HIGHER-ORDER THEORIES

- THIRD-ORDER SHEAR DEFORMATION PLATE THEORY
- LAYERWISE LAMINATE THEORY

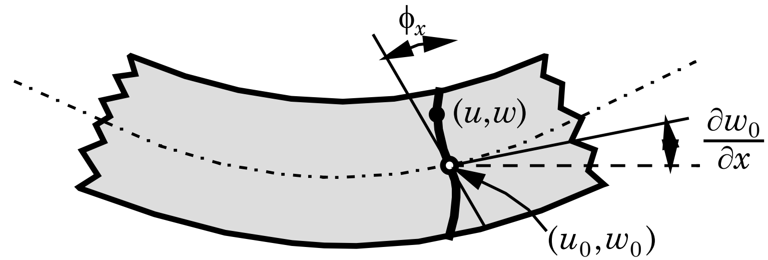
Third-Order Shear Deformation Plate Theory

Assumed Displacement Field

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t) - c_1 z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right)$$

$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t) - c_1 z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right)$$

$$w(x,y,z,t) = w_0(x,y,t)$$



Transverse Shear Strains

$$\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = \begin{Bmatrix}\gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\end{Bmatrix} + z^2\begin{Bmatrix}\gamma_{yz}^{(2)}\\ \gamma_{xz}^{(2)}\end{Bmatrix}, \quad c_2 = 3c_1$$

$$\begin{Bmatrix}\gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\end{Bmatrix} = \begin{Bmatrix}\phi_y + \frac{\partial w_0}{\partial y}\\ \phi_x + \frac{\partial w_0}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}\gamma_{yz}^{(2)}\\ \gamma_{xz}^{(2)}\end{Bmatrix} = -c_2\begin{Bmatrix}\left(\phi_y + \frac{\partial w_0}{\partial y}\right)\\ \left(\phi_x + \frac{\partial w_0}{\partial x}\right)\end{Bmatrix}$$

Transverse Shear Stresses

$$\sigma_{yz}^{(k)} = \bar{Q}_{44}^{(k)}\left(\gamma_{yz}^{(0)} + z^2\gamma_{yz}^{(2)}\right) + \bar{Q}_{45}^{(k)}\left(\gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)}\right)$$

$$\sigma_{xz}^{(k)} = \bar{Q}_{45}^{(k)}\left(\gamma_{yz}^{(0)} + z^2\gamma_{yz}^{(2)}\right) + \bar{Q}_{44}^{(k)}\left(\gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)}\right)$$

$$\sigma_{xz}(x,y,\pm\frac{h}{2}) = 0, \quad \sigma_{yz}(x,y,\pm\frac{h}{2}) = 0 \;\rightarrow\; c_2 = 4/h^2$$

Third-Order Shear Deformation Plate Theory (TSDT) (Continued)

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 + J_1 \ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v}_0 + J_1 \ddot{\phi}_y - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y}$$

$$\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right)$$

$$+ c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}\right) + q = I_0 \ddot{w}_0 - c_1^2 I_6\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right)$$

$$+ c_1\left[I_3\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) + J_4\left(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}\right)\right]$$

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = J_1 \ddot{u}_0 + K_2 \ddot{\phi}_x - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial x}$$

$$\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = J_1 \ddot{v}_0 + K_2 \ddot{\phi}_y - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial y}$$

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta}, \quad \bar{Q}_\alpha = Q_\alpha - c_2 R_\alpha$$

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, (z)^i \, dz \quad (i = 0, 1, 2, \cdots, 6)$$

Third-Order Shear Deformation Plate Theory (TSDT) (Continued)

Higher-Order Stress Resultants

$$\left\{\begin{matrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{matrix}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{matrix}\right\} z^3 dz, \quad \left\{\begin{matrix} R_x \\ R_y \end{matrix}\right\} = \int_{-h/2}^{h/2} \left\{\begin{matrix} \sigma_{yz} \\ \sigma_{xz} \end{matrix}\right\} z^2 dz$$

