TUTORIAL/QUIZ 3

Exercise Problem 1

Assuming that Young's modulus varies with z, (i.e., $E = E(z)$), derive the governing equations of the Euler-Bernoulli beam with the von Karman nonlinearity in terms of the displacements u and w.

Exercise Problem 2

Assuming that Young's modulus varies with z, (i.e., $E = E(z)$), derive the governing equations of the Timoshenko beam with the von Karman nonlinearity in terms of the generalized displacements (u, w, ϕ_x).

Exercise Problem 3:

Derive the governing equations of functionally graded plates using the first-order plate theory with the von Karman nonlinearity.

Exercise Problem 4

Determine the Navier solution for the case in which Young's modulus varies with z, (i.e., $E = E(z)$). Use the Euler-Bernoulli beam theory.

Exercise Problem 5

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Determine the Navier solution for the case in which Young's modulus varies with z, (i.e., $E = E(z)$). Use the Timoshenko beam theory.

EULER-BERNOULLI BEAM THEORY
Equilibrium Equations (nonlinear)

Equilibrium equations

$$\delta u: \quad -\left(\frac{dN_{xx}}{dx} + f\right) = 0,$$

$$\delta w: \quad -\frac{d^2 M_{xx}}{dx^2} - \frac{d}{dx}\left(N_{xx}\frac{dw}{dx}\right) - q = 0$$

Boundary conditions

$$\text{Specify}: \quad u \quad \text{or} \quad N_{xx}$$

$$w \quad \text{or} \quad V_{xz} \equiv \left(N_{xx}\frac{dw}{dx} + \frac{dM_{xx}}{dx}\right)$$

$$\theta_x \equiv -\frac{dw}{dx} \quad \text{or} \quad M_{xx}$$

Euler-Bernoulli Beam Theory (EBT)
Constitutive Relations (for FGM)

$$N_{xx} = \int_A \sigma_{xx}\, dA = \int_A E\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2w}{dx^2}\right) dA$$

$$= A_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - B_{xx}\frac{d^2w}{dx^2}$$

$$M_{xx} = \int_A \sigma_{xx}\cdot z\, dA = \int_A E\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2w}{dx^2}\right) z\, dA$$

$$= B_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - D_{xx}\frac{d^2w}{dx^2}$$

$$A_{xx} = \int_A E(z,T)\, dA,\;\; B_{xx} = \int_A zE(z,T)\, dA,\;\; D_{xx} = \int_A z^2E(z,T)\, dA$$

EULER-BERNOULLI BEAM THEORY

Equations of Equilibrium in Terms of Displacements (nonlinear FGM case)

$$-\frac{d}{dx}\left(A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]-B_{xx}\frac{d^2w}{dx^2}\right)-f=0$$

$$\frac{d^2}{dx^2}\left\{B_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]-D_{xx}\frac{d^2w}{dx^2}\right\}$$

$$-\frac{d}{dx}\left(\frac{dw}{dx}\left\{A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]-B_{xx}\frac{d^2w}{dx^2}\right\}\right)-q=0$$

TIMOSHENKO BEAM THEORY
Equations of Equilibrium (nonlinear)

Equilibrium equations

$$\delta u: \quad -\left(\frac{dN_{xx}}{dx} + f\right) = 0$$

$$\delta w: \quad -\frac{dQ_{xz}}{dx} - \frac{d}{dx}\left(N_{xx}\frac{dw}{dx}\right) - q = 0$$

$$\delta \phi_x: \quad -\frac{dM_{xx}}{dx} + Q_{xz} = 0$$

Boundary conditions

Specify: u or N_{xx}

w or $V_{xz} \equiv \left(N_{xx}\frac{dw}{dx} + Q_{xz}\right)$

ϕ_x or M_{xx}

TIMOSHENKO BEAM THEORY

Constitutive Relations (nonlinear FGM)

$$N_{xx} = \int_A \sigma_{xx}\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] dA$$

$$= A_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{xx}\frac{d\phi_x}{dx}$$

$$M_{xx} = \int_A \sigma_{xx} z\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] z\, dA$$

$$= B_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{xx}\frac{d\phi_x}{dx}$$

$$Q_{xz} = K_s \int_A \sigma_{xz}\, dA = \int_A G K_s\left(\phi_x + \frac{dw}{dx}\right) dA = S_{xz}\left(\phi_x + \frac{dw}{dx}\right)$$

TIMOSHENKO BEAM THEORY

Governing equations (nonlinear) in terms of the generalized displacements

$$-\frac{d}{dx}\left\{A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+B_{xx}\frac{d\phi_x}{dx}\right\}-f=0$$

$$-\frac{d}{dx}\left[S_{xz}\left(\phi_x+\frac{dw}{dx}\right)\right]-\frac{d}{dx}\left[\frac{dw}{dx}\left\{A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+B_{xx}\frac{d\phi_x}{dx}\right\}\right]-q=0$$

$$-\frac{d}{dx}\left\{B_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+D_{xx}\frac{d\phi_x}{dx}\right\}+S_{xz}\left(\phi_x+\frac{dw}{dx}\right)=0$$

KINEMATICS OF THE FSDT

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$

Strain Field with the von Karman nonlinearity

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{yz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial w}{\partial y} + \phi_y \\ \frac{\partial w}{\partial x} + \phi_x \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

CONSTITUTIVE RELATIONS

Inplane stresses

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha\ \Delta T \\ \varepsilon_{yy} - \alpha\ \Delta T \\ \gamma_{xy} \end{Bmatrix}$$

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, \quad Q_{12} = \nu Q_{11}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1+\nu)} = G$$

Transverse stresses

$$\sigma_{xz} = G\gamma_{xz}, \quad \sigma_{yz} = G\gamma_{yz}$$

Property variation

$$G = \frac{E}{2(1+\nu)} \quad \mathbf{E(z,T)} = [E_c(T) - E_m(T)] \left(\frac{1}{2} + \frac{z}{h}\right)^n + E_m(T)$$

EQUATIONS OF MOTION
with the von Karaman Nonlinearity

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right)$$

$$+ \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2}$$

Stress Resultants and Mass Inertias

Stress resultants

$$\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right\} dz, \quad \left\{\begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right\} z\, dz$$

$$\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \end{array}\right\} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array}\right\} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array}\right\}$$

$$\left\{\begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right\} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array}\right\} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array}\right\}$$

FGM Plate Constitutive Relations

$$N_{xx} = A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} - N_{xx}^T$$

$$N_{yy} = A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} - N_{yy}^T$$

$$N_{xy} = A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + B_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{xx} = B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y} - M_{xx}^T$$

$$M_{yy} = B_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y} - M_{yy}^T$$

$$M_{xy} = B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$