NONLINEAR ANALYSIS OF PLATE BENDING

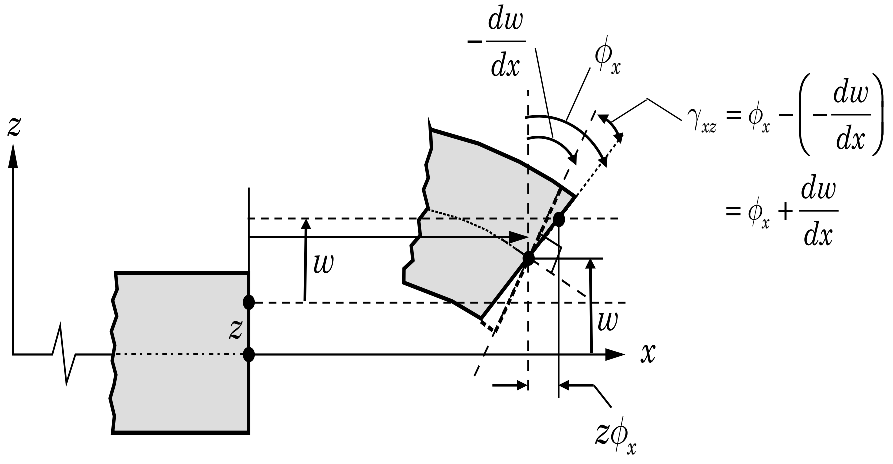
CONTENTS

- Governing Equations of the First-Order Shear Deformation theory (FSDT)
- Finite element models of FSDT
- Shear and membrane locking
- Computer implementation
- Stress calculation
- Numerical Examples

THE FIRST-ORDER SHEAR DEFORMATION THEORY

Displacement Field of the FSDT

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$



Nonlinear strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

Von Karman Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial u_m}{\partial x_1}\frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

NONLINEAR STARINS OF THE FSDT

Actual Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x} = \varepsilon_{xx}^0 + z\,\varepsilon_{xx}^1$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y} = \varepsilon_{yy}^0 + z\,\varepsilon_{yy}^1$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) = \gamma_{xy}^0 + z\,\gamma_{xy}^1$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x} = \gamma_{xz}^0, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y} = \gamma_{yz}^0$$

Virtual Nonlinear strains

$$\delta\varepsilon_{xx} = \frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + z\frac{\partial \delta\phi_x}{\partial x} = \delta\varepsilon_{xx}^0 + z\,\delta\varepsilon_{xx}^1$$

$$\delta\varepsilon_{yy} = \frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + z\frac{\partial \delta\phi_y}{\partial y} = \delta\varepsilon_{yy}^0 + z\,\delta\varepsilon_{yy}^1$$

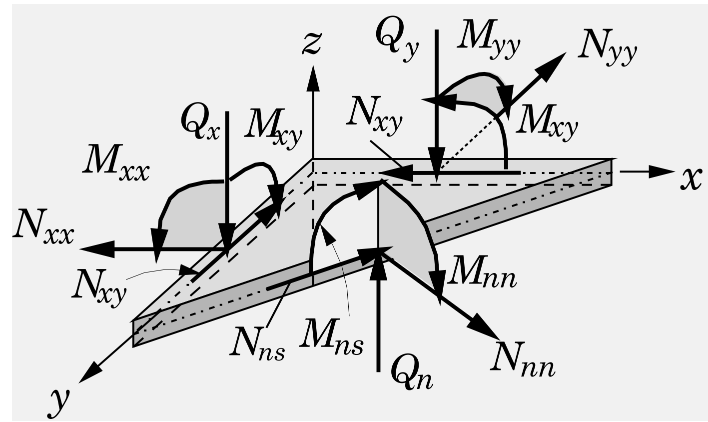
$$\delta\gamma_{xy} = \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + z\left(\frac{\partial \delta\phi_x}{\partial y} + \frac{\partial \delta\phi_y}{\partial x}\right) = \delta\gamma_{xy}^0 + z\,\delta\gamma_{xy}^1$$

$$\delta\gamma_{xz} = \delta\phi_x + \frac{\partial \delta w}{\partial x} = \delta\gamma_{xz}^0, \quad \delta\gamma_{yz} = \delta\phi_y + \frac{\partial \delta w}{\partial y} = \delta\gamma_{yz}^0$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

