$$\mathbf{T}\cdot\mathbf{S} = \left(T_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(S_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = T_{ij}S_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{jl}T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

We also note that (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors and $\mathbf{A}$ is a vector)

$$\mathbf{S}\times\mathbf{T} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(T_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{kl}\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p\hat{\mathbf{e}}_l$$

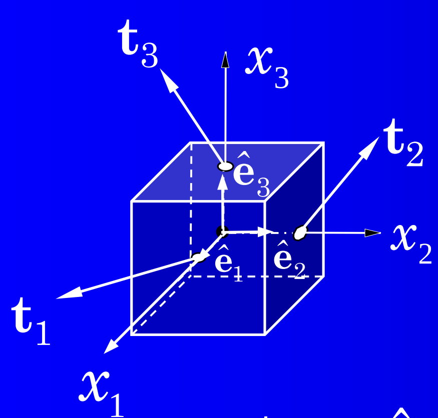
$$\mathbf{S}\cdot\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right) = S_{ij}A_j\hat{\mathbf{e}}_i$$

$$\mathbf{S}\times\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right) = S_{ij}A_k\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p$$

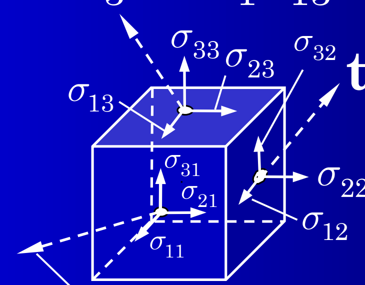
CAUCHY STRESS TENSOR

Stress tensor is a good example of a second-order tensor. The two basis vectors represent the direction and the plane on which they act. The Cauchy stress tensor is defined by the Cauchy formula (to be established):

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \quad \text{or} \quad t_i = \sigma_{ij} n_j$$



$$\mathbf{t}_3 = \hat{\mathbf{e}}_1\sigma_{13} + \hat{\mathbf{e}}_2\sigma_{23} + \hat{\mathbf{e}}_3\sigma_{33}$$



$$\mathbf{t}_2 = \hat{\mathbf{e}}_1\sigma_{12} + \hat{\mathbf{e}}_2\sigma_{22} + \hat{\mathbf{e}}_3\sigma_{32}$$

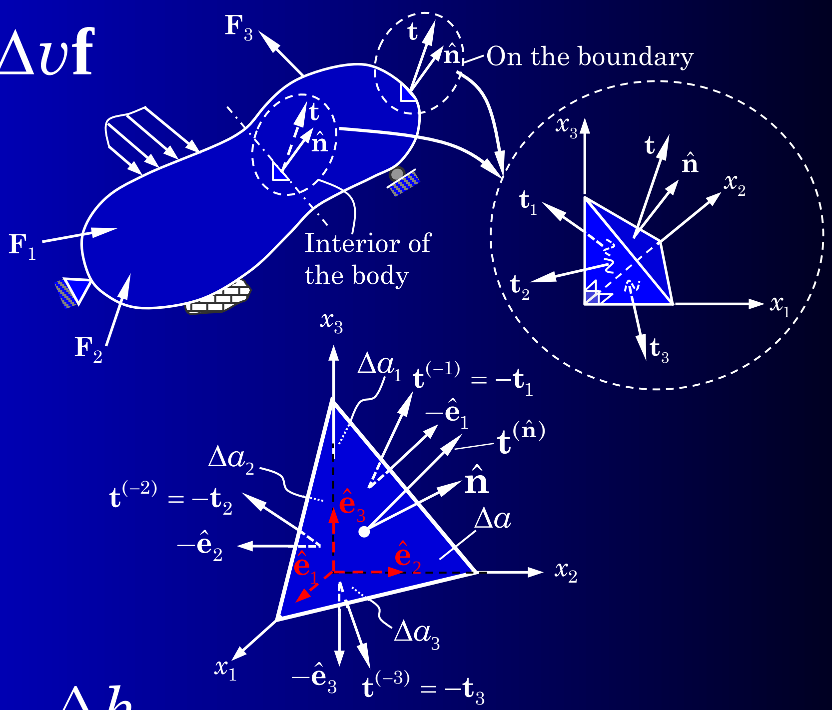
$$\mathbf{t}_1 = \hat{\mathbf{e}}_1\sigma_{11} + \hat{\mathbf{e}}_2\sigma_{21} + \hat{\mathbf{e}}_3\sigma_{31}$$

