SOLUTION TO TUTORIAL/QUIZ 1

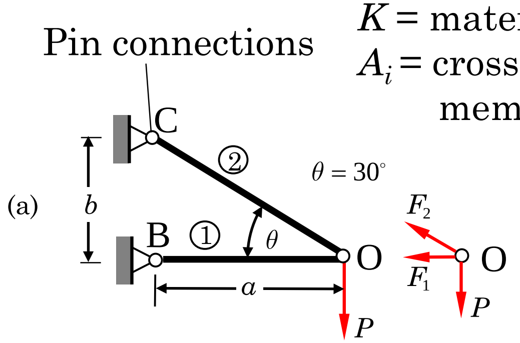
STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

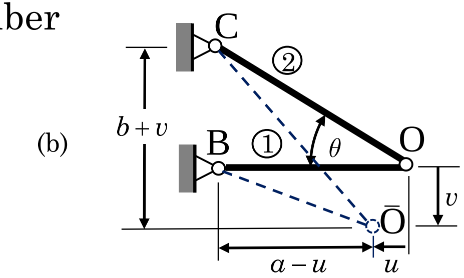
AN EXAMPLE

K = material constant

A_i = cross-sectional area of the i^{th} member

$$\sigma = \begin{cases} K\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -K\sqrt{-\varepsilon}, & \varepsilon \leq 0 \end{cases}$$





Find the (1) strain energy and (2) complementary strain energy of the truss. Assume linear strains.

(1) From Fig. (b), we have the following strains:

$$\varepsilon^{(1)} = \frac{\mathbf{B\bar{O}} - \mathbf{BO}}{\mathbf{BO}} = \left[\frac{(a-u)^2 + v^2}{a^2}\right]^{\frac{1}{2}} - 1 = \left[\frac{a^2 + u^2 + v^2 - 2au}{a^2}\right]^{\frac{1}{2}} - 1 \approx \left(1 - 2\frac{u}{a}\right)^{\frac{1}{2}} - 1 \approx -\frac{u}{a}$$

$$\varepsilon^{(2)} = \frac{\mathbf{C\bar{O}} - \mathbf{CO}}{\mathbf{CO}} = \left[\frac{(a-u)^2 + (b+v)^2}{a^2 + b^2}\right]^{\frac{1}{2}} - 1 = \left[\frac{a^2 + b^2 + u^2 + v^2 + 2(bv - au)}{a^2 + b^2}\right]^{\frac{1}{2}} - 1$$

$$= \left(1 + 2\frac{bv - au}{a^2 + b^2}\right)^{\frac{1}{2}} - 1 \approx \frac{bv - au}{a^2 + b^2} = \frac{\sqrt{3}v - 3u}{4a}$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

(1) continued [note that the stress in member 1 is compressive and it is tensile in member 2; see part (b) to confirm this]

The strain energy densities of each member is

$$U_0^{(1)} = \int_0^{\varepsilon^{(1)}} \sigma^{(1)} d\varepsilon^{(1)} = \int_0^{\varepsilon^{(1)}} \left(-K\sqrt{-\varepsilon^{(1)}}\right) d\varepsilon^{(1)} = \frac{2K}{3}\left(-\varepsilon^{(1)}\right)^{\frac{3}{2}} = \frac{2K}{3}\left(\frac{u^3}{a^3}\right)^{\frac{1}{2}},$$

$$U_0^{(2)} = \int_0^{\varepsilon^{(2)}} \sigma^{(2)}\, d\varepsilon^{(2)} = \int_0^{\varepsilon^{(2)}} \left(K\sqrt{\varepsilon^{(2)}}\right) d\varepsilon^{(2)} = \frac{2K}{3}\left(\varepsilon^{(2)}\right)^{\frac{3}{2}} = \frac{2K}{3}\left(\frac{\sqrt{3}v - 3u}{4a}\right)^{\frac{3}{2}}$$

The total strain energy of the truss is

$$U = \int_{V_1} U_0^{(1)} d\Omega + \int_{V_2} U_0^{(2)} d\Omega = A_1 L_1 U_0^{(1)} + A_2 L_2 U_0^{(2)}$$

$$= \frac{2KA_1 a}{3}\left(\frac{u^3}{a^3}\right)^{\frac{1}{2}} + \frac{4KA_2 a}{3\sqrt{3}}\left(\frac{\sqrt{3}v - 3u}{4a}\right)^{\frac{3}{2}}$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF TRUSSES - AN EXAMPLE

(2) From Fig. (a), we have

$$F_2 \sin\theta = P,\ \ F_1 + F_2 \cos\theta = 0,\ \theta = 30^\circ,\ F_1 = -\sqrt{3}P,\ \ F_2 = 2P$$