$$0 = \delta W^e \equiv \delta W_I^e + \delta W_E^e$$

$$\delta W_I^e = \int_{\Omega^e} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{xx} \left(\delta\varepsilon_{xx}^0 + z\delta\varepsilon_{xx}^1 \right) + \sigma_{yy} \left(\delta\varepsilon_{yy}^0 + z\delta\varepsilon_{yy}^1 \right) + \sigma_{xy} \left(\delta\gamma_{xy}^0 + z\delta\gamma_{xy}^1 \right) + K_s\sigma_{xz}\delta\gamma_{xz}^0 + K_s\sigma_{yz}\delta\gamma_{yz}^0 \right] dz \right\} dx\,dy$$



$$\delta W_E^e = -\left\{ \oint_{\Gamma^e} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{nn} \left(\delta u_n + z\delta\phi_n \right) + \sigma_{ns} \left(\delta u_s + z\delta\phi_s \right) + \sigma_{nz}\delta w \right] dz\,ds + \int_{\Omega^e} (q - kw)\delta w\; dx\,dy \right\}$$

$$0 = \int_{\Omega^e} \left[N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + N_{yy}\delta\varepsilon_{yy}^0 + M_{yy}\delta\varepsilon_{yy}^1 + N_{xy}\delta\gamma_{xy}^0 + M_{xy}\delta\gamma_{xy}^1 + Q_x\delta\gamma_{xz}^0 + Q_y\delta\gamma_{yz}^0 - q\delta w \right] dx\,dy - \oint_{\Gamma^e} \left(N_{nn}\delta u_n + N_{ns}\delta u_s + M_{nn}\delta\phi_n + M_{ns}\delta\phi_s + Q_n\delta w \right) ds$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY

Equations of equilibrium

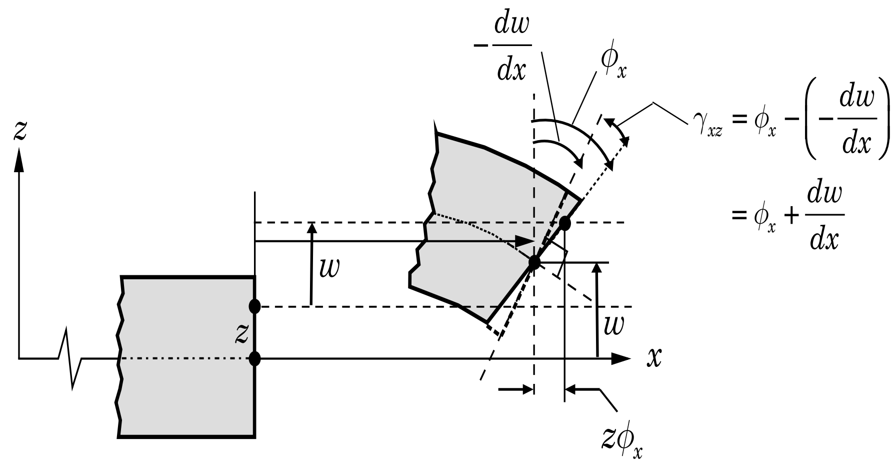
$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) - q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2}$$



Stress resultants

$$Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz} dz = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)$$

$$Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz} dz = K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

$$M_{xx} = D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}$$

$$M_{yy} = D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y}$$

$$M_{xy} = D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

Weak forms

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} + I_0 \delta u \frac{\partial^2 u}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta u N_n \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_{xy} + \frac{\partial \delta v}{\partial y} N_{yy} + I_0 \delta v \frac{\partial^2 v}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta v N_{ns} \, ds$$

$$\begin{aligned} 0 = \int_{\Omega^e} \Bigg[& \frac{\partial \delta w}{\partial x} \left(Q_x + N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \\ & + \frac{\partial \delta w}{\partial y} \left(Q_y + N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) \\ & + I_0 \delta w \frac{\partial^2 w}{\partial t^2} - \delta w q \Bigg] dxdy - \oint_{\Gamma^e} \delta w Q_n \, ds \end{aligned}$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_x}{\partial x} M_{xy} + \delta \phi_x Q_x + I_2 \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta \phi_x M_n \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_y}{\partial x} M_{yy} + \delta \phi_y Q_y + I_2 \delta \phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta \phi_y M_{ns} \, ds$$

Finite Element Models of The First-order Plate Theory (FSDT) (Continued)