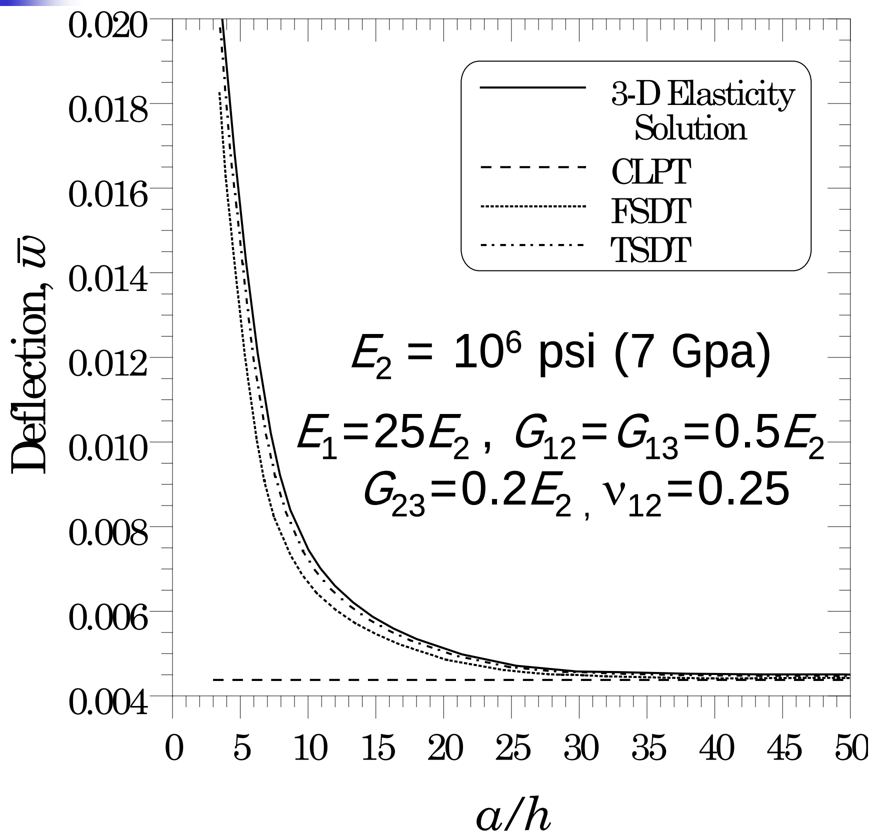
$$\text{Primary Variables}: \quad u_n, \ u_s, \ w_0, \ \frac{\partial w_0}{\partial n}, \ \phi_n, \ \phi_s$$

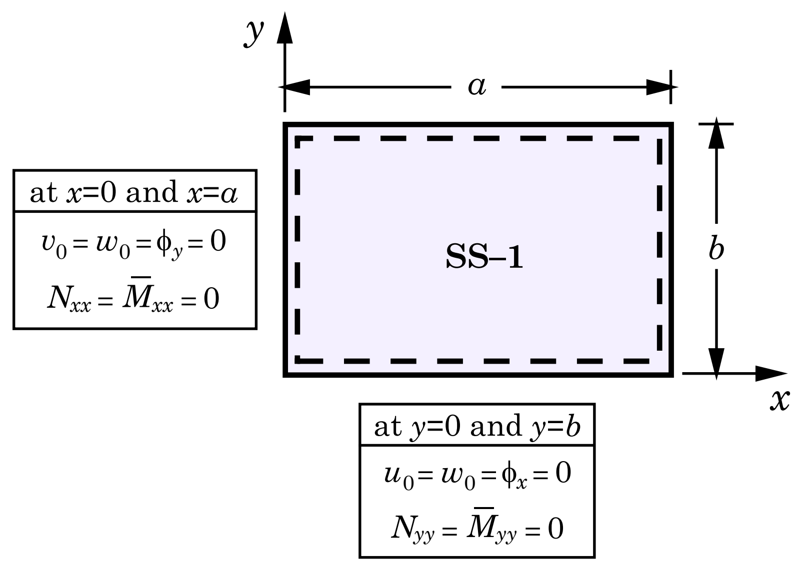
$$\text{Secondary Variables}: \quad N_{nn}, \ N_{ns}, \ \bar{V}_n, \ P_{nn}, \ \bar{M}_{nn}, \ \bar{M}_{ns}$$

$$\begin{aligned} \bar{V}_n \equiv c_1 & \left[\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y}\right) n_x + \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y}\right) n_y\right] \\ & - c_1 \left[\left(I_3 \ddot{u}_0 + J_4 \ddot{\phi}_x - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial x}\right) n_x + \left(I_3 \ddot{v}_0 + J_4 \ddot{\phi}_y - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial y}\right) n_y\right] \\ & + (\bar{Q}_x n_x + \bar{Q}_y n_y) + \mathcal{P}(w_0) + c_1 \frac{\partial P_{ns}}{\partial s} \end{aligned}$$

$$\mathcal{P}(w_0) = \left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y}\right) n_x + \left(N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y}\right) n_y$$