$$\mathbf{t}_i = \hat{\mathbf{e}}_j \sigma_{ji} = \sigma_{ji} \hat{\mathbf{e}}_j$$

$$\sigma = \mathbf{t}_i \hat{\mathbf{e}}_i = \sigma_{ji} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i = \sigma_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$$

CAUCHY'S FORMULA

$$\mathbf{t}\Delta a - \mathbf{t}_1\Delta a_1 - \mathbf{t}_2\Delta a_2 - \mathbf{t}_3\Delta a_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$



$$\Delta a\hat{\mathbf{n}} - \Delta a_1\hat{\mathbf{e}}_1 - \Delta a_2\hat{\mathbf{e}}_2 - \Delta a_3\hat{\mathbf{e}}_3 = \mathbf{0}$$

$$\Delta a_1 = (\hat{\mathbf{e}}_1\cdot\hat{\mathbf{n}})\Delta a, \quad \Delta a_2 = (\hat{\mathbf{e}}_2\cdot\hat{\mathbf{n}})\Delta a$$

$$\Delta a_3 = (\hat{\mathbf{e}}_3\cdot\hat{\mathbf{n}})\Delta a, \quad \Delta v = \frac{\Delta h}{3}\Delta a$$

$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3\cdot\hat{\mathbf{n}}) + \rho\frac{\Delta h}{3}(\mathbf{a}-\mathbf{f})$$

As $\Delta h \to 0$, we obtain

$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3\cdot\hat{\mathbf{n}}) = \mathbf{t}_i\hat{\mathbf{e}}_i\cdot\hat{\mathbf{n}} \equiv \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$$

HIGHER-ORDER TENSORS

A n^{th}-order tensor is one that has n basis vectors standing next to each other, and they satisfy the same rules as those of a vector. A n^{th}-order tensor **T** can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{T} = T_{\underbrace{ijk\ldots p}_{n \text{ subs}}} \underbrace{\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \cdots \hat{\mathbf{e}}_p}_{n \text{ base vectors}};$$

The permutation tensor is a <u>third-order</u> tensor

$$\boldsymbol{\varepsilon} = \varepsilon_{ijk} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k$$

The elasticity tensor is a <u>fourth-order</u> tensor

$$\mathbf{C} = C_{ijkl} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \hat{\mathbf{e}}_l$$

TRANSFORMATION OF TENSOR COMPONENTS

A second-order Cartesian tensor **S** (i.e., tensor with Cartesian components) may be represented in barred $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and unbarred (x_1, x_2, x_3) Cartesian coordinate systems as

$$\mathbf{S} = s_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = \bar{s}_{mn}\hat{\bar{\mathbf{e}}}_m\hat{\bar{\mathbf{e}}}_n$$

The unit base vectors in the unbarred and barred systems are related by

$$\hat{\mathbf{e}}_j = \ell_{ij}\hat{\bar{\mathbf{e}}}_i \quad \text{and} \quad \hat{\bar{\mathbf{e}}}_i = \ell_{ij}\hat{\mathbf{e}}_j, \quad \ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j$$

Thus the components of a second-order tensor transform according to

$$\boxed{\bar{s}_{ij} = \ell_{im}\ell_{jn}s_{mn}}$$

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

"Del" operator: $\nabla \equiv \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3}$

"Laplace" operator:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

"Gradient" operation:

$$\nabla F \equiv \hat{\mathbf{e}}_i \frac{\partial F}{\partial x_i}, \text{ where } F \text{ is a scalar function}$$

Grad F defines both the direction and magnitude of the maximum rate of increase of F at any point.

