INTRODUCTION TO THE THEORY OF SHELLS

CONTENTS

- Geometry of shells
- Kinematics of shells
- Displacements and strains
- Strain-displacement equations
- Stress resultants
- Equations of motion
- Shell constitutive relations
- Specialization to cylindrical shells
- Examples

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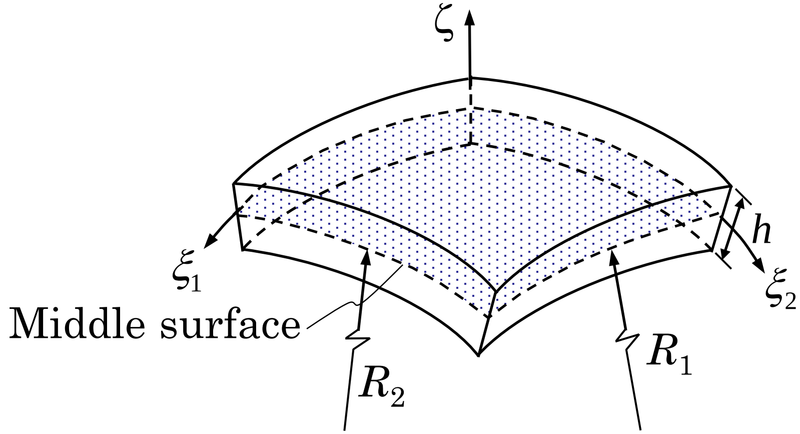
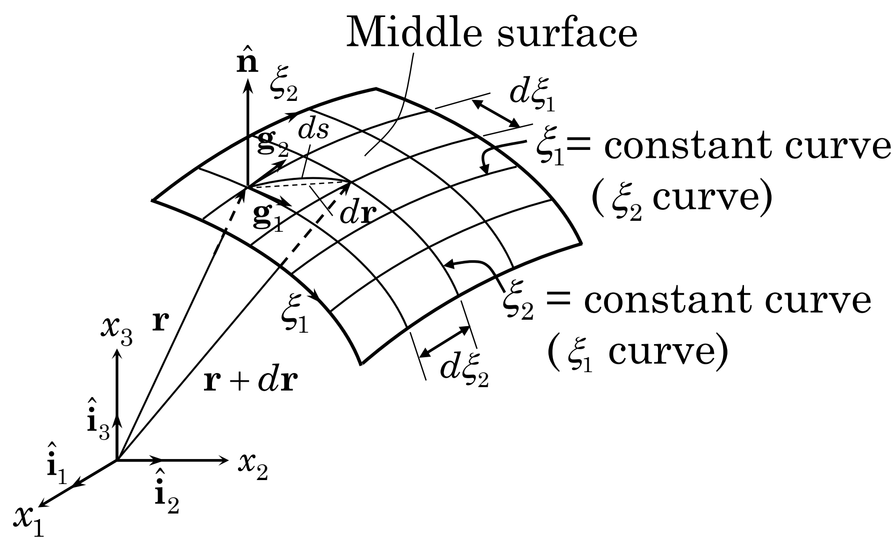
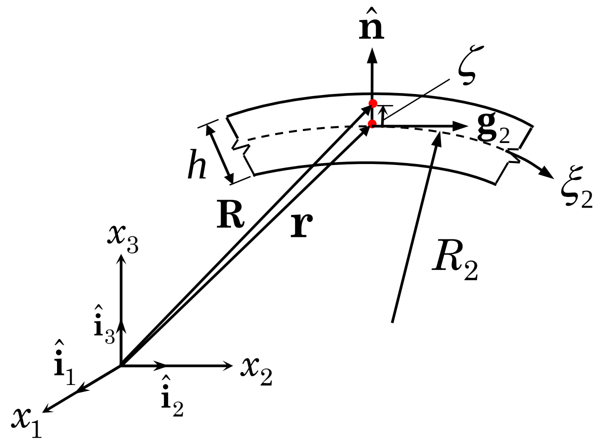
GEOMETRY OF SHELLS





GEOMETRY OF SHELLS

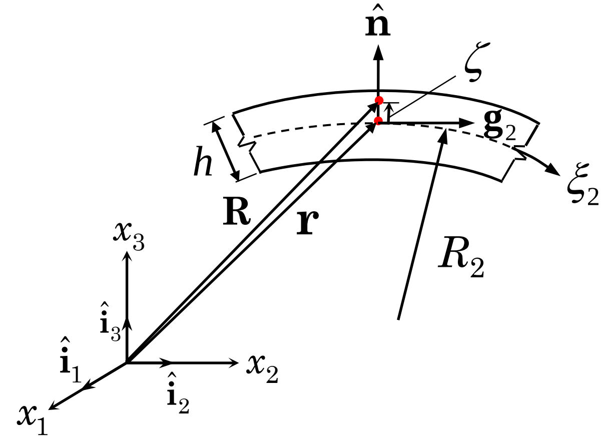


$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_\alpha} d\xi_\alpha = \mathbf{g}_\alpha d\xi_\alpha, \quad \mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi_\alpha} \quad (\alpha = 1,2)$$

$$\mathbf{g}_\alpha \cdot \mathbf{g}_\beta \equiv g_{\alpha\beta}, \; a_1 = \sqrt{g_{11}}, \quad a_2 = \sqrt{g_{22}}, \; \mathbf{g}_1 \cdot \mathbf{g}_2 = a_1 a_2 \cos\chi$$

KINEMATICS OF SHELLS

$$\hat{\mathbf{n}} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{a_1 a_2 \sin\chi}$$



$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = g_{\alpha\beta}\, d\xi_\alpha\, d\xi_\beta$$

$$= a_1^2 (d\xi_1)^2 + a_2^2 (d\xi_2)^2 + 2a_1 a_2 \cos\chi\, d\xi_1\, d\xi_2$$

$$\mathbf{R} = \mathbf{r} + \zeta \hat{\mathbf{n}}$$

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial \xi_i} d\xi_i = \frac{\partial \mathbf{R}}{\partial \xi_\alpha} d\xi_\alpha + \frac{\partial \mathbf{R}}{\partial \zeta} d\zeta = \mathbf{G}_\alpha\, d\xi_\alpha + \hat{\mathbf{n}} d\zeta$$

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{R}}{\partial \xi_\alpha} = \frac{\partial \mathbf{r}}{\partial \xi_\alpha} + \zeta \frac{\partial \hat{\mathbf{n}}}{\partial \xi_\alpha} = \left(1 + \frac{\zeta}{R_\alpha}\right)\mathbf{g}_\alpha \text{ (no sum on } \alpha)$$

KINEMATICS OF SHELLS

$$G_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta, \;\; A_1 = \sqrt{G_{11}}, \;\; A_2 = \sqrt{G_{22}}, \;\; A_3 = 1$$

$$A_\alpha = a_\alpha \left(1 + \frac{\zeta}{R_\alpha}\right) \text{ (no sum on } \alpha)$$

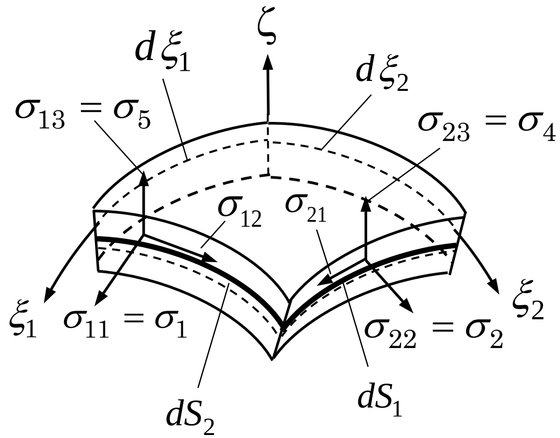
$$(dS)^2 = d\mathbf{R} \cdot d\mathbf{R} = G_{\alpha\beta}\, d\xi_\alpha d\xi_\beta + (d\zeta)^2 = A_1^2 (d\xi_1)^2 + A_2^2 (d\xi_2)^2 + A_3^2 (d\zeta)^2$$

$$\frac{\partial}{\partial \xi_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial \xi_2}\right) = -\frac{a_1 a_2}{R_1 R_2}$$

$$\frac{\partial^2 A_1}{\partial \xi_2 \partial \zeta} = \frac{1}{A_2}\frac{\partial A_1}{\partial \xi_2}\frac{\partial A_2}{\partial \zeta}, \quad \frac{\partial^2 A_2}{\partial \xi_1 \partial \zeta} = \frac{1}{A_1}\frac{\partial A_2}{\partial \xi_1}\frac{\partial A_1}{\partial \zeta}$$

$$\frac{1}{A_2}\frac{\partial A_1}{\partial \xi_2} = \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}, \qquad \frac{1}{A_1}\frac{\partial A_2}{\partial \xi_1} = \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}$$