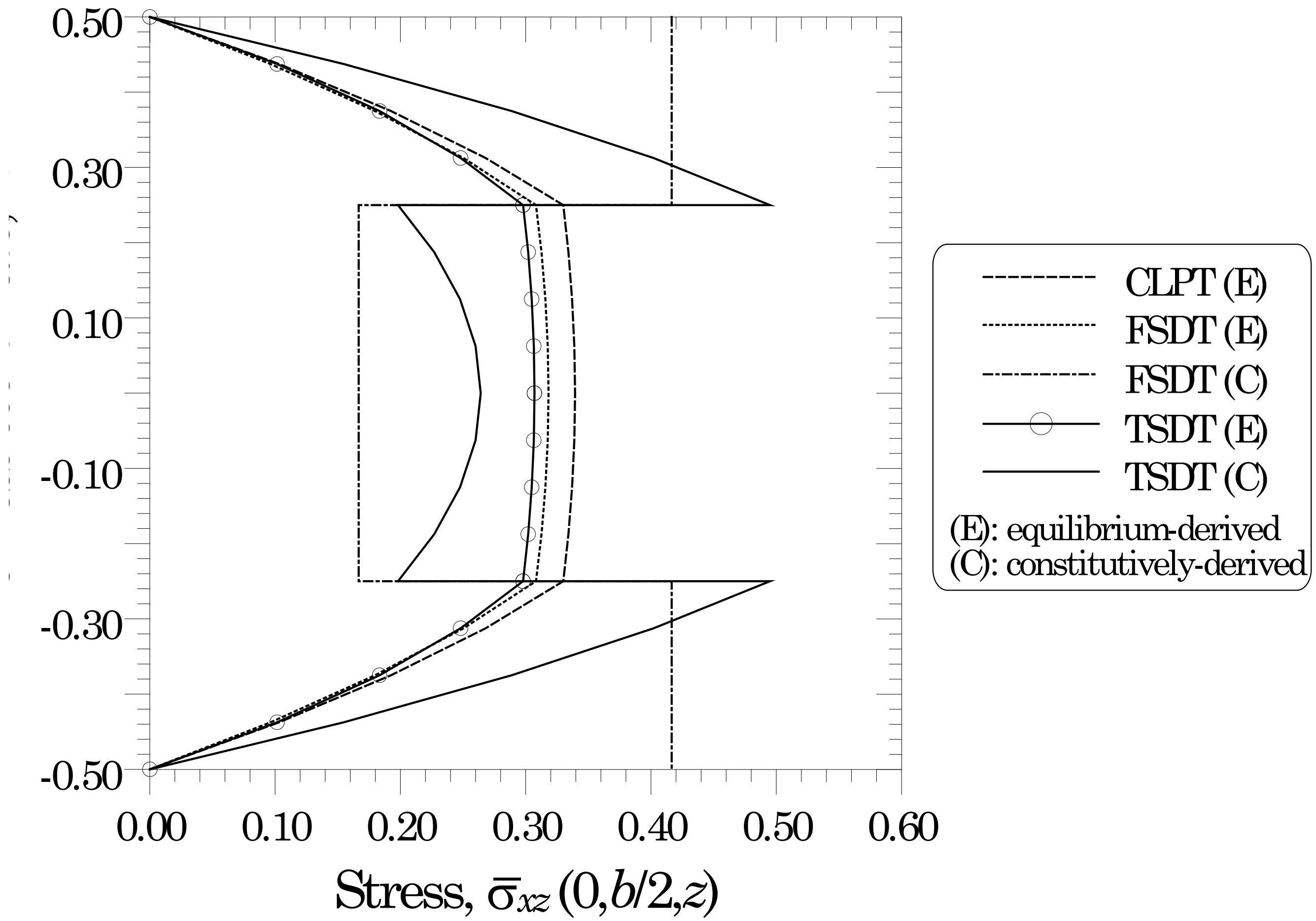
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