Finite Element Approximation

$$u(x,y,t) = \sum_{j=1}^{m} u_j(t)\psi_j(x,y), \quad v(x,y,t) = \sum_{j=1}^{m} v_j(t)\psi_j(x,y)$$

$$w(x,y,t) = \sum_{j=1}^{n} w_j(t)\psi_j(x,y)$$

$$\phi_x(x,y,t) = \sum_{j=1}^{p} S_j^1(t)\psi_j(x,y), \quad \phi_y(x,y,t) = \sum_{j=1}^{p} S_j^2(t)\psi_j(x,y)$$

Finite Element Model

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{55} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \\ \ddot{\mathbf{w}} \\ \ddot{\mathbf{S}}_x \\ \ddot{\mathbf{S}}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ \mathbf{S}_x \\ \mathbf{S}_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \mathbf{F}^4 \\ \mathbf{F}^5 \end{Bmatrix}$$

Fully Dicretized Model and Iterative Scheme

Fully Discretized Finite Element Model

$$[\hat{K}]_{s+1}\{\Delta\}_{s+1} = \{\hat{F}\}_{s,s+1}$$

$$[\hat{K}]_{s+1} = [K]_{s+1} + a_3[M]_{s+1}$$

$$\{\hat{F}\}_{s,s+1} = \{F\}_{s+1} + [M]_{s+1}(a_3\{\Delta\}_s + a_4\{\dot{\Delta}\}_s + a_5\{\ddot{\Delta}\}_s)$$

$$a_3 = \frac{2}{\gamma(\Delta t)^2}, \quad a_4 = \frac{2}{\gamma\Delta t}, \quad a_5 = \frac{1}{\gamma} - 1$$

Accelerations and Velocities

$$\{\ddot{\Delta}\}_{s+1} = a_3(\{\Delta\}_{s+1} - \{\Delta\}_s) - a_4\{\dot{\Delta}\} - a_5\{\ddot{\Delta}\}_s$$

$$\{\dot{\Delta}\}_{s+1} = \{\dot{\Delta}\}_s + a_2\{\ddot{\Delta}\}_s + a_1\{\ddot{\Delta}\}_{s+1}$$

where $a_1 = \alpha\Delta t$ and $a_2 = (1-\alpha)\Delta t$.

Newton-Raphson Iterative Scheme

$$\{\Delta\}_{r+1}^{s+1} = -[\hat{K}^T]_r^{-1}\{\hat{R}\}_r^{s+1}, \quad [\hat{K}^T]_r \equiv \left[\frac{\partial\{\hat{R}\}}{\partial\{\Delta\}}\right]_r^{s+1}$$

Stiffness Coefficients (typical)

$$K_{ij}^{11} = \int_{\Omega^e} \left(A_{11} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \left(A_{12} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) dxdy = K_{ji}^{21}$$

$$K_{ij}^{13} = \frac{1}{2} \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial x} \left(A_{11} \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{12} \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) + A_{66} \frac{\partial \psi_i^e}{\partial y} \left(\frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dxdy$$

$$K_{ij}^{22} = \int_{\Omega^e} \left(A_{66} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{22} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$

$$K_{ij}^{23} = \frac{1}{2} \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial y} \left(A_{12} \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{22} \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) + A_{66} \frac{\partial \psi_i^e}{\partial x} \left(\frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dxdy$$

$$K_{ij}^{31} = \int_{\Omega^e} \left[\frac{\partial \varphi_i^e}{\partial x} \left(A_{11} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial \varphi_i^e}{\partial y} \left(A_{66} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{12} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) \right] dxdy$$

Tangent Stiffness Coefficients (typical)

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad R_i^\alpha = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma}\Delta_k^\gamma - F_i^\alpha, \; \Delta_i^1 = u_i, \Delta_i^2 = v_i, \; \Delta_i^3 = w_i, \; \Delta_i^4 = S_i^1, \Delta_i^5 = S_i^2$$

$$T_{ij}^{\alpha\beta} = \frac{\partial}{\partial \Delta_j^\beta}\Big(\sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma}\Delta_k^\gamma - F_i^\alpha\Big) = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_k^\gamma + K_{ij}^{\alpha\beta}$$

$$\mathbf{T}^{\alpha 1} = \mathbf{K}^{\alpha 1} = (\mathbf{K}^{1\alpha})^{\mathrm{T}}, \quad \mathbf{T}^{\alpha 2} = \mathbf{K}^{\alpha 2} = (\mathbf{K}^{2\alpha})^{\mathrm{T}}$$