The complementary strain energy densities of each member are

$$U_0^{*(1)} = \int_0^{\sigma^{(1)}} \varepsilon^{(1)} d\sigma^{(1)} = \int_0^{\sigma^{(1)}} \left[-\left(\frac{\sigma^{(1)}}{K} \right)^2 \right] d\sigma^{(1)} = -\frac{1}{3K^2} \left(\sigma^{(1)} \right)^3 = \frac{\sqrt{3}P^3}{A_1^3 K^2},$$

$$U_0^{*(2)} = \int_0^{\sigma^{(2)}} \varepsilon^{(2)}\ d\sigma^{(2)} = \int_0^{\sigma^{(2)}} \left[\frac{\left(\sigma^{(2)} \right)^2}{K^2} \right] d\sigma^{(2)} = \frac{1}{3K^2} \left(\sigma^{(2)} \right)^3 = \frac{1}{3} \left(\frac{8P^3}{A_2^3 K^2} \right)$$

The total complementary strain energy of the truss is

$$U^* = U_0^{*(1)} A_1 L_1 + U_0^{*(2)} A_2 L_2 = \frac{1}{3} \left[\left(\frac{3\sqrt{3}P^3 a}{A_1^2 K^2} \right) + \left(\frac{16 P^3 a}{\sqrt{3} A_2^2 K^2} \right) \right]$$

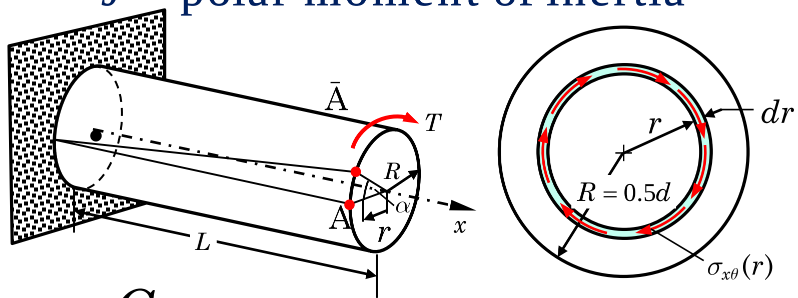
EXERCISE 2: Find the (1) strain energy and (2) complementary strain energy of a torsional member.

The shear stress in a shaft subjected to torque is given by

$$\sigma_{x\theta}(r) = \frac{Tr}{J}$$

d = diameter of the shaft

J = polar moment of inertia



The strain energy density and strain energy are

$$\gamma_{x\theta} = 2\varepsilon_{x\theta} = \frac{r\alpha}{L}, \quad \alpha = \frac{TL}{GJ}$$

$$U_0 = 2\int_0^{\varepsilon_{x\theta}} \sigma_{x\theta}\, d\varepsilon_{x\theta} = 4\int_0^{\varepsilon_{x\theta}} G\varepsilon_{x\theta}\, d\varepsilon_{x\theta} = 2G\varepsilon_{x\theta}^2 \equiv \frac{G}{2}\gamma_{x\theta}^2$$

$$U = \int_\Omega \frac{G}{2}\gamma_{x\theta}^2\, d\Omega = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_0^{d/2} G\left(\frac{r\alpha}{L}\right)^2 dr\, rd\theta\, dx = \frac{1}{2}\int_0^L GJ\frac{\alpha^2}{L^2}dx$$

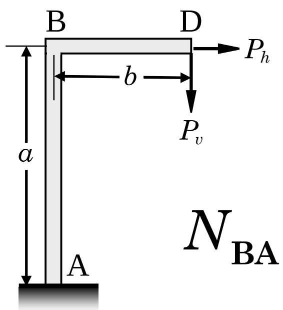
The complementary strain energy density is

$$U_0^* = 2\int_0^{\sigma_{x\theta}} \varepsilon_{x\theta}\, d\sigma_{x\theta} = \frac{1}{2G}\sigma_{x\theta}^2$$

$$U^* = \int_v \frac{1}{2G}\sigma_{x\theta}^2\, d\Omega = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_0^{d/2} \frac{1}{G}\left(\frac{Tr}{J}\right)^2 dr\, rd\theta\, dx = \frac{1}{2}\int_0^L \frac{T^2}{GJ}dx$$

EXERCISE 3:

Determine the complementary strain energy of the (determinate) frame structure.