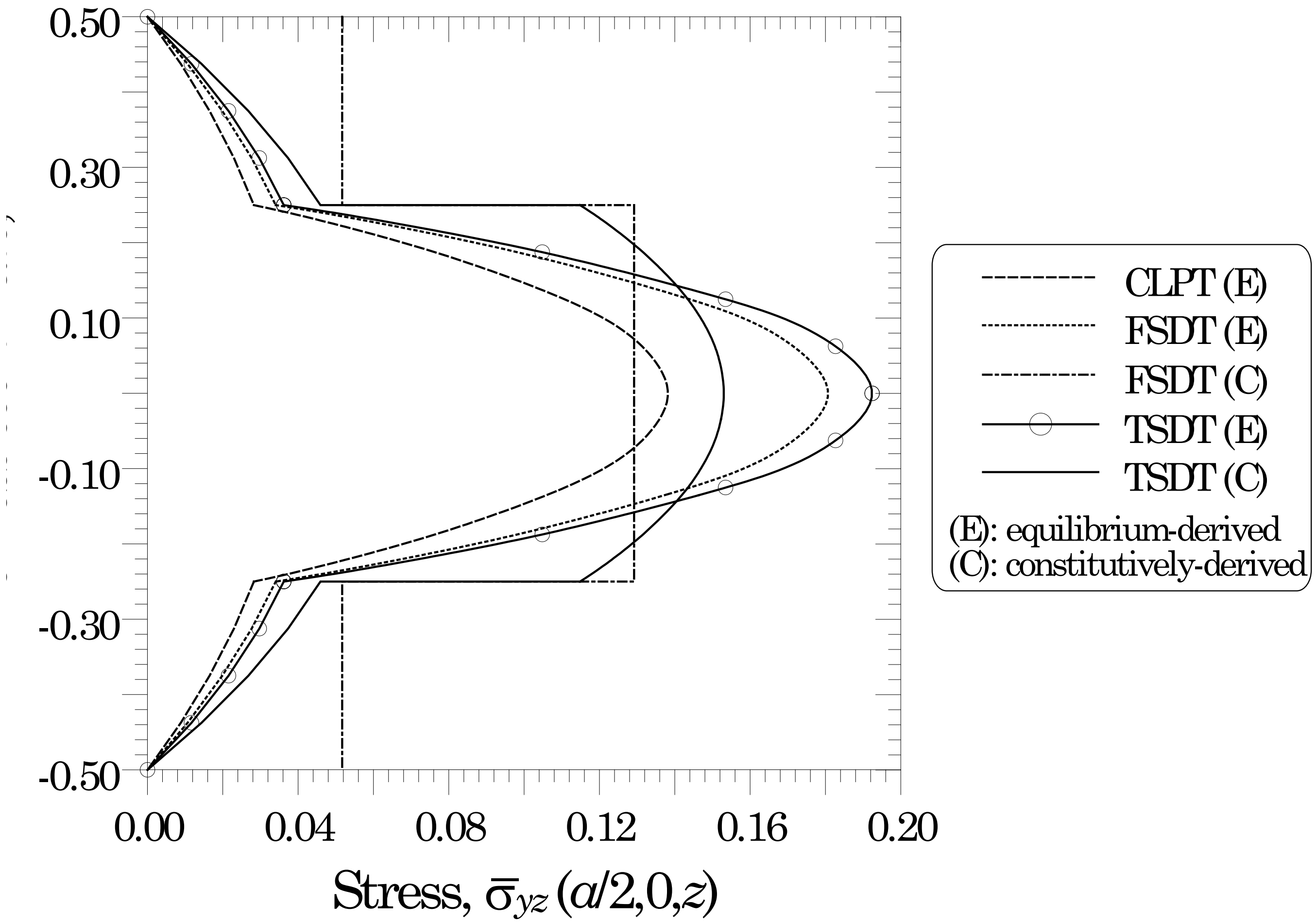


SS-1 Boundary Conditions

Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load

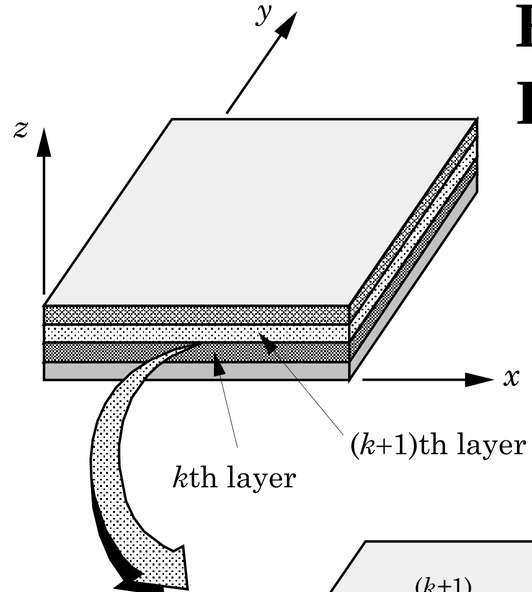


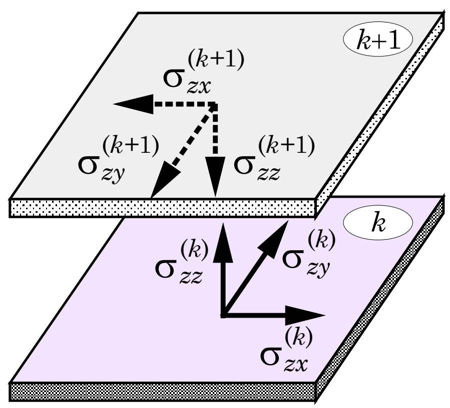
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



LAYERWISE THEORY

Equilibrium of Interlaminar Stresses

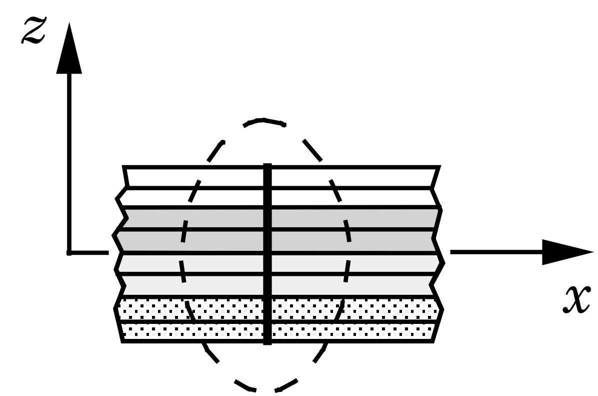




$$\sigma_{zx}^{(k+1)} = \sigma_{zx}^{(k)}$$

$$\sigma_{zy}^{(k+1)} = \sigma_{zy}^{(k)}$$

$$\sigma_{zz}^{(k+1)} = \sigma_{zz}^{(k)}$$



Layerwise Laminate Theory

Equilibrium Requirements

$$\left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\}^{(k)} \neq \left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\}^{(k+1)}, \quad \left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k)} = \left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k+1)}$$

$$\left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k)} = \left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k+1)} \rightarrow \left\{\begin{matrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{matrix}\right\}^{(k)} \neq \left\{\begin{matrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{matrix}\right\}^{(k+1)}$$

Single-Layer Theories

$$\left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\}^{(k)} \neq \left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\}^{(k+1)}, \quad \left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k)} \neq \left\{\begin{matrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{matrix}\right\}^{(k+1)}$$