$$\mathbf{T}^{\alpha 4} = \mathbf{K}^{\alpha 4} = (\mathbf{K}^{4\alpha})^{\mathrm{T}}, \quad \mathbf{T}^{\alpha 5} = \mathbf{K}^{\alpha 5} = (\mathbf{K}^{5\alpha})^{\mathrm{T}}$$

$$T_{ij}^{13} = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{1\gamma}}{\partial w_j}\Delta_k^\gamma + K_{ij}^{13} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{13}}{\partial w_j} w_k + K_{ij}^{13}$$

$$= \frac{1}{2}\int_{\Omega^e}\left[\frac{\partial \psi_i^{(1)}}{\partial x}\left(A_{11}\frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{12}\frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\right.$$

$$\left. + A_{66}\frac{\partial \psi_i^{(1)}}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right] dx\,dy + K_{ij}^{13}$$

$$= K_{ij}^{13} + K_{ij}^{13} = 2K_{ij}^{13} = T_{ji}^{31}$$

Tangent Stiffness Coefficients (typical)

$$T_{ij}^{33} = K_{ij}^{33} + \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{3\gamma}}{\partial w_j}\Delta_k^{\gamma} = K_{ij}^{33} + \sum_{k=1}^{n(\gamma)}\left(\frac{\partial K_{ik}^{31}}{\partial w_j}u_k + \frac{\partial K_{ik}^{32}}{\partial w_j}v_k + \frac{\partial K_{ik}^{33}}{\partial w_j}w_k\right)$$

$$= K_{ij}^{33} + \int_{\Omega^e}\left[\left(A_{11}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{12}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\frac{\partial u}{\partial x} + A_{66}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\frac{\partial u}{\partial y}\right]dx\,dy$$

$$+\int_{\Omega^e}\left[\left(A_{12}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{22}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\frac{\partial v}{\partial y} + A_{66}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\frac{\partial v}{\partial x}\right]dx\,dy$$

$$+\int_{\Omega^e}\left\{\left[A_{11}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{22}\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + A_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right]\right.$$

$$\left.+\frac{1}{2}(A_{12}+A_{66})\left[\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right]\right\}dx\,dy$$

Tangent Stiffness Coefficients (typical)

$$T_{ij}^{33} = K_{ij}^{33} + \int_{\Omega^e} \left\{ A_{11}\left[\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\frac{\partial v}{\partial y} + \frac{1}{2}(A_{12}+A_{66})\left(\frac{\partial w}{\partial y}\right)^2 \right\} \frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} dx\,dy$$

$$+\int_{\Omega^e} \left\{ A_{22}\left[\frac{\partial v}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2\right] + A_{12}\frac{\partial u}{\partial x} + \frac{1}{2}(A_{12}+A_{66})\left(\frac{\partial w}{\partial x}\right)^2 \right\} \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} dx\,dy$$

$$+\int_{\Omega^e} \left[A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + \frac{1}{2}(A_{12}+A_{66})\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right]\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \right) dx\,dy$$

$$T_{ij}^{33} = \int_{\Omega^e} \left(A_{55}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{44}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + k\psi_i^{(2)}\psi_j^{(2)} \right) dx\,dy$$

$$+\int_{\Omega^e} \left\{ N_{xx}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + N_{yy}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + N_{xy}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + N_{xy}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \right.$$

$$+ (A_{12}+A_{66})\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \right)$$

$$\left. + \left[A_{11}\left(\frac{\partial w}{\partial x}\right)^2 + A_{66}\left(\frac{\partial w}{\partial y}\right)^2 \right]\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + \left[A_{66}\left(\frac{\partial w}{\partial x}\right)^2 + A_{22}\left(\frac{\partial w}{\partial y}\right)^2 \right]\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} \right\} dx\,dy$$

Shear and Membrane Locking (Revisit)

Shear Locking

Use reduced integration to evaluate all *shear* stiffnesses (i.e., all K_{ij} that contain transverse shear terms)

Membrane Locking

Use reduced integration to evaluate all *membrane* stiffnesses (i.e., all K_{ij} that contain von Kármán nonlinear terms)