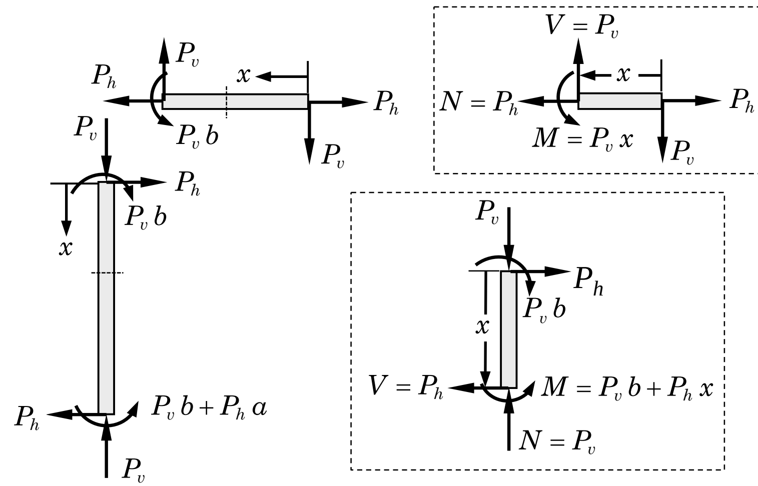


All members have the same *E*, *A*, and *I*

$$N_{\mathbf{BA}} = P_v,\ \ V_{\mathbf{BA}} = P_h,$$

$$M_{\mathbf{BA}} = P_v \cdot b + P_h \cdot x$$

$$N_{\mathbf{DB}} = P_h,\ \ V_{\mathbf{DB}} = P_v,\ \ M_{\mathbf{DB}} = P_v$$



$$U^*_{\mathbf{DB}} = \int_0^b \left[\frac{P_h^2}{2EA} + \frac{1}{2EI}(P_v x)^2 + \frac{f_s P_v^2}{2GA}\right] dx = \frac{P_h^2 b}{2EA} + \frac{P_v^2 b^3}{6EI} + \frac{f_s P_v^2 b}{2GA},$$

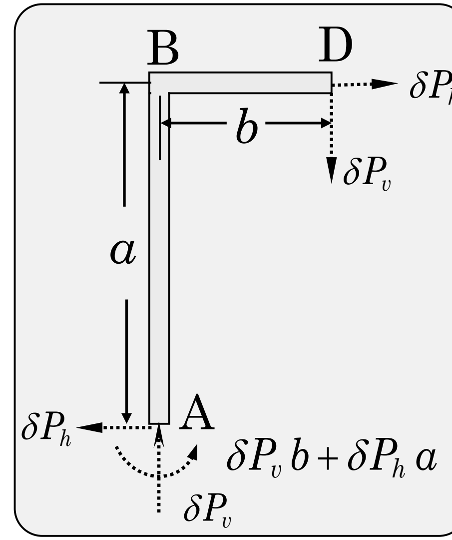
$$U^*_{\mathbf{BA}} = \int_0^a \left[\frac{P_v^2}{2EA} + \frac{1}{2EI}(P_v b + P_h x)^2 + \frac{f_s P_h^2}{2GA}\right] dx$$

$$= \frac{P_v^2 a}{2EA} + \frac{1}{2EI}\left(P_v^2 a b^2 + P_v P_h a^2 b + \frac{1}{3} P_h^2 a^3\right) + \frac{f_s P_h^2 a}{2GA}$$

EXERCISE 4: Determine the complementary strain energy of the (determinate) frame structure.

$$\delta W_I^* = \delta U^* = \int_0^L \left[\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{V}{K_s GA}\delta V \right] dx,$$

$$\delta W_E^* = \delta V_E^* = -\left(v\delta P_v + u\delta P_h \right)$$



$$\delta N_{\mathbf{DB}} = \delta P_h,\ \delta N_{\mathbf{BA}} = \delta P_v, \delta V_{\mathbf{DB}} = \delta P_v,\ \delta V_{\mathbf{BA}} = \delta P_h$$

$$\delta M_{\mathbf{DB}} = \delta P_v \cdot x,\ \delta M_{\mathbf{BA}} = \delta P_v \cdot b + \delta P_h \cdot x$$