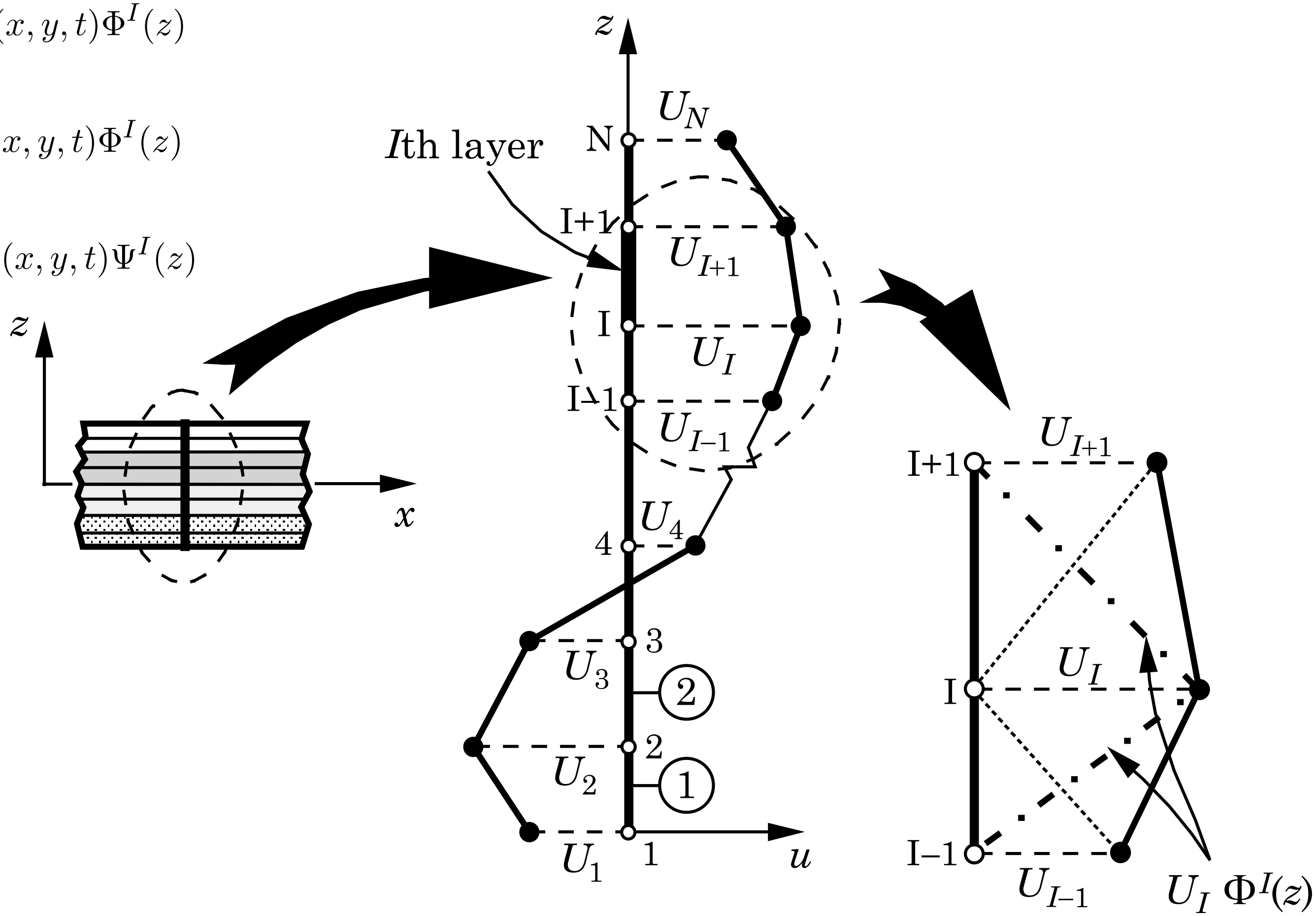
$$\left\{\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{matrix}\right\}^{(k)} = \left\{\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{matrix}\right\}^{(k+1)}, \quad \left\{\begin{matrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{matrix}\right\}^{(k)} = \left\{\begin{matrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{matrix}\right\}^{(k+1)}$$

Layerwise Kinematic Model

$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$



Layerwise Displacement Field, Governing Equations, and FEM Approximation

$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

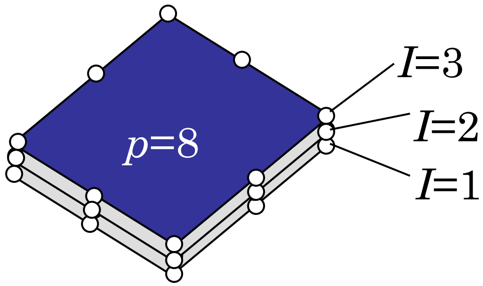
$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$

$$\frac{\partial N_{xx}^I}{\partial x} + \frac{\partial N_{xy}^I}{\partial y} - Q_x^I = \sum_{J=1}^{N} I^{IJ}\frac{\partial^2 U_J}{\partial t^2}$$

$$\frac{\partial N_{xy}^I}{\partial x} + \frac{\partial N_{yy}^I}{\partial y} - Q_y^I = \sum_{J=1}^{N} I^{IJ}\frac{\partial^2 V_J}{\partial t^2}$$

$$\frac{\partial \tilde{Q}_x^I}{\partial x} + \frac{\partial \tilde{Q}_y^I}{\partial y} - \tilde{Q}_z^I + \tilde{\mathcal{N}}^I + q_b\delta_{I1} + q_t\delta_{IM} = \sum_{J=1}^{M} \tilde{I}^{IJ}\frac{\partial^2 W_J}{\partial t^2}$$