DISPLACEMENT FIELD AND STRAINS

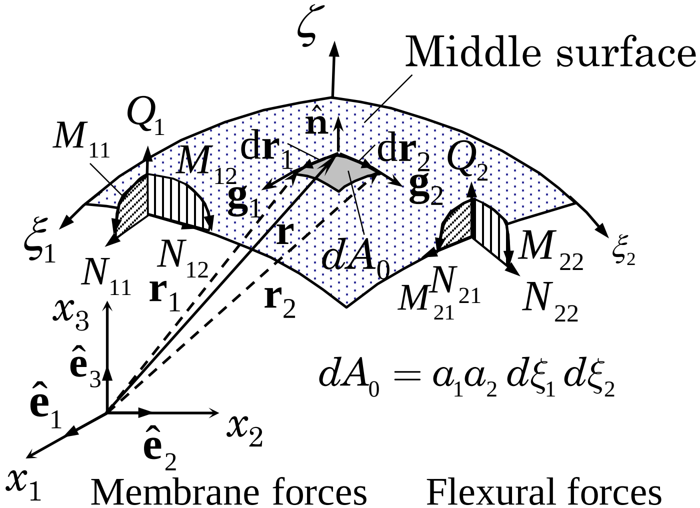


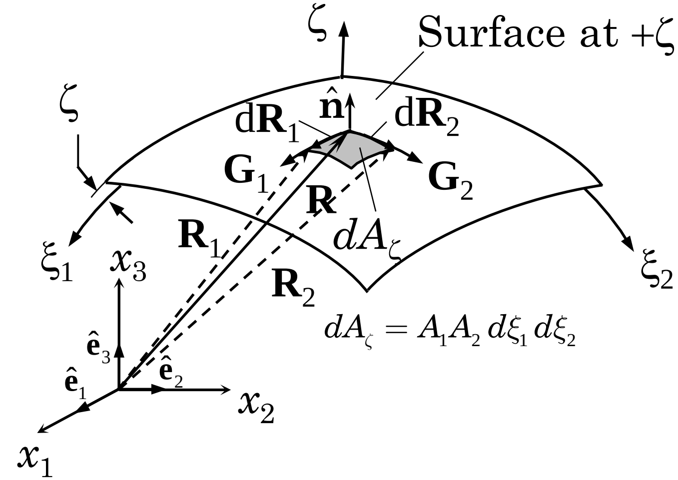
$$dS_1 = a_1\left(1+\frac{\zeta}{R_1}\right)d\xi_1 = A_1 d\xi_1, \quad dS_2 = a_2\left(1+\frac{\zeta}{R_2}\right)d\xi_2 = A_2\, d\xi_2$$

$$dS_1 d\zeta = A_1\, d\xi_1 d\zeta = a_1\left(1+\frac{\zeta}{R_1}\right)d\xi_1\, d\zeta$$

$$dS_2 d\zeta = A_2\, d\xi_2 d\zeta = a_2\left(1+\frac{\zeta}{R_2}\right)d\xi_2\, d\zeta$$

AREAS AND VOLUME OF A SHELL ELEMENT





Membrane forces: $N_{11},\ N_{12},\ N_{21},\ N_{22}$

Flexural forces: $M_{11},\ M_{12},\ M_{21},\ M_{22}$, $Q_1,\ Q_2$

$$dA_0 = d\mathbf{r}_1 \times d\mathbf{r}_2 \cdot \hat{\mathbf{n}} = \left(\frac{\partial \mathbf{r}}{\partial \xi_1} \times \frac{\partial \mathbf{r}}{\partial \xi_2} \cdot \hat{\mathbf{n}}\right) d\xi_1\, d\xi_2 = a_1 a_2\, d\xi_1\, d\xi_2$$

$$dA_\zeta = d\mathbf{R}_1 \times d\mathbf{R}_2 \cdot \hat{\mathbf{n}} = \left(\frac{\partial \mathbf{R}}{\partial \xi_1} \times \frac{\partial \mathbf{R}}{\partial \xi_2} \cdot \hat{\mathbf{n}}\right) d\xi_1\, d\xi_2 = A_1 A_2\, d\xi_1\, d\xi_2$$

$$dV = d\mathbf{R}_1 \times d\mathbf{R}_2 \cdot \hat{\mathbf{n}}\, d\zeta = dA_\zeta\, d\zeta = A_1 A_2\, d\xi_1\, d\xi_2\, d\zeta$$

$$= a_1 a_2 \left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\xi_1\, d\xi_2\, d\zeta$$

STRAIN-DISPLACEMENT RELATIONS

$$\varepsilon_i = \frac{\partial}{\partial \xi_i}\left(\frac{u_i}{A_i}\right) + \frac{1}{A_i}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_i}{\partial \xi_k} + \frac{1}{2}\left[\frac{\partial}{\partial \xi_i}\left(\frac{u_i}{A_i}\right) + \frac{1}{A_i}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_i}{\partial \xi_k}\right]^2$$

$$+ \frac{1}{2A_i^2}\sum_{k=1,k\neq i}^{3}\left(\frac{\partial u_k}{\partial \xi_i} - \frac{u_i}{A_k}\frac{\partial A_i}{\partial \xi_k}\right)^2 \quad (i=1,2,3)$$

$$\gamma_{ij} = \frac{A_i}{A_j}\frac{\partial}{\partial \xi_j}\left(\frac{u_i}{A_i}\right) + \frac{A_j}{A_i}\frac{\partial}{\partial \xi_i}\left(\frac{u_j}{A_j}\right) + \sum_{k\neq i,k\neq j}^{3}\frac{1}{A_iA_j}\left(\frac{\partial u_k}{\partial \xi_i} - \frac{u_i}{A_k}\frac{\partial A_i}{\partial \xi_k}\right)\left(\frac{\partial u_k}{\partial \xi_j} - \frac{u_j}{A_k}\frac{\partial A_j}{\partial \xi_k}\right)$$

$$+ \frac{1}{A_j}\left(\frac{\partial u_i}{\partial \xi_j} - \frac{u_j}{A_i}\frac{\partial A_j}{\partial \xi_i}\right)\left[\frac{\partial}{\partial \xi_i}\left(\frac{u_i}{A_i}\right) + \frac{1}{A_i}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_i}{\partial \xi_k}\right]$$

$$+ \frac{1}{A_i}\left(\frac{\partial u_j}{\partial \xi_i} - \frac{u_i}{A_j}\frac{\partial A_i}{\partial \xi_j}\right)\left[\frac{\partial}{\partial \xi_j}\left(\frac{u_j}{A_j}\right) + \frac{1}{A_j}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_j}{\partial \xi_k}\right]$$

$$\xi_3 = \zeta, \quad A_1 = a_1\left(1 + \frac{\zeta}{R_1}\right), \quad A_2 = a_2\left(1 + \frac{\zeta}{R_2}\right), \quad A_3 = a_3 = 1$$

STRAIN-DISPLACEMENT RELATIONS (simplified relations)

$$\varepsilon_1 = \varepsilon_{11} = \frac{1}{A_1}\left(\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \xi_2} + \frac{a_1}{R_1}u_3\right) + \frac{1}{2A_1^2}\left[\left(\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \xi_2} + \frac{a_1}{R_1}u_3\right)^2 + \left(\frac{\partial u_2}{\partial \xi_1} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial u_3}{\partial \xi_1} - \frac{a_1}{R_1}u_1\right)^2\right]$$

$$\varepsilon_2 = \varepsilon_{22} = \frac{1}{A_2}\left(\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \xi_1} + \frac{a_2}{R_2}u_3\right) + \frac{1}{2A_2^2}\left[\left(\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \xi_1} + \frac{a_2}{R_2}u_3\right)^2 + \left(\frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \xi_1}\right)^2 + \left(\frac{\partial u_3}{\partial \xi_2} - \frac{a_2}{R_2}u_2\right)^2\right]$$

$$\varepsilon_3 = \varepsilon_{33} = \frac{\partial u_3}{\partial \zeta} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial \zeta}\right)^2 + \left(\frac{\partial u_2}{\partial \zeta}\right)^2 + \left(\frac{\partial u_3}{\partial \zeta}\right)^2\right]$$