Post-Computation of Stress Components

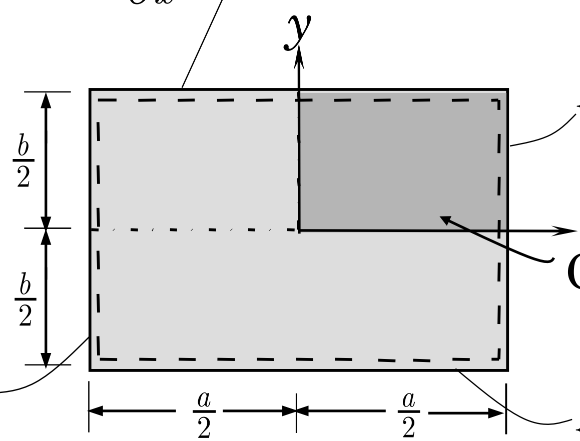
$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix},\quad \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\end{Bmatrix}=\begin{bmatrix}Q_{55} & 0\\ 0 & Q_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}$$

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{66}=G_{12},\ Q_{44}=G_{23},\ Q_{55}=G_{13},$$

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\end{Bmatrix}-z\begin{Bmatrix}\dfrac{\partial \phi_x}{\partial x}\\ \dfrac{\partial \phi_y}{\partial y}\\ \dfrac{\partial \phi_x}{\partial y}+\dfrac{\partial \phi_y}{\partial x}\end{Bmatrix},\quad \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\begin{Bmatrix}\phi_x+\dfrac{\partial w}{\partial x}\\ \phi_y+\dfrac{\partial w}{\partial y}\end{Bmatrix}$$

TYPICAL SIMPLY SUPPORT CONDITIONS for Pure Bending case

CPT: $w = \dfrac{\partial w}{\partial x} = 0$; FSDT: $w = \phi_x = 0$



$$\begin{cases} \text{CPT:} \quad w = \dfrac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{cases}$$

$$\begin{cases} \text{CPT:} \quad w = \dfrac{\partial w}{\partial x} = 0 \\ \text{FSDT:} \quad w = \phi_x = 0 \end{cases}$$

$$\left. \begin{array}{l} \text{CPT:} \quad w = \dfrac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{array} \right\}$$

$$\text{Symmetry conditions:} \begin{cases} \text{CPT:} \quad \dfrac{\partial w}{\partial x} = 0 \text{ at } x = 0; \ \dfrac{\partial w}{\partial y} = 0 \text{ at } y = 0 \\ \text{FSDT:} \ \phi_x = 0 \text{ at } x = 0; \quad \phi_y = 0 \text{ at } y = 0 \end{cases}$$

The effect of reduced integration, thickness, and mesh refinement

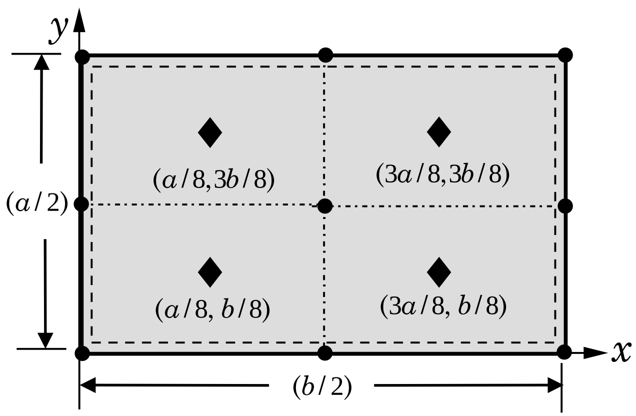
on the linear center deflections and stresses of a simply supported, isotropic (ν = 0.25) square plate under a uniform transverse load of intensity q_0.