Finite element approximation



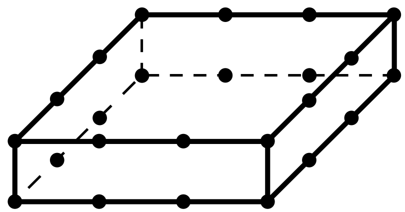
$$U_I(x,y,t) = \sum_{j=1}^{p} U_I^j(t)\psi_j(x,y)$$

$$V_I(x,y,t) = \sum_{j=1}^{p} V_I^j(t)\psi_j(x,y)$$

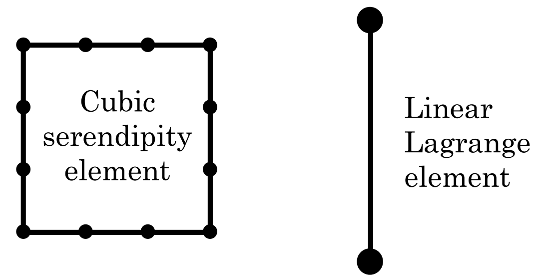
$$W_I(x,y,t) = \sum_{j=1}^{q} W_I^j(t)\varphi_j(x,y)$$

Layerwise Kinematic Model

Conventional 3D	Layerwise 2D + 1D

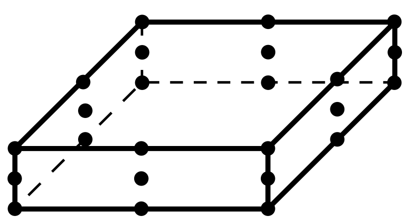


(1a)

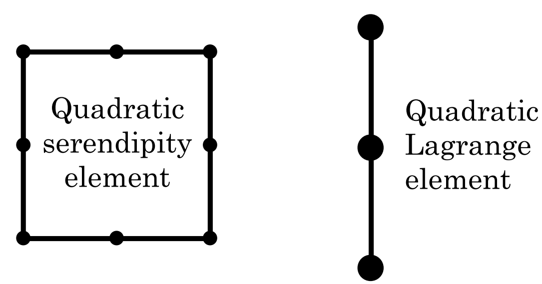


(in-plane) (through thickness)

(1b)



(2a)



(in-plane) (through thickness)

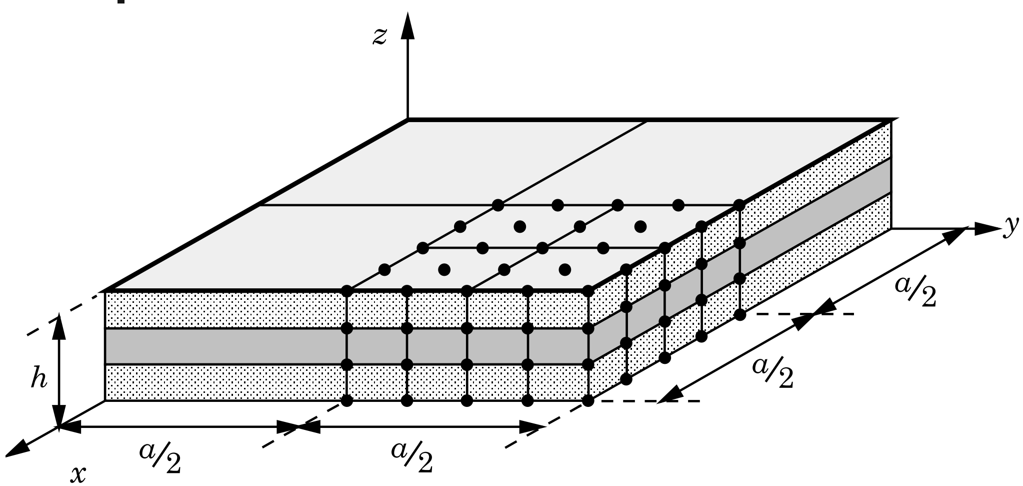
(2b)

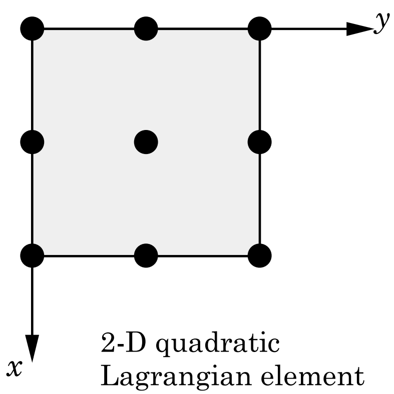
Table: Comparison of the number of operations needed to form the element stiffness matrices for equivalent elements in the conventional 3-D format and the layerwise 2-D format. Full quadrature is used in all.