STRAIN-DISPLACEMENT RELATIONS (simplified relations - continued)

$$\varepsilon_4 = 2\varepsilon_{23} = \frac{1}{A_2}\frac{\partial u_3}{\partial \xi_2} + A_2\frac{\partial}{\partial \zeta}\left(\frac{u_2}{A_2}\right) + \frac{1}{A_2}\left[\frac{\partial u_2}{\partial \zeta}\left(\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \xi_1} + \frac{a_2}{R_2}u_3\right)\right.$$
$$\left. + \frac{\partial u_1}{\partial \zeta}\left(\frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \xi_1}\right) + \frac{\partial u_3}{\partial \zeta}\left(\frac{\partial u_3}{\partial \xi_2} - \frac{a_2}{R_2}u_2\right)\right]$$

$$\varepsilon_5 = 2\varepsilon_{13} = \frac{1}{A_1}\frac{\partial u_3}{\partial \xi_1} + A_1\frac{\partial}{\partial \zeta}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1}\left[\frac{\partial u_1}{\partial \zeta}\left(\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \xi_2} + \frac{a_1}{R_1}u_3\right)\right.$$
$$\left. + \frac{\partial u_2}{\partial \zeta}\left(\frac{\partial u_2}{\partial \xi_1} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) + \frac{\partial u_3}{\partial \zeta}\left(\frac{\partial u_3}{\partial \xi_1} - \frac{a_1}{R_1}u_1\right)\right]$$

$$\varepsilon_6 = 2\varepsilon_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1 A_2}\left[\left(\frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \xi_1}\right)\left(\frac{\partial u_1}{\partial \xi_1} + \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}u_2 + \frac{a_1}{R_1}u_3\right)\right.$$
$$\left. + \left(\frac{\partial u_2}{\partial \xi_1} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \xi_2}\right)\left(\frac{\partial u_2}{\partial \xi_2} + \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}u_1 + \frac{a_2}{R_2}u_3\right) + \left(\frac{\partial u_3}{\partial \xi_1} - \frac{a_1}{R_1}u_1\right)\left(\frac{\partial u_3}{\partial \xi_2} - \frac{a_2}{R_2}u_2\right)\right]$$

STRESS RESULTANTS

$$\int_{-h/2}^{h/2} \sigma_{11}\, dS_2\, d\zeta = a_2 \left[\int_{-h/2}^{h/2} \sigma_{11} \left(1 + \frac{\zeta}{R_2} \right) d\zeta \right] d\xi_2 \equiv N_{11}\, a_2\, d\xi_2$$

$$\int_{-h/2}^{h/2} \sigma_{11}\, dS_2\, \zeta\, d\zeta = a_2 \left[\int_{-h/2}^{h/2} \sigma_{11} \left(1 + \frac{\zeta}{R_2} \right) \zeta\, d\zeta \right] d\xi_2 \equiv M_{11}\, a_2\, d\xi_2$$

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ K_s \sigma_{13} \end{Bmatrix} \left(1 + \frac{\zeta}{R_2} \right) d\zeta, \begin{Bmatrix} N_{22} \\ N_{21} \\ Q_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ K_s \sigma_{23} \end{Bmatrix} \left(1 + \frac{\zeta}{R_1} \right) d\zeta$$

$$\begin{Bmatrix} M_{11} \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \end{Bmatrix} \left(1 + \frac{\zeta}{R_2} \right) \zeta\, d\zeta, \quad \begin{Bmatrix} M_{22} \\ M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \end{Bmatrix} \left(1 + \frac{\zeta}{R_1} \right) \zeta\, d\zeta$$

ASSUMPTIONS

1. The transverse normal is inextensible (i.e., $\varepsilon_3 \approx 0$) and the transverse normal stress is small compared with the other normal stress components and may be neglected.
2. Normals to the undeformed middle surface of the shell before deformation remain straight, but not necessarily normal after deformation.
3. The deflections and strains are sufficiently small so that the quantities of second- and higher-order magnitude, except for second-order rotations about the transverse normals, may be neglected in comparison with the first-order terms.
4. The rotations about the ξ_1 and ξ_2 axes are moderate so that we retain second-order terms (i.e., terms that are products and squares of the terms $\dfrac{\partial u_3}{\partial \xi_\alpha} - \dfrac{a_\alpha u_\alpha}{R_\alpha}$) in the strain-displacement relations (the von K'arm'an nonlinearity).

DISPLACEMENTS AND STRAINS

$$u_1(\xi_1,\xi_2,\zeta,t) = u_1^0(\xi_1,\xi_2,t) + \zeta\varphi_1(\xi_1,\xi_2,t)$$
$$u_2(\xi_1,\xi_2,\zeta,t) = u_2^0(\xi_1,\xi_2,t) + \zeta\varphi_2(\xi_1,\xi_2,t)$$
$$u_3(\xi_1,\xi_2,\zeta,t) = u_3^0(\xi_1,\xi_2,t)$$

$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial u_1^0}{\partial \xi_1} + \frac{u_2^0}{a_2}\frac{\partial a_1}{\partial \xi_2} + \frac{a_1 u_3^0}{R_1} + \frac{1}{2A_1}\left(\frac{\partial u_3^0}{\partial \xi_1} - \frac{a_1 u_1^0}{R_1}\right)^2 + \zeta\left(\frac{\partial \varphi_1}{\partial \xi_1} + \frac{\varphi_2}{a_2}\frac{\partial a_1}{\partial \xi_2}\right)\right]$$

$$\varepsilon_2 = \frac{1}{A_2}\left[\frac{\partial u_2^0}{\partial \xi_2} + \frac{u_1^0}{a_1}\frac{\partial a_2}{\partial \xi_1} + \frac{a_2 u_3^0}{R_2} + \frac{1}{2A_2}\left(\frac{\partial u_3^0}{\partial \xi_2} - \frac{a_2 u_2^0}{R_2}\right)^2 + \zeta\left(\frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{a_1}\frac{\partial a_2}{\partial \xi_1}\right)\right]$$

$$\varepsilon_3 = 0,\ \ \varepsilon_4 = \frac{a_2}{A_2}\left(\frac{1}{a_2}\frac{\partial u_3^0}{\partial \xi_2} + \varphi_2 - \frac{u_2^0}{R_2}\right),\ \ \varepsilon_5 = \frac{a_1}{A_1}\left(\frac{1}{a_1}\frac{\partial u_3^0}{\partial \xi_1} + \varphi_1 - \frac{u_1^0}{R_1}\right)$$

$$\varepsilon_6 = \frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2^0}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1^0}{A_1}\right) + \frac{1}{A_1 A_2}\left(\frac{\partial u_3^0}{\partial \xi_1} - \frac{a_1}{R_1}u_1^0\right)\left(\frac{\partial u_3^0}{\partial \xi_2} - \frac{a_2}{R_2}u_2^0\right)$$
$$+ \zeta\left[\frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{\varphi_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{\varphi_1}{A_1}\right)\right]$$

EQUATIONS OF MOTION
(obtained through the use of virtual work principle)

$$\frac{1}{a_2 a_1}\left[\frac{\partial}{\partial \xi_1}\left(a_2 N_{11}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 N_{21}\right)+N_{12}\frac{\partial a_1}{\partial \xi_2}-N_{22}\frac{\partial a_2}{\partial \xi_1}\right]+\frac{Q_1}{R_1}+f_1$$

$$+\frac{\hat{N}_{11}}{a_1 R_1}\left(\frac{\partial u_3^0}{\partial \xi_1}-\frac{a_1 u_1^0}{R_1}\right)+\frac{\tilde{N}_{12}}{a_2 R_1}\left(\frac{\partial u_3^0}{\partial \xi_2}-\frac{a_2 u_2^0}{R_2}\right)=I_0\frac{\partial^2 u_1^0}{\partial t^2}+I_1\frac{\partial^2 \varphi_1}{\partial t^2}$$

$$\frac{1}{a_1 a_2}\left[\frac{\partial}{\partial \xi_2}\left(a_1 N_{22}\right)+\frac{\partial}{\partial \xi_1}\left(a_2 N_{12}\right)+N_{21}\frac{\partial a_2}{\partial \xi_1}-N_{11}\frac{\partial a_1}{\partial \xi_2}\right]+\frac{Q_2}{R_2}+f_2$$

$$+\frac{\hat{N}_{22}}{a_2 R_2}\left(\frac{\partial u_3^0}{\partial \xi_2}-\frac{a_2 u_2^0}{R_2}\right)+\frac{\tilde{N}_{12}}{a_1 R_2}\left(\frac{\partial u_3^0}{\partial \xi_1}-\frac{a_1 u_1^0}{R_1}\right)=I_0\frac{\partial^2 u_2^0}{\partial t^2}+I_1\frac{\partial^2 \varphi_2}{\partial t^2}$$

$$\frac{1}{a_1 a_2}\left\{\frac{\partial}{\partial \xi_1}\left[\frac{a_2}{a_1}\hat{N}_{11}\left(\frac{\partial u_3^0}{\partial \xi_1}-\frac{a_1 u_1^0}{R_1}\right)+\tilde{N}_{12}\left(\frac{\partial u_3^0}{\partial \xi_2}-\frac{a_2 u_2^0}{R_2}\right)\right]\right.$$

$$\left.+\frac{\partial}{\partial \xi_2}\left[\frac{a_1}{a_2}\hat{N}_{22}\left(\frac{\partial u_3^0}{\partial \xi_2}-\frac{a_2 u_2^0}{R_2}\right)+\tilde{N}_{12}\left(\frac{\partial u_3^0}{\partial \xi_1}-\frac{a_1 u_1^0}{R_1}\right)\right]+\frac{\partial}{\partial \xi_1}\left(a_2 Q_1\right)+\frac{\partial}{\partial \xi_2}\left(a_1 Q_2\right)\right\}$$

$$-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)+f_3=I_0\frac{\partial^2 u_3^0}{\partial t^2}$$

