SOLUTION TO THE TEST

Note: For each problem at least describe the procedure to obtain the required solution. When possible, give the equations you use to solve the problem.

Problem 1: Find the linear strains associated with the displacements

$$u_1 = \left[X_1 X_2 (2 - X_1) - c_1 X_2 + c_2 X_2^3\right],$$
$$u_2 = -\left[c_3 X_2^2 (1 - X_1) + (3 - X_1)\frac{X_1^2}{3} + c_1 X_1\right].$$

Solution: Various derivatives of u_1 and u_2 are

$$\frac{\partial u_1}{\partial X_1} = 2(1 - X_1)X_2, \quad \frac{\partial u_1}{\partial X_2} = X_1\,(2 - X_1) - c_1 + 3c_2 X_2^2,$$
$$\frac{\partial u_2}{\partial X_1} = -c_1 + c_3 X_2^2 - X_1\,(2 - X_1)\,, \quad \frac{\partial u_2}{\partial X_2} = -2c_3(1 - X_1)X_2.$$

The linearized strains are

$$e_{11} = \frac{\partial u_1}{\partial X_1} = 2(1 - X_1)X_2,$$
$$2e_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = (3c_2 + c_3)X_2^2 - 2c_1,$$
$$e_{22} = -2c_3(1 - X_1)X_2.$$

All other linearized stain components are zero.

Problem 2: Determine the equations of equilibrium of 3D elastic solid (see the figure below) using the vector approach.

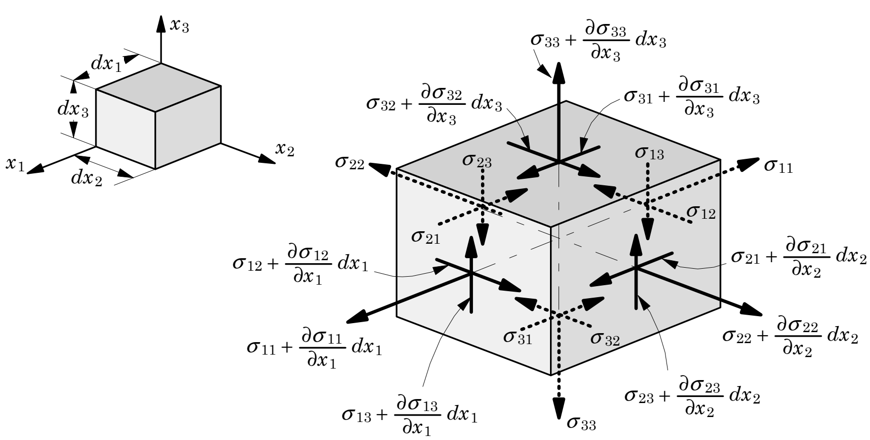

Figure 1: Stresses on a parallelepiped element.

Solution: The sum of all forces in the x_1-direction is given by

$$\Big(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\Big) dx_2 dx_3 - \sigma_{11} dx_2 dx_3 + \Big(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\Big) dx_1 dx_3$$
$$- \sigma_{21} dx_1 dx_3 + \Big(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\Big) dx_1 dx_2 - \sigma_{31} dx_1 dx_2 + \rho f_1 dx_1 dx_2 dx_3$$
$$= \Big(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1\Big) dx_1 dx_2 dx_3.$$

By Newton's second law of motion, the sum of the forces is equal to the time rate of change of linear momentum in the x_1-direction:

$$(\rho dx_1 dx_2 dx_3) \frac{\partial^2 u_1}{\partial t^2},$$

where ρ is the density (assumed to be independent of time t). Thus, upon dividing throughout by $dx_1 dx_2 dx_3$, we obtain

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1 = \rho \frac{\partial^2 u_1}{\partial t^2} \quad \text{or} \quad \frac{\partial \sigma_{j1}}{\partial x_j} + \rho f_1 = \rho \frac{\partial^2 u_1}{\partial t^2}.$$

Similarly, the application of Newton's second law in the x_2- and x_3-directions yields

$$\frac{\partial \sigma_{j2}}{\partial x_j} + \rho f_2 = \rho \frac{\partial^2 u_2}{\partial t^2}, \quad \frac{\partial \sigma_{j3}}{\partial x_j} + \rho f_3 = \rho \frac{\partial^2 u_3}{\partial t^2},$$

Problem 3: Determine the (a) strain energy and (b) complementary strain energy of the truss shown in Fig. 2. Assume elastic behavior of the form $\sigma = K\sqrt{\varepsilon}$. Express your answer in terms of displacements in the former case and applied loads in the latter case. *Hint:* You may have to use a displacement constraint to determine the reactions.