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

$$\nabla F = \hat{\mathbf{n}} \frac{\partial F}{\partial n}, \text{ where } \hat{\mathbf{n}} \text{ is a unit vector normal to the surface } F = \text{constant}$$

$$\text{We also have } \hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} \text{ and } \frac{\partial F}{\partial n} = \hat{\mathbf{n}} \cdot \nabla F$$

"Divergence" operation:

$$\nabla \cdot \mathbf{G} \equiv \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\hat{\mathbf{e}}_j G_j \right) = \frac{\partial G_i}{\partial x_i}, \text{ where } \mathbf{G} \text{ is a } \textit{vector} \text{ function}$$

The divergence of a vector function represents the volume density of the outward flux of the vector field.

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

"Curl" operation:

$$\nabla \times \mathbf{G} \equiv \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \times \left(\hat{\mathbf{e}}_j G_j \right) = \varepsilon_{ijk} \frac{\partial G_j}{\partial x_i} \hat{\mathbf{e}}_k,$$

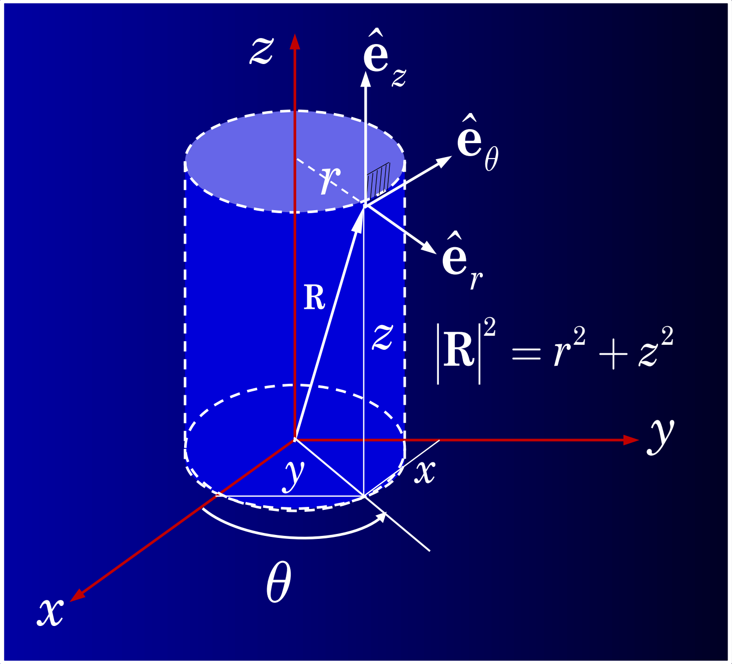
where $\mathbf{G}$ is a *vector* function.

The curl of a vector function represents its rotation. If the vector field is the velocity of a fluid, curl of the velocity represents the rotation of the fluid at the point.

CYLINDRICAL COORDINATE SYSTEM

$$\begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$



$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$

"Del" operator in cylindrical coordinates

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

CYLINDRICAL COORDINATE SYSTEM

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\left[\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r\frac{\partial A_z}{\partial z}\right]$$

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\right]$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{e}}_z$$

$$\nabla \mathbf{A} = \frac{\partial A_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{\partial A_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - A_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \frac{\partial A_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \frac{\partial A_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r$$

$$+ \frac{1}{r}\left(A_r + \frac{\partial A_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial A_z}{\partial \theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial A_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial A_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z$$

Here **A** is a vector:

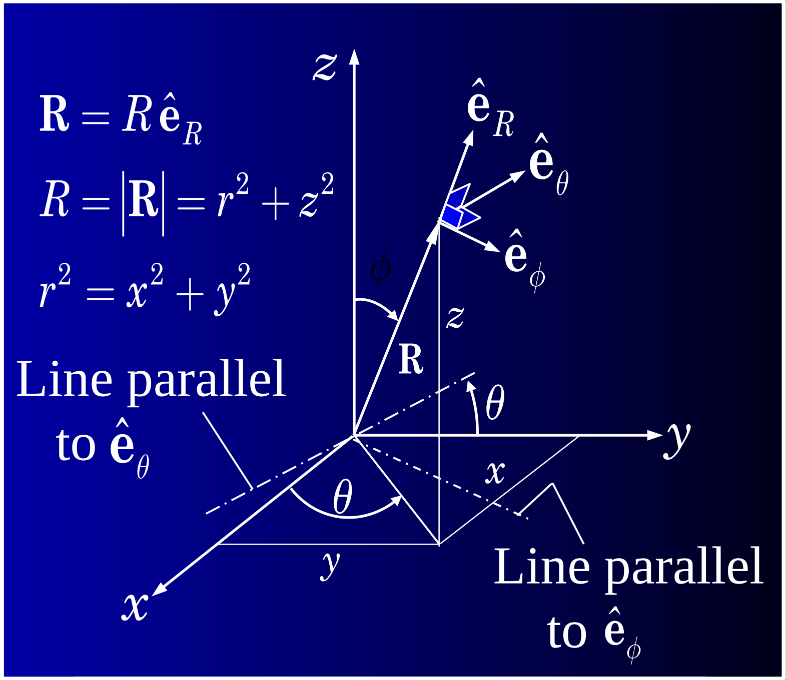
$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$$

Verify these relations to yourself

SPHERICAL COORDINATE SYSTEM

$$\begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix}$$



$$\mathbf{A} = A_R\hat{\mathbf{e}}_R + A_\phi\hat{\mathbf{e}}_\phi + A_\theta\hat{\mathbf{e}}_\theta$$

$$\frac{\partial\hat{\mathbf{e}}_R}{\partial\phi} = \hat{\mathbf{e}}_\phi, \quad \frac{\partial\hat{\mathbf{e}}_R}{\partial\theta} = \sin\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_R, \quad \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\theta} = \cos\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\sin\phi\,\hat{\mathbf{e}}_R - \cos\phi\,\hat{\mathbf{e}}_\phi$$

"Del" operator

$$\nabla = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta}$$

SPHERICAL COORDINATE SYSTEM

$$\nabla^2 = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right]$$

$$\nabla\cdot\mathbf{A} = \frac{2A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{A} = \frac{1}{R\sin\phi}\left[\frac{\partial(\sin\phi A_\theta)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_R + \left[\frac{1}{R\sin\phi}\frac{\partial A_R}{\partial\theta} - \frac{1}{R}\frac{\partial(RA_\theta)}{\partial R}\right]\hat{\mathbf{e}}_\phi + \frac{1}{R}\left[\frac{\partial(RA_\phi)}{\partial R} - \frac{\partial A_R}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$$

$$\nabla\mathbf{A} = \frac{\partial A_R}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_R + \frac{\partial A_\phi}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\phi + \frac{1}{R}\left(\frac{\partial A_R}{\partial\phi} - A_\phi\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_R + \frac{\partial A_\theta}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_R}{\partial\theta} - A_\theta\sin\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_R$$
$$+ \frac{1}{R}\left(A_R + \frac{\partial A_\phi}{\partial\phi}\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial A_\theta}{\partial\phi}\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_\phi}{\partial\theta} - A_\theta\cos\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\phi$$
$$+ \frac{1}{R\sin\phi}\left(A_R\sin\phi + A_\phi\cos\phi + \frac{\partial A_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$$

EXERCISES ON VECTOR CALCULUS

Establish the following identities (using rectangular Cartesian components and index notation):

1. $\nabla(r) = \dfrac{\mathbf{r}}{r}$

2. $\nabla(r^n) = nr^{n-2}\mathbf{r}$

3. $\nabla \times (\nabla F) = \mathbf{0}$

4. $\nabla \cdot (\nabla F \times \nabla G) = 0$

5. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

6. $\nabla \cdot \left(\mathbf{A} \times \mathbf{B}\right) = \nabla \times \mathbf{A} \cdot \mathbf{B} - \nabla \times \mathbf{B} \cdot \mathbf{A}$

7. $\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A} \cdot \nabla \mathbf{A}$

