

J.T. Oden J. N. Reddy

# Variational Methods in Theoretical Mechanics

Second Edition

Universitext



Springer-Verlag  
Berlin Heidelberg New York

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Second Edition

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and Applied Mathematics  
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**Springer-Verlag, Berlin, 1983**

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# Preface to the First Edition

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This is a textbook written for use in a graduate-level course for students of mechanics and engineering science. It is designed to cover the essential features of modern variational methods and to demonstrate how a number of basic mathematical concepts can be used to produce a unified theory of variational mechanics. As prerequisite to using this text, we assume that the student is equipped with an introductory course in functional analysis at a level roughly equal to that covered, for example, in Kolmogorov and Fomin (*Functional Analysis, Vol. I*, Graylock, Rochester, 1957) and possibly a graduate-level course in continuum mechanics. Numerous references to supplementary material are listed throughout the book.

We are indebted to Professor Jim Douglas of the University of Chicago, who read an earlier version of the manuscript and whose detailed suggestions were extremely helpful in preparing the final draft. We also gratefully acknowledge that much of our own research work on variational theory was supported by the U.S. Air Force Office of Scientific Research. We are indebted to Mr. Ming-Goei Sheu for help in proofreading. Finally, we wish to express thanks to Mrs. Marilyn Gude for her excellent and pains-taking job of typing the manuscript.

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