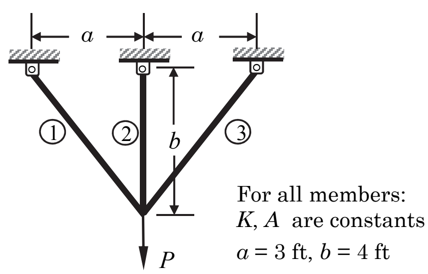

Figure 2: A three member truss

Solution: (a) The strain energy is given by

$$U = \sum_{i=1}^{3} \int_{V_i} \int_0^{\varepsilon_i} \sigma d\varepsilon \, dV = \frac{2}{3} \sum_{i=1}^{3} AL_i E \left(\varepsilon_i\right)^{\frac{3}{2}}$$

where

$$\varepsilon_1 = \varepsilon_3 = \sqrt{\frac{a^2 + (b+v)^2}{a^2 + b^2}} - 1 \approx \frac{bv}{a^2 + b^2} = \frac{4v}{25}, \quad \varepsilon_2 = \frac{v}{b} = \frac{v}{4}. \tag{1}$$

Then we have

$$
\begin{aligned}
U &= \frac{2}{3}AE\left[2\times 5\left(\frac{4v}{25}\right)^{\frac{3}{2}} + 4\left(\frac{v}{4}\right)^{\frac{3}{2}}\right] \\
&= \frac{2}{3}AE\left(\frac{16}{25}+\frac{1}{2}\right)v\sqrt{v} = \frac{19}{25}AEv\sqrt{v}
\end{aligned} \tag{2}
$$

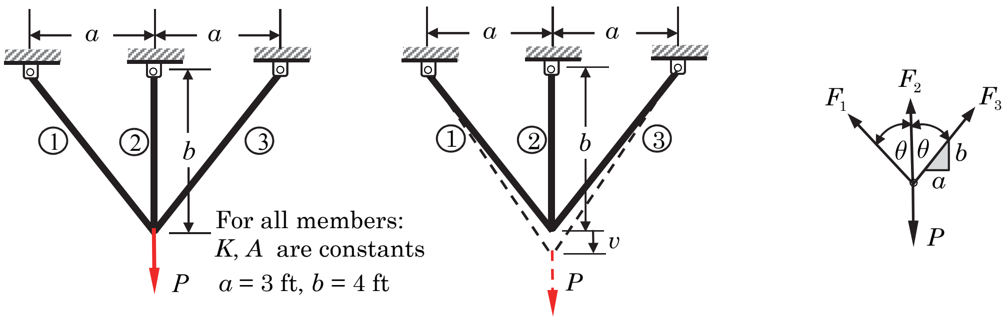

Figure 3: A three member truss

(b) The complementary strain energy is given by

$$
U^* = \sum_{i=1}^{3}\int_{V_i}\int_0^{\sigma_i}\varepsilon d\sigma\, dV = \frac{A}{3E^2}\sum_{i=1}^{3}L_i\left(\sigma_i\right)^3 = \frac{1}{3E^2A^2}\sum_{i=1}^{3}L_i\left(F_i\right)^3, \tag{3}
$$

where $\sigma_i = \frac{F_i}{A}$, F_i being the axial force in the ith member.

From the equilibrium of the truss, we obtain

$$
F_1 - F_3 = 0, \qquad (F_1+F_3)\cos\theta + F_2 = P, \qquad \cos\theta = \frac{4}{5}. \tag{4}
$$

We have only two equations in three unknowns, F_i, $(i = 1,2,3)$; hence, the truss is indeterminate. We must use an additional relationship provided by the kinematics of deformation to determine the member forces. Equation (1) can be used to express the strains (hence, the vertical displacement v) in terms of the member forces ($\varepsilon_i = \sigma_i^2/E^2 = F_i^2/E^2A^2$, $i = 1,2$). Thus, F_2 and $F_1 = F_3$ are related by

$$
\frac{4v}{25} = \left(\frac{F_1}{EA}\right)^2, \quad \frac{v}{4} = \left(\frac{F_2}{EA}\right)^2 \quad \text{or} \quad \frac{25}{4}(F_1)^2 = 4(F_2)^2 \to F_1 = \frac{4}{5}F_2. \tag{5}
$$

Then, from Eqs. (4) and (5), we have

$$
F_2 = \frac{25}{57}P, \qquad F_1 = F_3 = \frac{20}{57}P.
$$

The complementary strain energy becomes

$$
U^* = \frac{1}{3E^2A^2}\sum_{i=1}^{3}L_i\left(F_i\right)^3 = \frac{P^3}{3E^2A^2}\left[2\times 5\left(\frac{20}{57}\right)^3 + 4\left(\frac{25}{57}\right)^3\right] = \left(\frac{50}{57}\right)^2\frac{P^3}{3E^2A^2}. \tag{6}
$$