EQUATIONS OF MOTION

$$\frac{1}{a_1 a_2}\left[\frac{\partial}{\partial \xi_1}\left(a_2 M_{11}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 M_{21}\right)+M_{12}\frac{\partial a_1}{\partial \xi_2}-M_{22}\frac{\partial a_2}{\partial \xi_1}\right]-Q_1$$

$$=I_1\frac{\partial^2 u_1^0}{\partial t^2}+I_2\frac{\partial^2 \varphi_1}{\partial t^2}$$

$$\frac{1}{a_2 a_1}\left[\frac{\partial}{\partial \xi_1}\left(a_2 M_{12}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 M_{22}\right)+M_{21}\frac{\partial a_2}{\partial \xi_1}-M_{11}\frac{\partial a_1}{\partial \xi_2}\right]-Q_2$$

$$=I_1\frac{\partial^2 u_2^0}{\partial t^2}+I_2\frac{\partial^2 \varphi_2}{\partial t^2}$$

CONSTITUTIVE RELATIONS

Stress-Strain Relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, Q_{66} = G_{12}, Q_{44} = G_{23}, Q_{55} = G_{13},$$

Resultant-Strain Relations

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix}, \quad \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^1 \\ \varepsilon_2^1 \\ \varepsilon_6^1 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} = K_s \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix}$$

$$A_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}\, d\zeta = Q_{ij} h, \quad D_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}\zeta^2\, d\zeta = Q_{ij}\frac{h^3}{12}$$

STRAIN-DISPLACEMENT RELATIONS

$$\begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_4^0 \\ \varepsilon_5^0 \\ \varepsilon_6^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{a_1}\left(\dfrac{\partial u_1^0}{\partial \xi_1} + \dfrac{u_2^0}{a_2}\dfrac{\partial a_1}{\partial \xi_2} + \dfrac{a_1 u_3^0}{R_1}\right) \\ \dfrac{1}{a_2}\left(\dfrac{\partial u_2^0}{\partial \xi_2} + \dfrac{u_1^0}{a_1}\dfrac{\partial a_2}{\partial \xi_1} + \dfrac{a_2 u_3^0}{R_2}\right) \\ \dfrac{1}{a_2}\dfrac{\partial u_3^0}{\partial \xi_2} - \dfrac{u_2^0}{R_2} + \varphi_2 \\ \dfrac{1}{a_1}\dfrac{\partial u_3^0}{\partial \xi_1} - \dfrac{u_1^0}{R_1} + \varphi_1 \\ \dfrac{1}{a_1}\left(\dfrac{\partial u_2^0}{\partial \xi_1} - \dfrac{u_1^0}{a_2}\dfrac{\partial a_1}{\partial \xi_2}\right) + \dfrac{1}{a_2}\left(\dfrac{\partial u_1^0}{\partial \xi_2} - \dfrac{u_2^0}{a_1}\dfrac{\partial a_2}{\partial \xi_1}\right) \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1^1 \\ \varepsilon_2^1 \\ \varepsilon_6^1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{a_1}\left(\dfrac{\partial \varphi_1}{\partial \xi_1} + \dfrac{\varphi_2}{a_2}\dfrac{\partial a_1}{\partial \xi_2}\right) \\ \dfrac{1}{a_2}\left(\dfrac{\partial \varphi_2}{\partial \xi_2} + \dfrac{\varphi_1}{a_1}\dfrac{\partial a_2}{\partial \xi_1}\right) \\ \dfrac{1}{a_1}\left(\dfrac{\partial \varphi_2}{\partial \xi_1} - \dfrac{\varphi_1}{a_2}\dfrac{\partial a_1}{\partial \xi_2}\right) + \dfrac{1}{a_2}\left(\dfrac{\partial \varphi_1}{\partial \xi_2} - \dfrac{\varphi_2}{a_1}\dfrac{\partial a_2}{\partial \xi_1}\right) \end{Bmatrix}$$

SPECIALIZATION TO CYLINDRICAL SHELLS

$$(1/R_1)=0,\ \ R_2=R,\ \ a_1\xi_1=x_1,\ \ a_2\xi_2=x_2,\ \ a_1=1, a_2=R$$

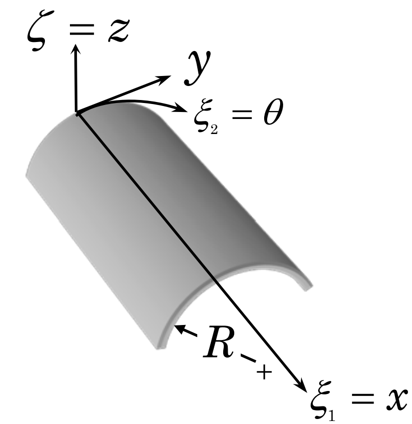
$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial}{\partial x_2}(N_{21}+C_0\tilde{M}_{12})+f_1=I_0\frac{\partial^2 u_1^0}{\partial t^2}+I_1\frac{\partial^2\varphi_1}{\partial t^2}$$

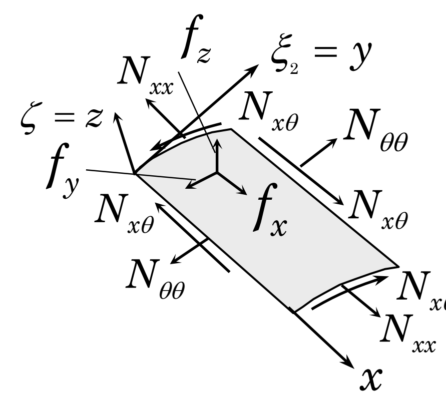
$$\frac{\partial}{\partial x_1}(N_{12}-C_0\tilde{M}_{12})+\frac{\partial N_{22}}{\partial x_2}+\frac{Q_2}{R}+f_2=I_0\frac{\partial^2 u_2^0}{\partial t^2}+I_1\frac{\partial^2\varphi_2}{\partial t^2}$$

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}-\frac{N_{22}}{R}+f_3=I_0\frac{\partial^2 u_3^0}{\partial t^2}$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{21}}{\partial x_2}-Q_1=I_1\frac{\partial^2 u_1^0}{\partial t^2}+I_2\frac{\partial^2\varphi_1}{\partial t^2}$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-Q_2=I_1\frac{\partial^2 u_2^0}{\partial t^2}+I_2\frac{\partial^2\varphi_2}{\partial t^2}$$





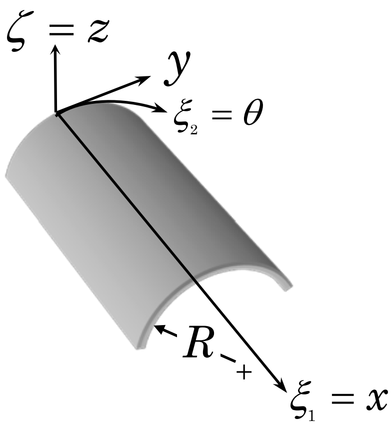
JN Reddy

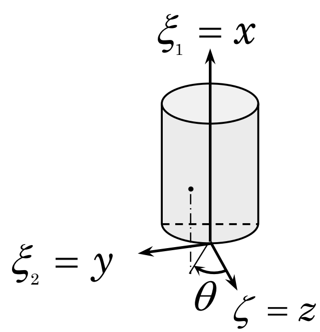
THIN CYLINDRICAL SHELLS

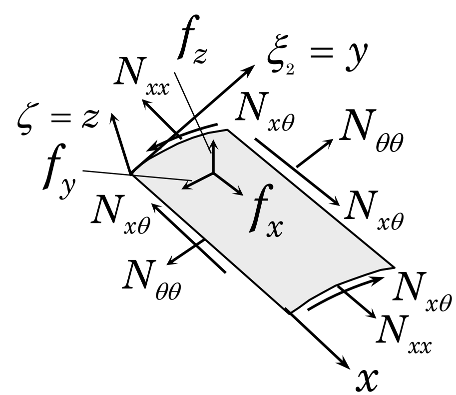
$$a_1\xi_1 = x_1 = x,\ a_2\xi_2 = x_2 = R\theta,\ C_0 = -1/2R,\ N_{11} = N_{xx},\ N_{12} = N_{x\theta},\ N_{22} = N_{\theta\theta},$$

$$M_{11} = M_{xx},\ M_{12} = M_{x\theta},\ M_{22} = M_{\theta\theta},\ Q_1 = Q_x,\ Q_2 = Q_\theta,$$

$$u_1^0 = u_0,\ u_2^0 = v_0,\ u_3^0 = w_0, \varphi_1 = \varphi_x, \varphi_2 = \varphi_\theta$$