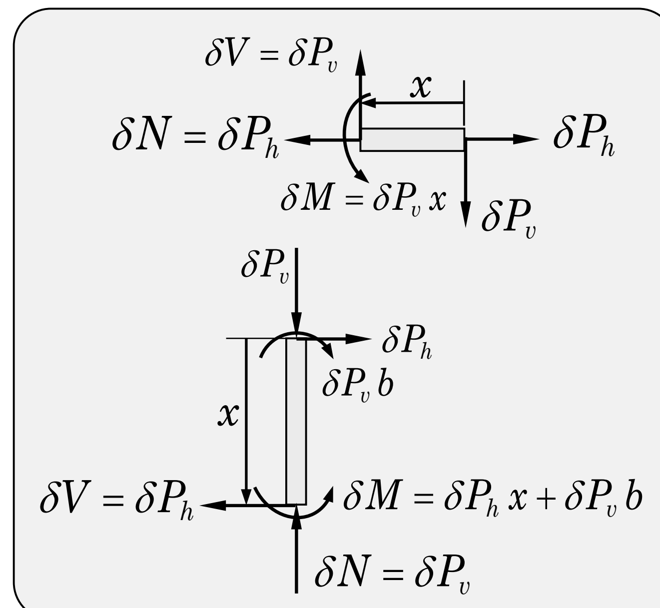
$$\delta U_{\mathbf{DB}}^* = \int_0^b \left[\frac{N_{\mathbf{DB}}}{EA}\delta N_{\mathbf{DB}} + \frac{M_{\mathbf{DB}}}{EI}\delta M_{\mathbf{DB}} + \frac{V_{DB}}{K_s GA}\delta V_{\mathbf{DB}} \right] dx$$

$$= \frac{P_h b}{EA}\delta P_h + \left(\frac{P_v b^3}{3EI} + \frac{P_v b}{K_s GA} \right)\delta P_v,$$

$$\delta U_{\mathbf{BA}}^* = \int_0^b \left[\frac{N_{\mathbf{BA}}}{EA}\delta N_{\mathbf{BA}} + \frac{M_{\mathbf{BA}}}{EI}\delta M_{\mathbf{BA}} + \frac{V_{BA}}{K_s GA}\delta V_{\mathbf{BA}} \right] dx$$

$$= \left[\frac{P_v a}{EA} + \frac{1}{EI}\left(P_v b^2 a + P_h b \frac{a^2}{2} \right) \right] \delta P_v$$

$$+ \left[\frac{1}{EI}\left(P_v b \frac{a^2}{2} + P_h \frac{a^3}{3} \right) + \frac{P_h a}{K_s GA} \right] \delta P_h$$



Exercise 5: PRINCIPLE OF VIRTUAL DISPLACEMENTS

Displacement field and the von Karman nonlinear strains

$$u_1(x,z) = u(x) + z\phi_x(x),\ u_2 = 0,\ u_3 = w(x)$$

$$\varepsilon_{11}(x,z) = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx},\ 2\varepsilon_{xz}(x) = \phi_x + \frac{dw}{dx}$$

Principle of virtual displacements

$$0 = \int_0^L \int_A \left(\sigma_{11}\,\delta\varepsilon_{11} + 2\sigma_{13}\,\delta\varepsilon_{13}\right) dA dx - \left[\int_0^L f \cdot \delta u\, dx + \int_0^L q \cdot \delta w\, dx\right]$$

$$= \int_0^L \int_A \left[\sigma_{11}\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} + z\frac{d\delta\phi_x}{dx}\right) + \sigma_{13}\left(\delta\phi_x + \frac{d\delta w}{dx}\right)\right] dA dx - \int_0^L \left(f\delta u + q\delta w\right) dx$$

$$= \int_0^L \left[N\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) + M\frac{d\delta\phi_x}{dx} + Q\left(\delta\phi_x + \frac{d\delta w}{dx}\right) - \left(f\delta u + q\delta w\right)\right] dx$$

where $N = \int_A \sigma_{11} dA,\ M = \int_A z\sigma_{11} dA,\ Q = \int_A \sigma_{13} dA$

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

Simplifying, we obtain

$$0 = \int_0^L \left[N\frac{d\delta u}{dx} + \left(N\frac{dw}{dx} + Q \right)\frac{d\delta w}{dx} + M\frac{d\delta\phi_x}{dx} + Q\,\delta\phi_x - \left(f\delta u + q\delta w \right) \right] dx$$

$$= \int_0^L \left\{ -\left(\frac{dN}{dx} + f \right)\delta u - \left[\frac{d}{dx}\left(N\frac{dw}{dx} + Q \right) + q \right]\delta w + \left(-\frac{dM}{dx} + Q \right)\delta\phi_x \right\} dx$$

$$+ \left[N\delta u + \left(N\frac{dw}{dx} + Q \right)\delta w + M\delta\phi_x \right]_0^L$$

Using the fundamental lemma, we obtain the following equations of equilibrium and the natural boundary conditions:

$$-\frac{dN}{dx} - f = 0, \quad -\frac{d}{dx}\left(N\frac{dw}{dx} + Q \right) - q = 0, \quad -\frac{dM}{dx} + Q = 0$$

Specify

$$N \text{ or } u; \quad N\frac{dw}{dx} + Q \text{ or } w; \quad M \text{ or } \phi_x$$