Problem 4: Use the principle of virtual displacements to derive the governing equations of the Bernoulli–Euler beam theory using the von Kámán nonlinear strain. The displacement filed and the only nonzero strain are

$$u_1(x,y,z) = u(x) + z\theta_x, \quad u_2 = 0, \quad u_3(x,y,z) = w(x); \quad \theta_x \equiv -\frac{dw}{dx}. \tag{1}$$

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\theta_x}{dx}. \tag{2}$$

Assume that the beam rests on a linear elastic foundation with foundation modulus k and subjected to a distributed longitudinal load $f(x)$ and distributed transverse load $q(x)$ at the top (see Fig. 4).

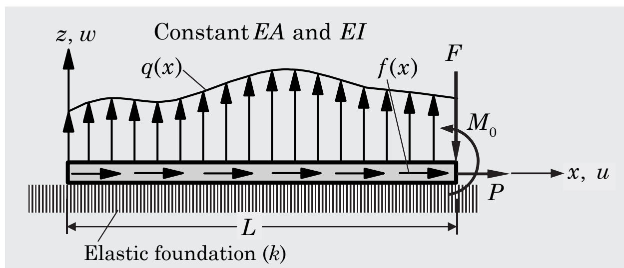

Figure 4: A beam on elastic foundation.

Solution: Let the virtual displacements be δu and δw, which are completely arbitrary because there are no specified geometric boundary conditions for the problem at hand. Then the virtual strain $\delta\varepsilon_{xx}$ is given by

$$\delta\varepsilon_{xx} = \delta\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\theta_x}{dx}\right] = \frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} + z\frac{d\delta\theta_x}{dx}. \tag{3}$$

Then the internal and external virtual works due to the virtual displacements δu and δw are given by

$$\begin{aligned}\delta W_E = -\Bigg\{&\int_0^L [f(x)\delta u + q(x)\delta w(x,h_t)]\,dx + \int_0^L (-F_s)\delta w(x,h_b)\,dx \\ &+ P\delta u(L) + (-F)\delta w(L) + (-M_0)\left(-\frac{d\delta w}{dx}\right)_{x=L}\Bigg\}\end{aligned} \tag{4}$$

$$\begin{aligned}= -\Bigg\{&\int_0^L [f(x)\delta u + q(x)\delta w(x)]\,dx - \int_0^L kw(x)\delta w(x)\,dx \\ &+ P\delta u(L) - F\delta w(L) + M_0\left(\frac{d\delta w}{dx}\right)_{x=L}\Bigg\},\end{aligned} \tag{5}$$

$$\begin{aligned}\delta W_I &= \int_0^L\int_A \sigma_{xx}\delta\varepsilon_{xx}\,dxdA \\ &= \int_0^L\int_A \sigma_{xx}\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} - z\frac{d^2\delta w}{dx^2}\right)dxdA,\end{aligned} \tag{6}$$

where L is the length, h_t is the distance from the x-axis to the top of the beam, h_b is the distance from the x-axis to the bottom of the beam, and A is the cross-sectional area of the beam. The foundation reaction force F_s (acting downward) is replaced with $F_s = kw(x)$ using the linear elastic constitutive equation for the foundation.

The principle of virtual displacements requires that $\delta W = \delta W_I + \delta W_E = 0$, which gives

$$\begin{aligned}
0 &= \int_0^L \int_A \sigma_{xx} \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} - z\frac{d^2\delta w}{dx^2} \right) dAdx \\
&\quad - \int_0^L \left[f\delta u + (q - kw)\,\delta w \right] dx - P\delta u(L) + F\delta w(L) - M_0 \left(\frac{d\delta w}{dx} \right)_{x=L} \\
&= \int_0^L \left[N \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) - M\frac{d^2\delta w}{dx^2} \right] dx \\
&\quad\quad - \int_0^L \left[f\delta u + (q - kw)\,\delta w \right] dx - P\delta u(L) + F\delta w(L) - M_0 \left(\frac{d\delta w}{dx} \right)_{x=L} \\
&= \int_0^L \left[-\frac{dN}{dx}\delta u - \frac{d}{dx}\left(\frac{dw}{dx}N \right)\delta w - \frac{d^2 M}{dx^2}\delta w \right] dx - \int_0^L \left[f\delta u + (q - kw)\,\delta w \right] dx \\
&\quad - P\delta u(L) + F\delta w(L) - M_0 \left(\frac{d\delta w}{dx} \right)_{x=L} \\
&\quad + \left[N\delta u + \left(\frac{dw}{dx}N + \frac{dM}{dx} \right)\delta w - M\frac{d\delta w}{dx} \right]_0^L,
\end{aligned} \tag{7}$$