THIN CYLINDRICAL SHELLS

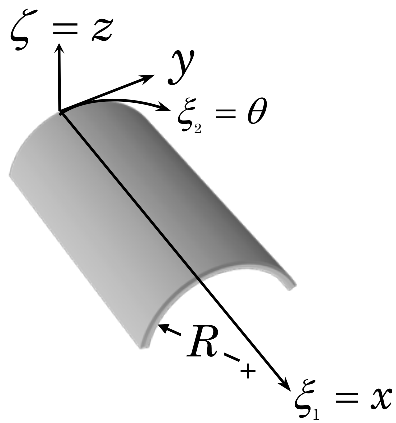
$$\frac{\partial N_{xx}}{\partial x}+\frac{1}{R}\frac{\partial}{\partial \theta}\left(N_{x\theta}-\frac{1}{2R}M_{x\theta}\right)+f_x=0 \quad (1)$$

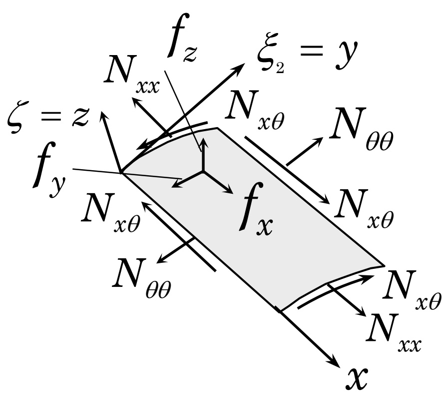
$$\frac{\partial}{\partial x}\left(N_{x\theta}+\frac{1}{2R}M_{x\theta}\right)+\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta}+\frac{Q_\theta}{R}+f_\theta=0 \quad (2)$$

$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{\partial Q_\theta}{\partial \theta}-\frac{N_{\theta\theta}}{R}+f_z=0 \quad (3)$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta}-Q_x=0 \quad (4)$$

$$\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial \theta}-Q_\theta=0 \quad (5)$$



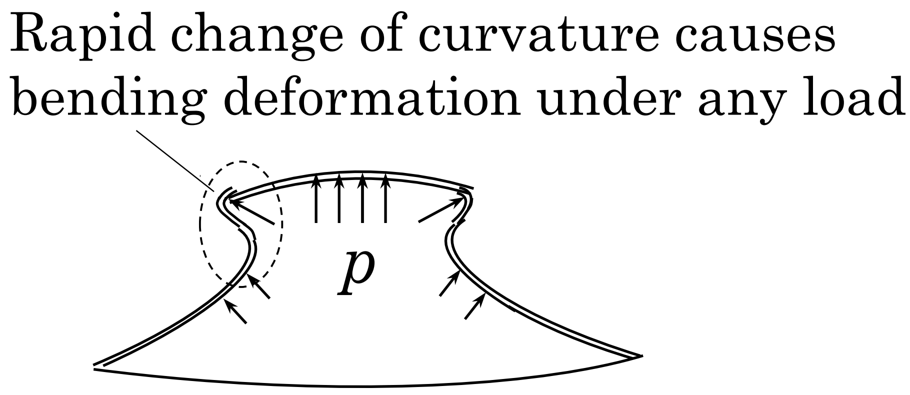


CONSTITUTIVE EQUATIONS FOR CYLINDRICAL SHELLS

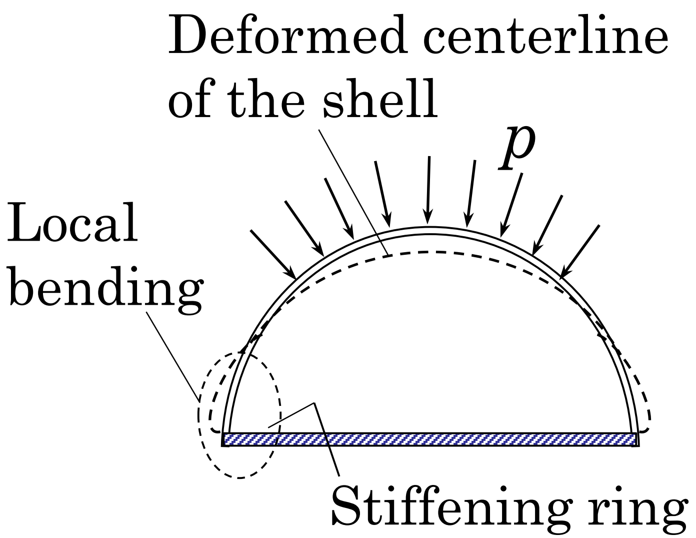
$$\begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \end{Bmatrix} = \frac{Eh}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{w_0}{R} \\ \frac{\partial v_0}{\partial x} + \frac{1}{R}\frac{\partial u_0}{\partial \theta} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial \varphi_x}{\partial x} \\ \frac{1}{R}\frac{\partial \varphi_\theta}{\partial \theta} \\ \frac{\partial \varphi_\theta}{\partial x} + \frac{1}{R}\frac{\partial \varphi_x}{\partial \theta} \end{Bmatrix}, \quad \begin{Bmatrix} Q_\theta \\ Q_x \end{Bmatrix} = K_s Gh \begin{Bmatrix} \frac{1}{R}\frac{\partial w_0}{\partial \theta} - \frac{v_0}{R} + \varphi_\theta \\ \frac{\partial w_0}{\partial x} + \varphi_x \end{Bmatrix}$$

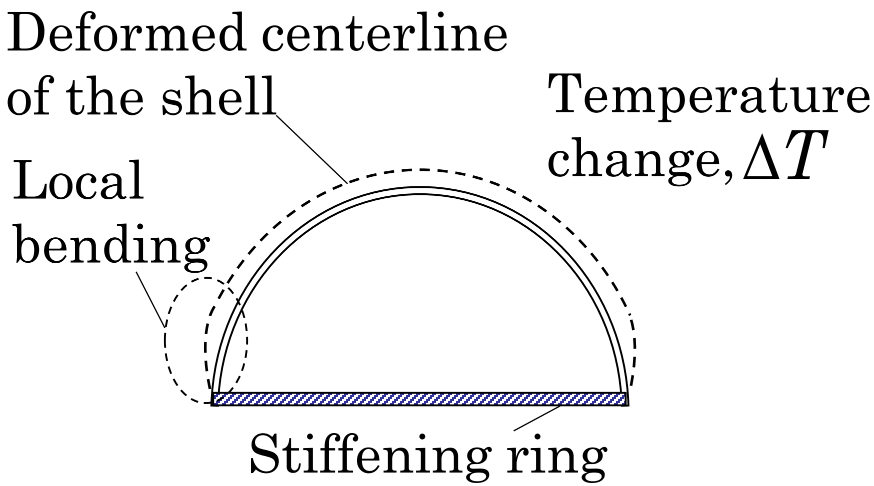
MEMBRANE AND BENDING STATES



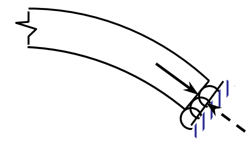
Internal pressure



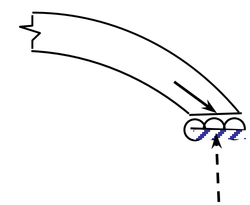
MEMBRANE AND BENDING STATES



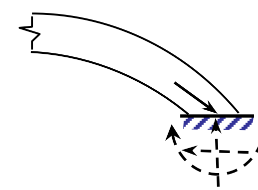
Allows pure membrane state of stress



Membrane state of stress is only approximate



Bending state of stress exists



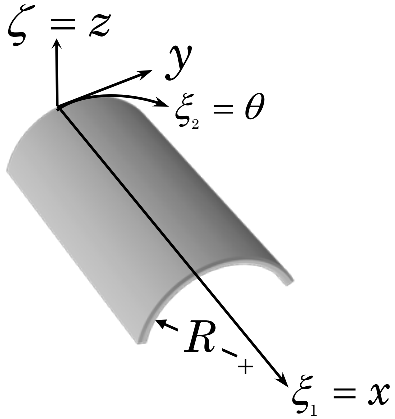
MEMBRANE THEORY OF CYLINDRICAL SHELLS

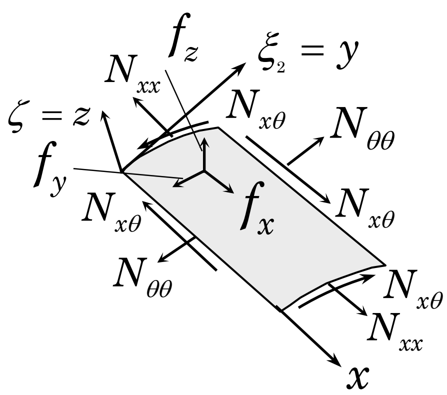
The equilibrium equations governing the membrane state of deformation and stress, called the **membrane theory**, are obtained by setting bending moments and transverse shear forces to zero:

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} + f_x = 0$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + f_\theta = 0$$

$$-\frac{N_{\theta\theta}}{R} + f_z = 0$$





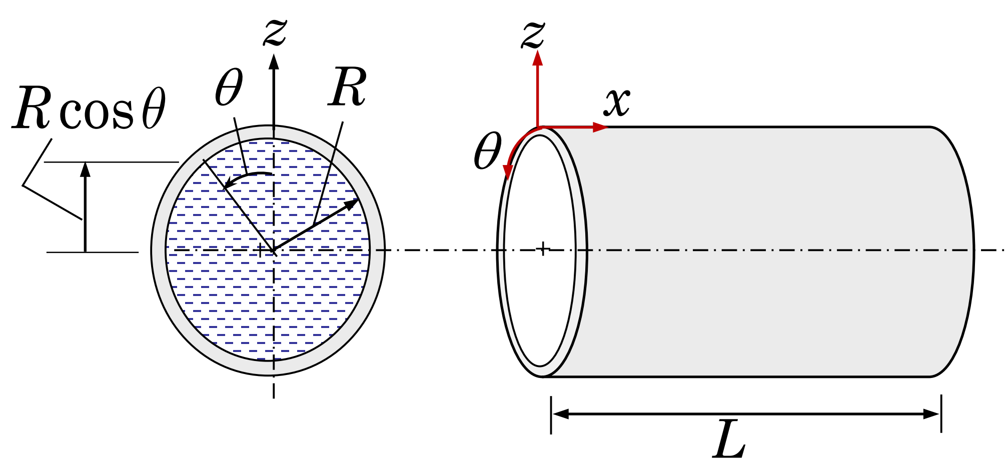
ANALYTICAL SOLUTIONS of the membrane theory of shells

EXAMPLE 1: *Consider a circular cylindrical shell of radius R and thickness h, filled with liquid, and simply supported at its ends. Determine* $N_{\theta\theta}$, $N_{x\theta}$, and N_{xx} *assuming that there are no axial forces at the ends of the shell and the bending deformation is negligible.*

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} + f_x = 0 \quad (1)$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + f_\theta = 0 \quad (2)$$

$$-\frac{N_{\theta\theta}}{R} + f_z = 0 \quad (3)$$



EXAMPLE 1: CYLINDRICAL SHELL FILLED WITH LIQUID

The components of load for this case are is the pressure at the axis of the tube and γ is the specific weight of the liquid. We have

$$f_x = f_\theta = 0 \text{ and } f_z = p_0 - \gamma R \cos\theta,$$

where p_0 is the pressure at the axis of the tube and γ is the specific weight of the liquid.

From the Eq. (3) we obtain

$$-\frac{N_{\theta\theta}}{R} + f_z = 0 \Rightarrow N_{\theta\theta} = p_0 R - \gamma R^2 \cos\theta$$

EXAMPLE 1 CONTINUED

From the Eqs. (1) and (2) we obtain

$$\frac{\partial N_{x\theta}}{\partial x} = -\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} = -\gamma R \sin\theta \Rightarrow N_{x\theta} = -x\gamma R\sin\theta + A(\theta)$$

$$\frac{\partial N_{xx}}{\partial x} = -\frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} = x\gamma\cos\theta - \frac{A'(\theta)}{R} \Rightarrow N_{xx} = \frac{x^2}{2}\gamma\cos\theta - \frac{x}{R}A'(\theta) + B(\theta)$$

Using the end conditions

$$N_{xx}(0,\theta) = 0,\ N_{xx}(L,\theta) = 0,\ N_{xx}(0,\theta) = 0:\ \ B(\theta) = 0$$

$$N_{xx}(L,\theta) = 0: \quad A'(\theta) = \frac{1}{2}\gamma RL\cos\theta \rightarrow A(\theta) = \frac{1}{2}\gamma RL\sin\theta + C$$

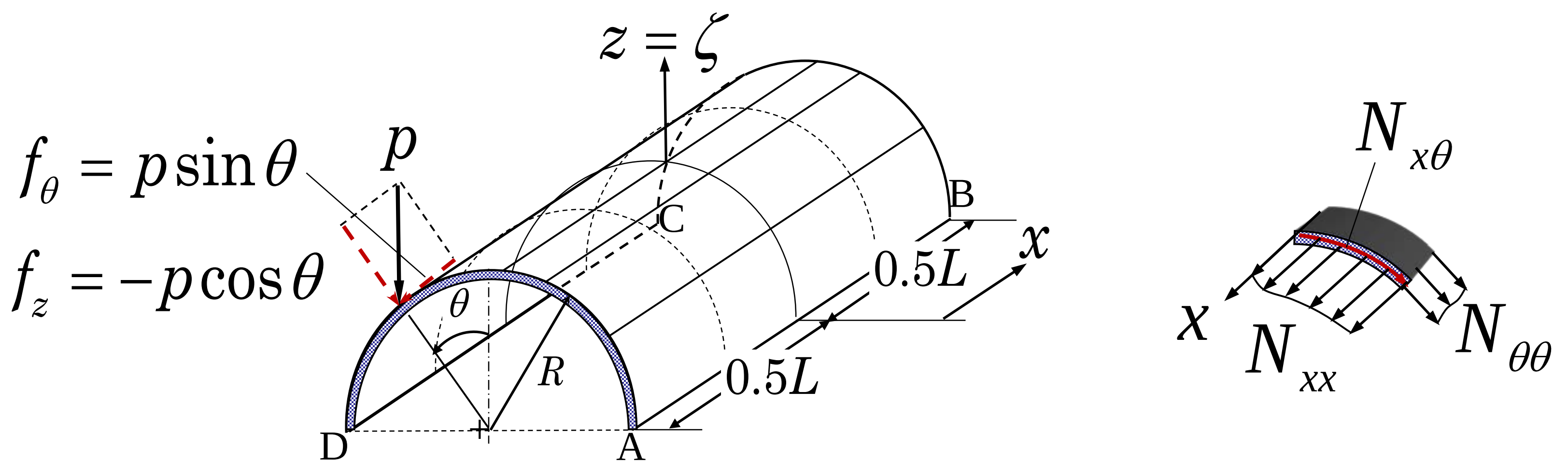
Thus, we have

$$N_{\theta\theta} = p_0 R - \gamma R^2\cos\theta,\ \ N_{x\theta} = \gamma R(0.5L - x)\sin\theta + C,\ \ N_{xx} = -\frac{\gamma}{2}(xL - x^2)\cos\theta$$

In the absence of any torsional moment, we have C=0

EXAMPLE 2: CYLINDRICAL PANEL UNDER ITS OWN WEIGHT

Consider a cylindrical shell of semicircular cross section supporting its own weight, which is assumed to be distributed uniformly over the surface of the shell. The shell is assumed to be supported at the four corners A, B, C, and D, but the edges AB and CD are free, as shown in the figure. Using the membrane theory of shells and assuming that there are no axial forces at the ends of the shell, $N_{xx}(-L/2)=0,\ N_{xx}(L/2)=0$, *determine the forces,* $(N_\theta, N_{x\theta}, N_{xx})$ *and displacements* $(u_0,\ v_0,\ w_0)$.



EXAMPLE 2: CYLINDRICAL PANEL UNDER ITS OWN WEIGHT

The body force components are $f_x = 0, \quad f_\theta = p\sin\theta, \quad f_z = -p\cos\theta$ where p is the weight per unit area. From Eq. (3), we obtain

$$N_\theta = -pR\cos\theta$$

Equations (2) and (1), respectively, yield

$$N_{x\theta} = -2xp\sin\theta + C_1(\theta), \quad N_{xx} = \frac{p}{R}x^2\cos\theta - \frac{x}{R}C_1'(\theta) + C_2(\theta)$$

Since $N_{x\theta}(-L/2,\theta) = -N_{x\theta}(L/2,\theta)$ (by symmetry), we must have. $C_1(\theta) = 0$. Also, the boundary conditions $N_{xx}(\pm L/2) = 0$ give