Element Type†	Multipli.	Addition	Assignments
1a (3-D)	1,116,000	677,000	511,000
1b (LWPT)	423,000	370,000	106,000
2a (3-D)	1,182,000	819,000	374,000
2b (LWPT)	284,000	270,000	69,000

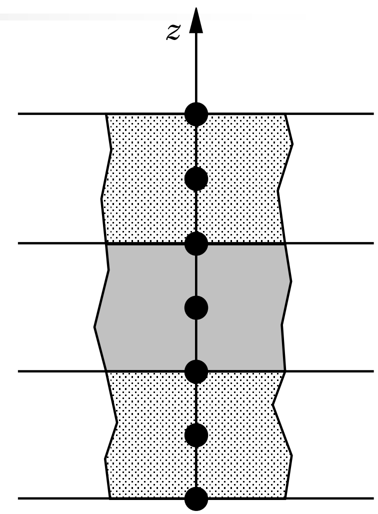
† *Element* 1*a*: 72 degrees of freedom, 24-node 3-D isoparametric hexahedron with cubic in-plane interpolation and linear transverse interpolation.
Element 1*b*: 72 degrees of freedom, E12–L1 layerwise element.
Element 2*a*: 81 degrees of freedom, 27-node 3-D isoparametric hexahedron with quadratic interpolation in all three directions.
Element 2*b*: 81 degrees of freedom, E9–Q1 layerwise element.

Layerwise Kinematic Model
3D modeling with 2D & 1D elements





2-D quadratic
Lagrangian element



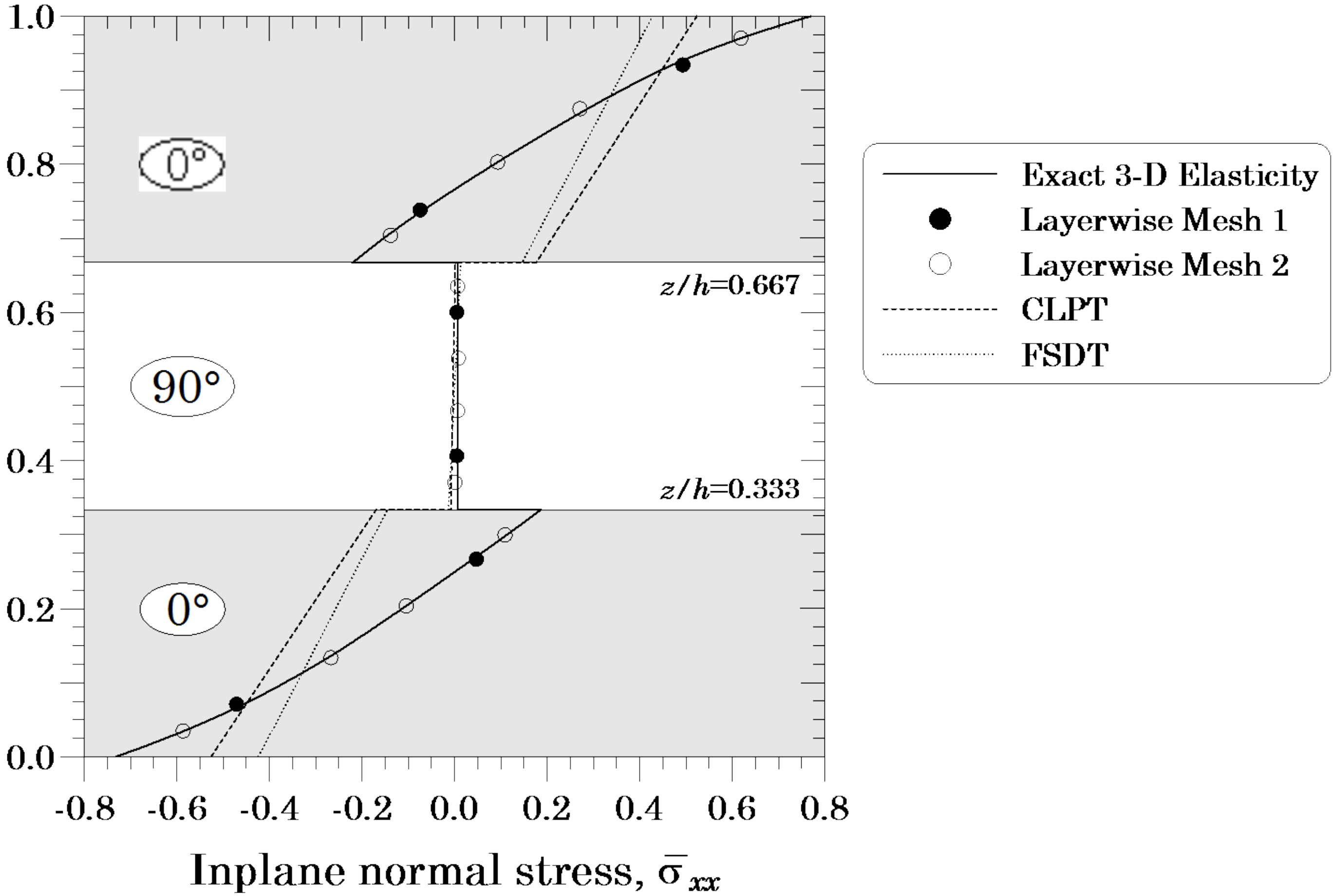
three quadratic layers
through the thickness