where N and M are the stress resultants defined by

$$N = \int_A \sigma_{xx} dA, \quad M = \int_A \sigma_{xx} z \; dA. \tag{8}$$

The Euler equations are obtained by setting the coefficients of δu and δw under the integral separately to zero:

$$\delta u: \quad -\frac{dN}{dx} = f(x), \tag{9}$$

$$\delta w: \quad -\frac{d^2 M}{dx^2} - \frac{d}{dx}\left(\frac{dw}{dx}N \right) + kw - q = 0. \tag{10}$$

Problem 5: In Problem 4 assume that $E = E(x,z)$ is a function of x and z and write the equilibrium equations in terms of the displacements (u,w).

Solution: The equilibrium equations are:

$$\frac{\partial N_{xx}}{\partial x} + f_x = 0; \quad \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} \right) - kw + q = 0$$

The stress resultants in the Bernoulli–Euler beam theory are

$$\begin{aligned}
N_{xx} &= \int_A \sigma_{xx}\, dA = A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{xx}\frac{\partial \theta_x}{\partial x}, \\
M_{xx} &= \int_A z\sigma_{xx}\, dA = B_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x} \right)^2 \right] + D_{xx}\frac{\partial \theta_x}{\partial x},
\end{aligned}$$

where A_{xx}, B_{xx}, and D_{xx} are the extensional, extensional-bending, and bending stiffness coefficients

$$(A_{xx}, B_{xx}, D_{xx}) = \int_A (1, z, z^2) E(z)\, dA$$

The equations of equilibrium of the Bernoulli–Euler beam theory in terms of the generalized displacements $(u, w, \theta_x = -\partial w/\partial x)$ are

$$\frac{\partial}{\partial x}\left\{ A_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{xx}\frac{\partial \theta_x}{\partial x}\right\} + f_x = 0$$

$$\frac{\partial^2}{\partial x^2}\left\{ B_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + D_{xx}\frac{\partial \theta_x}{\partial x}\right\}$$
$$+\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\left\{ A_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{xx}\frac{\partial \theta_x}{\partial x}\right\}\right] - kw + q = 0$$

Problem 6: The equations of equilibrium of the Timoshenko FGM beams for the linear case are

$$\frac{\partial}{\partial x}\left(A_{xx}\frac{\partial u}{\partial x} + B_{xx}\frac{\partial \phi_x}{\partial x}\right) + f_x = 0 \tag{1}$$

$$\frac{\partial}{\partial x}\left[K_s S_{xz}\left(\phi_x + \frac{\partial w}{\partial x}\right)\right] + q = 0, \tag{2}$$

$$\frac{\partial}{\partial x}\Big(B_{xx}\frac{\partial u}{\partial x} + D_{xx}\frac{\partial \phi_x}{\partial x}\Big) - \left[K_s S_{xz}\left(\phi_x + \frac{\partial w}{\partial x}\right)\right] = 0 \tag{3}$$

Find the Navier solution for the simply supported beams.

Solution: The boundary conditions of a simply supported beam are

$$N_{xx} = 0, \quad w = 0, \quad M_{xx} = 0, \quad \text{at } x = 0, L$$

The solution is assumed to be of the form

$$u(x) = \sum_{n=1}^{\infty} U_n \cos\alpha_n x, \quad w(x) = \sum_{n=1}^{\infty} W_n \sin\alpha_n x, \quad \phi_x(x) = \sum_{n=1}^{\infty} X_n \cos\alpha_n x,$$

and the mechanical forces $(f_x,\, q)$ are expanded in series as

$$f_x(x) = \sum_{n=1}^{\infty} F_n \cos\alpha_n x, \qquad q(x) = \sum_{n=1}^{\infty} Q_n \sin\alpha_n x.$$

We note that the assumed solution satisfies the boundary conditions.

Use of the expansions in the equilibrium equations results in

$$\begin{bmatrix} \alpha_n^2 A_{xx} & 0 & \alpha_n^2 B_{xx} \\ 0 & \alpha_n^2 K_s S_{xz} & \alpha_n K_s S_{xz} \\ \alpha_n^2 B_{xx} & \alpha_n K_s S_{xz} & \left(\alpha_n^2 D_{xx} + K_s S_{xz}\right) \end{bmatrix} \begin{Bmatrix} U_n \\ W_n \\ X_n \end{Bmatrix} = \begin{Bmatrix} F_n \\ Q_n \\ 0 \end{Bmatrix}. \tag{1}$$