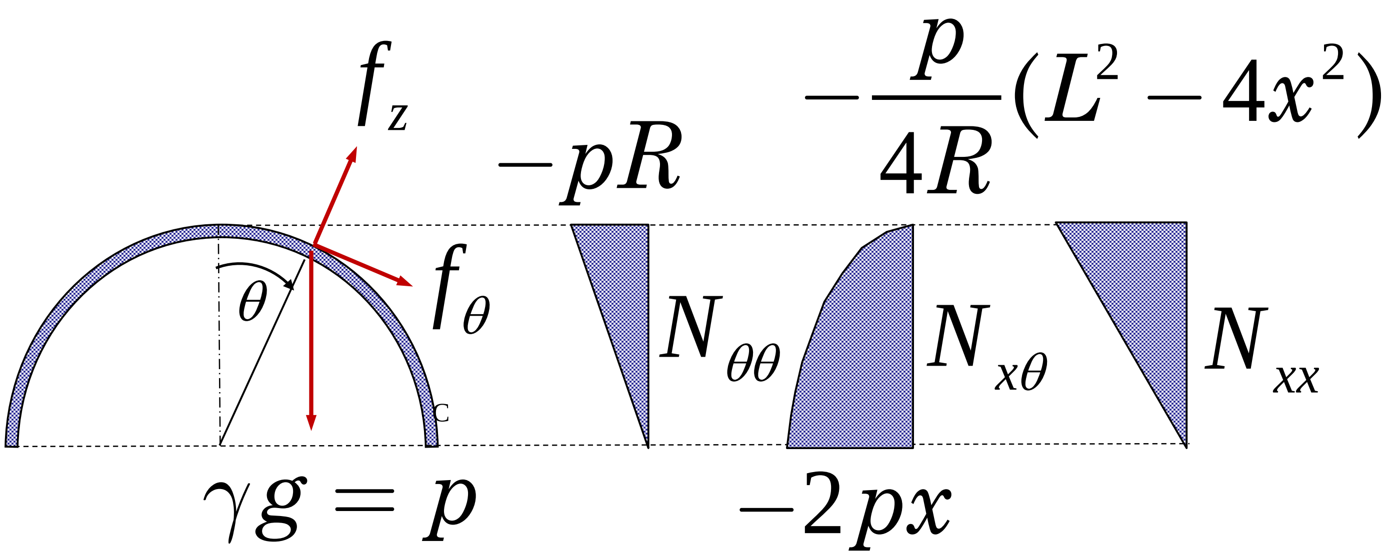
$$C_2(\theta) = -\frac{pL^2}{4R}\cos\theta$$

Thus, the complete solution is

$$N_\theta = -pR\cos\theta, \; N_{x\theta} = -2xp\sin\theta, N_{xx} = -\frac{p}{4R}\left(L^2 - 4x^2\right)\cos\theta$$

EXAMPLE 2: Cylindrical panel under its own weight (continued)

Plots of the variations of these forces with θ, for a fixed x, are shown in figure below.



EXAMPLE 2: Cylindrical panel under its own weight (continued)

The displacements can be determined using the constitutive relations

$$\begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \end{Bmatrix} = \frac{Eh}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{w_0}{R} \\ \frac{\partial v_0}{\partial x} + \frac{1}{R}\frac{\partial u_0}{\partial \theta} \end{Bmatrix}$$

Inverting the relations

$$\frac{1}{Eh} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{w_0}{R} \\ \frac{1}{R}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \end{Bmatrix}$$

EXAMPLE 2: Cylindrical shell filled with liquid (continued)

From the first equation, we have

$$\frac{\partial u_0}{\partial x} = \frac{1}{Eh}\left(N_{xx} - \nu N_{\theta\theta}\right) = \frac{p}{ERh}\left(x^2 - \frac{L^2}{4} + \nu R^2\right)\cos\theta$$

Integrating with respect to *x* gives

$$u_0(x,\theta) = \frac{px}{ERh}\left(\frac{x^2}{3} - \frac{L^2}{4} + \nu R^2\right)\cos\theta + C_3(\theta)$$

Using the boundary condition (by symmetry)

$$u_0(0,\theta) = 0,\; C_3(\theta) = 0$$

The third equation can now be expressed as

$$\frac{\partial v_0}{\partial x} = -\frac{1}{R}\frac{\partial u_0}{\partial \theta} + \frac{2(1+\nu)}{Eh}N_{x\theta} = \frac{px}{ER^2h}\left[\frac{x^2}{3} - \frac{L^2}{4} - (4+3\nu)R^2\right]\sin\theta$$

EXAMPLE 2: Cylindrical panel under its own weight (continued)

Upon integration, we have

$$v_0(x,\theta) = \frac{px^2}{2ER^2h}\left[\frac{x^2}{6} - \frac{L^2}{4} - (4+3\nu)R^2\right]\sin\theta + C_4(\theta)$$

The boundary condition $v_0(L/2,\theta) = 0$ allows us to calculate $C_4(\theta)$ as

$$C_4(\theta) = \frac{pL^2}{8ER^2h}\left[\frac{5L^2}{24} + (4+3\nu)R^2\right]\sin\theta$$

We have

$$v_0(x,\theta) = \frac{p}{192EhR^2}\left(L^2 - 4x^2\right)\left[\left(5L^2 - 4x^2\right) + 24(4+3\nu)R^2\right]\sin\theta$$

EXAMPLE 2: Cylindrical panel under its own weight (continued)

Finally, using the second equation, we can write

$$w_0 = -\frac{\partial v_0}{\partial \theta} + \frac{R}{Eh}\left(N_{\theta\theta} - \nu N_{xx}\right)$$

$$= -\frac{p}{192EhR^2}\left\{\left(L^2 - 4x^2\right)\left[\left(5L^2 - 4x^2\right) + 24(4+\nu)R^2\right] + 192R^4\right\}\cos\theta$$

Thus, the displacement field is given by

$$u_0(x,\theta) = \frac{px}{12ERh}\left(4x^2 - 3L^2 + 12\nu R^2\right)\cos\theta$$

$$v_0(x,\theta) = \frac{p}{192EhR^2}\left(L^2 - 4x^2\right)\left[\left(5L^2 - 4x^2\right) + 24(4+3\nu)R^2\right]\sin\theta$$

$$w_0(x,\theta) = -\frac{p}{192EhR^2}\left\{\left(L^2 - 4x^2\right)\left[\left(5L^2 - 4x^2\right) + 24(4+\nu)R^2\right] + 192R^4\right\}\cos\theta$$

We note that $w_0(L/2,\theta) \neq 0$, implying that there is some bending state of stress at $x = L/2$.

Flexural Theory for Axisymmetric Loads

If the cylindrical shell is axisymmetrically loaded, i.e., the shell is subjected to only forces normal to the surface, the deformation is independent of θ (i.e., $v_0 = 0$), $(N_{x\theta}, M_{x\theta}, Q_\theta)$ are zero, and $(N_{\theta\theta}, M_{\theta\theta})$ are constant. Then the second and fifth equations of equilibrium of cylindrical shells (slide 20) are trivially satisfied, and the remaining three equations take the form

$$N_{xx} = 0, \quad \frac{dQ_x}{dx} - \frac{N_{\theta\theta}}{R} = -f_z, \quad \frac{dM_{xx}}{dx} - Q_x = 0$$

There are two equations in three unknowns, requiring us to use kinematic relations. We consider here isotropic material:

$$A_{11} = A_{22} = A = Eh/(1-\nu^2), \; A_{12} = \nu A$$
$$D_{11} = D_{22} = D = Eh^3/12(1-\nu^2), \; D_{12} = \nu D$$

Flexural Theory of Thin Shells for Axisymmetric Loads

Stress resultant-displacement relations

$$N_{xx} = \frac{Eh}{1-\nu^2}\left(\frac{du_0}{dx} + \nu\frac{w_0}{R}\right) = 0 \rightarrow \frac{du_0}{dx} = -\nu\frac{w_0}{R}$$

$$N_{\theta\theta} = \frac{Eh}{1-\nu^2}\left(\nu\frac{du_0}{dx} + \frac{w_0}{R}\right) = \frac{Ehw_0}{R}$$

$$M_{xx} = \frac{Eh^3}{12(1-\nu^2)}\frac{d\varphi_1}{dx} = -D\frac{d^2w_0}{dx^2}$$

$$M_{\theta\theta} = \nu M_{xx} = -\nu D\frac{d^2w_0}{dx^2},\quad Q_x = \frac{dM_{xx}}{dx} = -D\frac{d^3w_0}{dx^3}$$

Equilibrium equation in terms of transverse deflection

$$\frac{d^2M_{xx}}{dx^2} - \frac{N_{\theta\theta}}{R} = -f_z \Rightarrow \frac{d^2}{dx^2}\left(D\frac{d^2w_0}{dx^2}\right) + \frac{Ehw_0}{R^2} = f_z$$

EXAMPLE 3: Flexure of Thin Shells for Axisymmetric Loads

Consider a <u>long</u> circular cylindrical shell of radius R, subjected to uniform bending moment M_0 and shearing force Q_0 at the end x = 0. Determine the deflection w_0.

The displacement field is given by

$$w_0(x) = e^{\alpha x}\left(K_1 \cos \alpha x + K_2 \sin \alpha x\right) + e^{-\alpha x}\left(K_3 \cos \alpha x + K_4 \sin \alpha x\right)$$

where $\alpha^4 = \dfrac{Eh}{4R^2 D}$.