$$E_1 = 25 \times 10^6 \text{ psi}, \quad E_2 = E_3 = 10^6 \text{ psi}$$

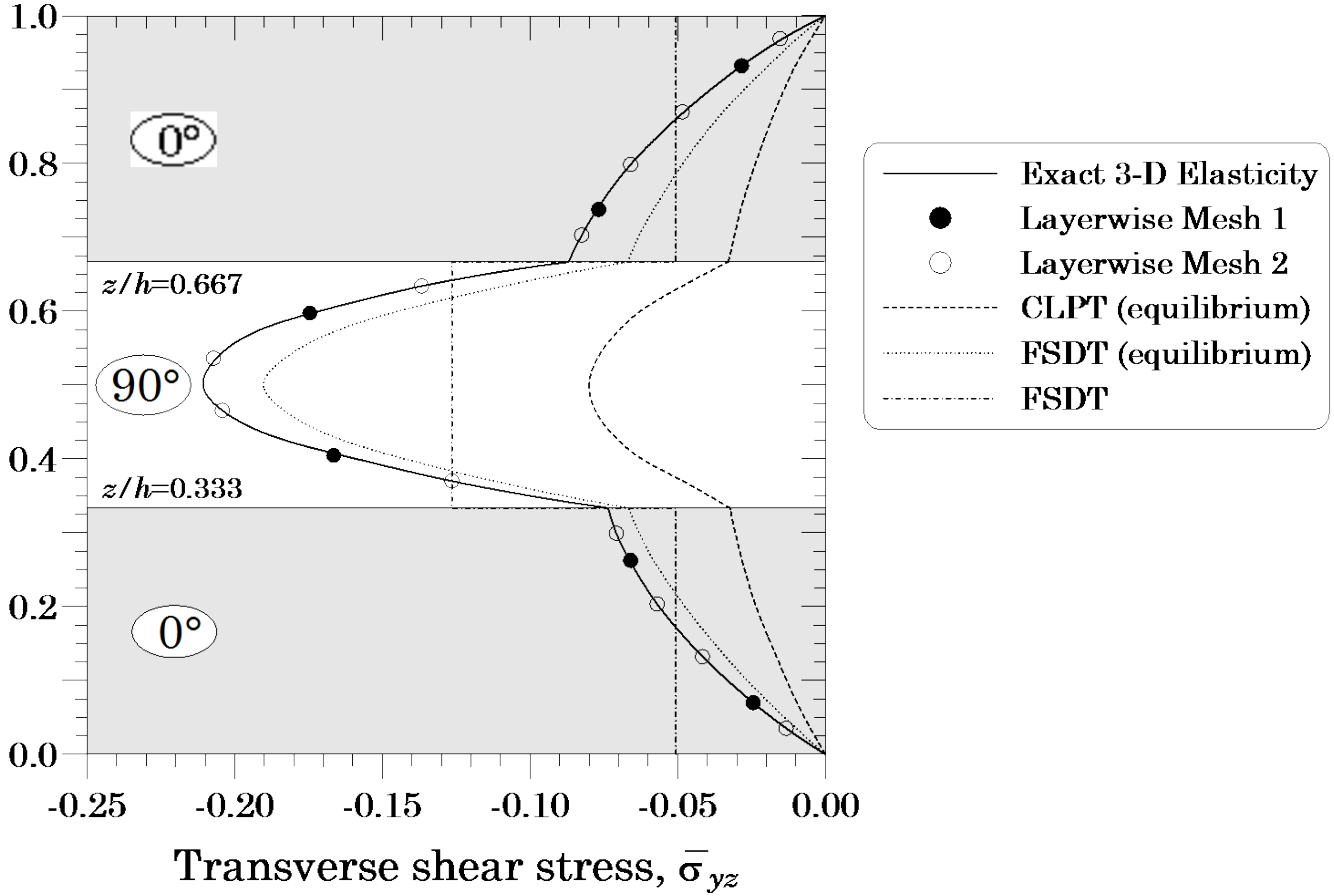
$$G_{12} = 0.5{\times}10^6 \text{ psi}, \; G_{13} = G_{23} = 0.2{\times}10^6 \text{ psi}, \; \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

$$\begin{aligned} u(x, a/2, z) &= u(a/2, y, z) = 0 \\ v(a/2, y, z) &= u(x, a/2, z) = 0 \\ w(x, a, z) &= u(a, y, z) = 0 \end{aligned}$$

Validation of the Layerwise Theory



Verification of the Layerwise Theory



SUMMARY

In this lecture, we have discussed the following topics:

Third-order Shear Deformation Plate Theory
- **Development of governing equations**
- **Numerical results**

Layerwise Laminate Theory
- **Development of governing equations**
- **Numerical results**