Exercise: Check appropriate box

f – scalar; $\mathbf{F}$ – vector

Quantity	Vector	Scalar	Nonsense
$\nabla \times (\nabla f)$	☐	☐	☐
$\nabla \cdot (\nabla \times \mathbf{F})$	☐	☐	☐
$\nabla \cdot (\nabla \times f)$	☐	☐	☐
$\nabla \times (\nabla \times \mathbf{F})$	☐	☐	☐
$\nabla (\nabla \cdot \mathbf{F})$	☐	☐	☐
$\nabla \times (\nabla \times f)$	☐	☐	☐

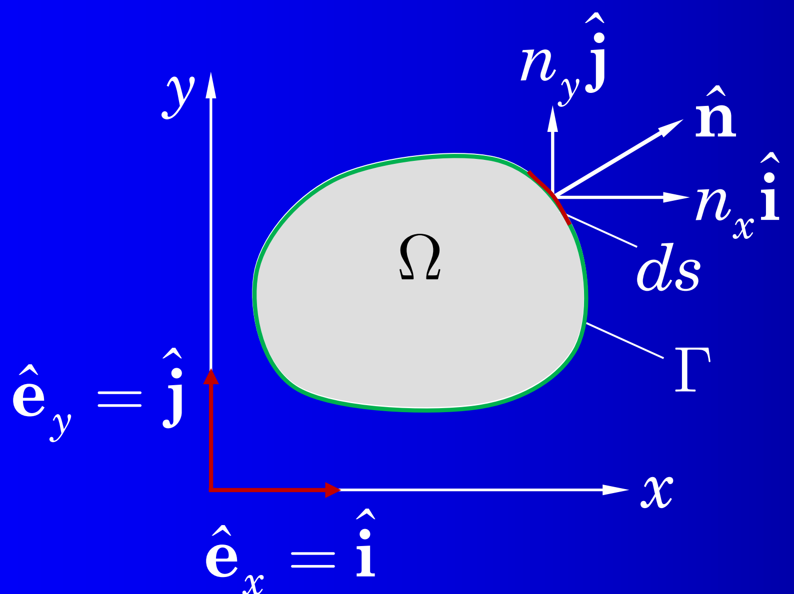
INTEGRAL THEOREMS

involving the del operator

$$\int_\Omega \nabla\phi \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\phi \, ds \quad \text{(Gradient theorem)}$$

$$\int_\Omega \nabla\cdot\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{A} \, ds \quad \text{(Divergence theorem)}$$

$$\int_\Omega \nabla\times\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\times\mathbf{A} \, ds \quad \text{(Curl theorem)}$$



$$\begin{aligned}\hat{\mathbf{n}} &= n_x\,\hat{\mathbf{i}} + n_y\,\hat{\mathbf{j}}\\ &= n_x\,\hat{\mathbf{e}}_x + n_y\,\hat{\mathbf{e}}_y\\ &= n_1\,\hat{\mathbf{e}}_1 + n_2\,\hat{\mathbf{e}}_2\end{aligned}$$

EXERCISES ON INTEGRAL IDENTITIES

Establish the following identities using the integral theorems:

1. volume $= \frac{1}{6}\oint_\Gamma \nabla(r^2)\cdot\hat{\mathbf{n}}\, d\Gamma = \frac{1}{3}\oint_\Gamma \mathbf{r}\cdot\hat{\mathbf{n}}\, d\Gamma$

2. $\int_\Omega \nabla^2\phi\, d\Omega = \oint_\Gamma \frac{\partial\phi}{\partial n}\, d\Gamma$

3. $\int_\Omega \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right) d\Omega = \oint_\Gamma \phi\frac{\partial\psi}{\partial n}\, d\Gamma$

4. $\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\Omega = \oint_\Gamma \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\Gamma$

5. $\int_\Omega \left(\phi\nabla^4\psi - \nabla^2\phi\nabla^2\psi\right) d\Omega = \oint_\Gamma \left[\phi\frac{\partial}{\partial n}(\nabla^2\psi) - \nabla^2\psi\frac{\partial\phi}{\partial n}\right] d\Gamma$