Since the applied loads M_0 and Q_0 are expected to produce local bending and shear and their influence on the solution w_0 is expected to die out rapidly with x increasing (St. Venant's principle), the constants K_1 and K_2 must be zero, giving the solution

EXAMPLE 3: Flexure of Thin Shells for Axisymmetric Loads (cont.)

$$w_0(x) = e^{-\alpha x}\left(K_3 \cos\alpha x + K_4 \sin\alpha x\right)$$

The remaining constants, K_3 and K_4, are determined using the boundary conditions at $x = 0$:

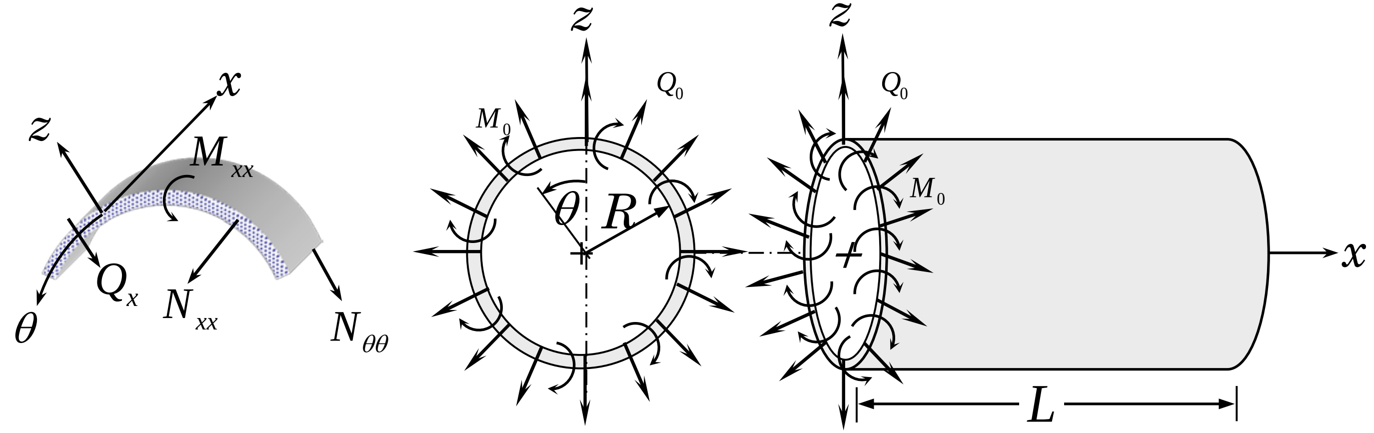
$$M_{xx}(0) = \left(-D\frac{d^2 w_0}{dx^2}\right)_{x=0} = -M_0, \quad Q_x(0) = \left(-D\frac{d^3 w_0}{dx^3}\right)_{x=0} = -Q_0$$

We obtain $K_3 = \dfrac{Q_0 + \alpha M_0}{2\alpha^3 D}, \quad K_4 = -\dfrac{M_0}{2\alpha^2 D}$

The solution becomes

$$w_0(x) = \frac{1}{2\alpha^3 D} e^{-\alpha x}\left[\alpha M_0\left(\cos\alpha x - \sin\alpha x\right) + Q_0 \cos\alpha x\right] \quad (1)$$

EXAMPLE 3: Flexure of Thin Shells for Axisymmetric Loads (cont.)



$$w_0 = \frac{1}{2\alpha^3 D}\left[\alpha M_0 f_2(\alpha x) + Q_0 f_3(\alpha x)\right], \; \frac{dw_0}{dx} = \frac{1}{2\alpha^2 D}\left[2\alpha M_0 f_3(\alpha x) + Q_0 f_1(\alpha x)\right]$$

$$M_{xx} = \frac{1}{\alpha}\left[\alpha M_0 f_1(\alpha x) + Q_0 f_4(\alpha x)\right], \; M_{\theta\theta} = \frac{\nu}{\alpha}\left[\alpha M_0 f_1(\alpha x) + Q_0 f_4(\alpha x)\right]$$

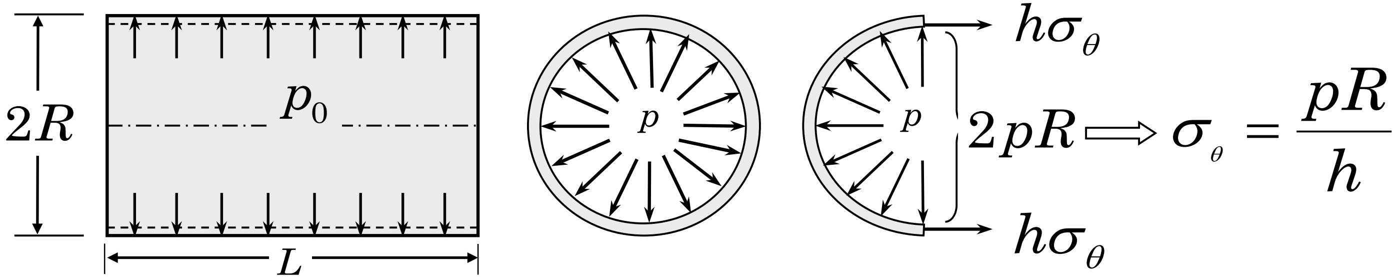
$$N_{\theta\theta} = \frac{Eh}{2R\alpha^3 D}\left[\alpha M_0 f_2(\alpha x) + Q_0 f_3(\alpha x)\right]$$

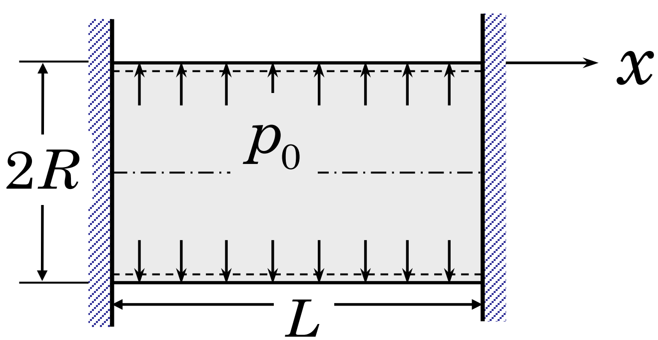
$$f_1(\alpha x) = e^{-\alpha x}\left(\cos\alpha x + \sin\alpha x\right), \quad f_2(\alpha x) = e^{-\alpha x}\left(\cos\alpha x - \sin\alpha x\right),$$

$$f_3(\alpha x) = e^{-\alpha x}\cos\alpha x, \quad f_4(\alpha x) = e^{-\alpha x}\sin\alpha x$$

EXAMPLE 4: Flexure of Thin Shells for Axisymmetric Loads

Consider a long isotropic circular cylindrical shell of radius R and thickness h, subjected to uniform internal pressure of intensity p. Determine the deflection w_0 *and bending moment* M_{xx} *when the edges are built-in.*





EXAMPLE 4: Flexure of Thin Shells for Axisymmetric Loads

If the shell is free of any geometric constraints, the shell experiences membrane forces and hoop and circumferential stresses of

$$N_{xx} = \frac{pR}{2},\ N_{\theta\theta} = pR, \sigma_{xx} = \frac{pR}{2h},\ \sigma_{\theta\theta} = \frac{pR}{h}$$

where h is the thickness of the cylinder. The cylinder experiences an increase in the radius of the cylinder by the amount

$$\delta_0 = R\varepsilon_{\theta\theta} = \frac{R}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{xx}\right) = \frac{pR^2}{2Eh}\left(2-\nu\right)$$

Since the ends of the cylinder are restrained from moving out, the shell develops local bending stresses at the edges. If the shell is sufficiently long, we can use the solution (1) of Example 3 to

EXAMPLE 4: Flexure of Thin Shells for Axisymmetric Loads (continued)

determine the bending moment M_0 and shear force Q_0 developed at the ends that produce zero deflection and slope there. Thus, the deflection for the present problem is the sum of the deflection in Eq. (1) and δ_0

$$w_0(x) = \frac{1}{2\alpha^3 D} e^{-\alpha x}\left[\alpha M_0\left(\cos\alpha x - \sin\alpha x\right) + Q_0 \cos\alpha x\right] + \delta_0$$

The boundary conditions $w_0 = 0, \frac{dw_0}{dx} = 0 \text{ at } x = 0$ yield

$$M_0 = 2\alpha^2 D\delta_0 = \frac{p}{2\alpha^2}, \quad Q_0 = -4\alpha^3 D\delta_0 = -\frac{p}{\alpha}$$

EXAMPLE 4: Flexure of Thin Shells for Axisymmetric Loads (continued)

The solution becomes

$$w_0(x) = \frac{pR^2}{Eh}\left[1 - e^{-\alpha x}\left(\cos\alpha x + \sin\alpha x\right)\right],$$

$$M_{xx}(x) = \frac{p}{2\alpha^2} e^{-\alpha x}\left(\sin\alpha x - \cos\alpha x\right),\ \ Q_x(x) = \frac{p}{\alpha} e^{-\alpha x}\cos\alpha x$$

The maximum deflection occurs for large values of x and it is equal to δ_0 ; the maximum bending and shear forces occur at x = 0 and they are

$$w_{\max} = \delta_0,\ \ M_{\max} = -\frac{p}{2\alpha^2},\ \ Q_{\max} = \frac{p}{\alpha}$$

More examples can be found in the author's book on plates and shells.