F – full integration
M – Mixed integration

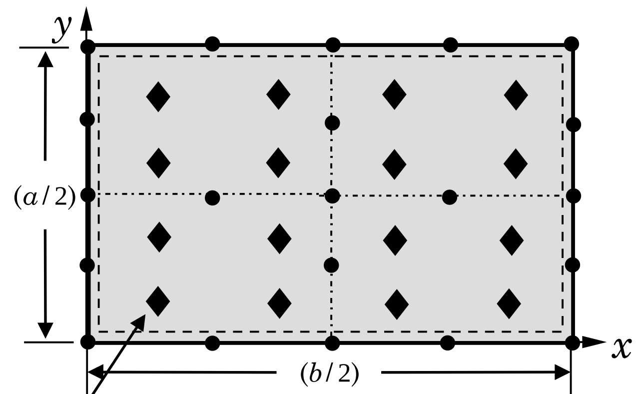
		1 × 1 linear		2 × 2 linear		4 × 4 linear		2 × 2 quadratic		Exact‡	
a/h	Integ.	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$
10	F	0.964	0.018	2.474	0.119	3.883	0.216	4.770	0.290	4.791	0.276
	M	3.950	0.095	4.712	0.235	4.773	0.266	4.799	0.272		
20	F	0.270	0.005	0.957	0.048	2.363	0.138	4.570	0.268	4.625	0.276
	M	3.669	0.095	4.524	0.235	4.603	0.266	4.633	0.272		
40	F	0.070	0.001	0.279	0.014	0.944	0.056	4.505	0.270	4.584	0.276
	M	3.599	0.095	4.375	0.235	4.560	0.266	4.592	0.271		
50	F	0.005	0.000	0.182	0.009	0.652	0.039	4.496	0.267	4.579	0.276
	M	3.590	0.095	4.472	0.235	4.555	0.266	4.587	0.271		
100	F	0.011	0.000	0.047	0.002	0.182	0.011	4.482	0.266	4.572	0.276
	M	3.579	0.095	4.465	0.235	4.548	0.266	4.580	0.272		
CPT(N)		5.643	0.260	4.857	0.274	4.643	0.276	—	—	4.570	0.276
CPT(C)		4.638	0.262	4.574	0.272	4.570	0.275	—	—	4.570	0.276

†$\bar{w} = wEh^3 \times 10^2/q_0a^4$, $\bar{\sigma}_x = \sigma_x(A, A, \pm h)h^2/q_0a^2$, $A = \frac{1}{4}a$ (1 × 1 linear), $\frac{1}{8}a$ (2 × 2 linear), $\frac{1}{16}a$ (4 × 4 linear), $0.05283a$ (2 × 2 quadratic).

Gauss Point Locations (based on reduced Integration Gauss points) for Stress Computation



2×2 Mesh of
4-node (linear) elements



2×2 Mesh of
9-node (quadratic) elements

$$\left(\frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}}, \frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}}\right) = (0.05283a, 0.05283b)$$

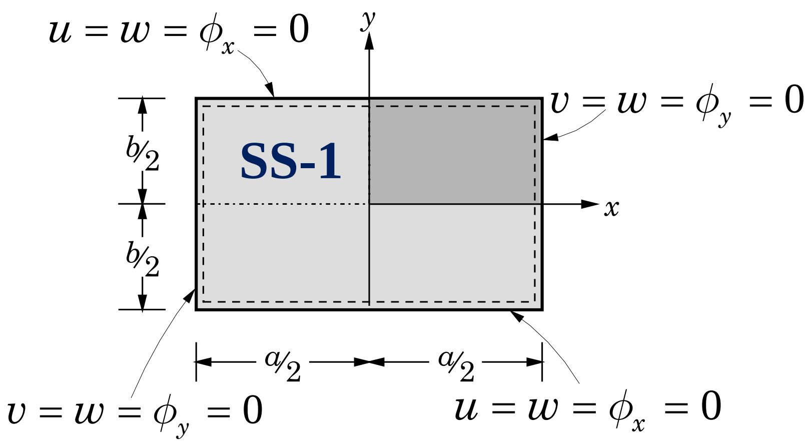
REMARKS

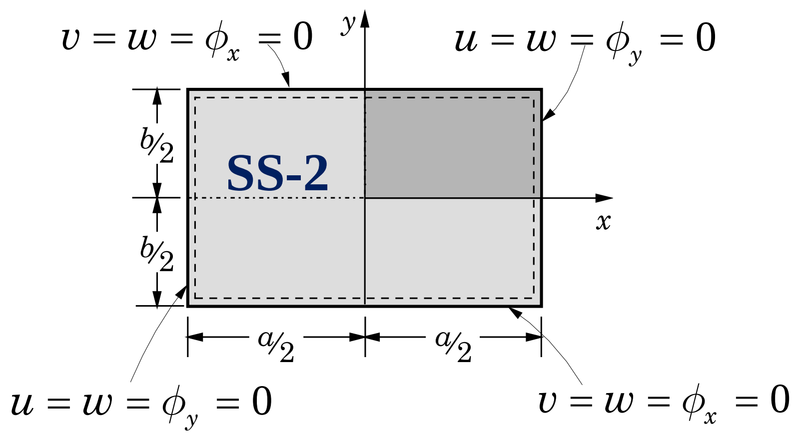
The nine-node element gives virtually the same results for full (3×3 Gauss rule) and mixed (3×3 and 2×2 Gauss rules for bending and shear terms, respectively) integrations. However, the results obtained using the mixed integration are closest to the exact solution.

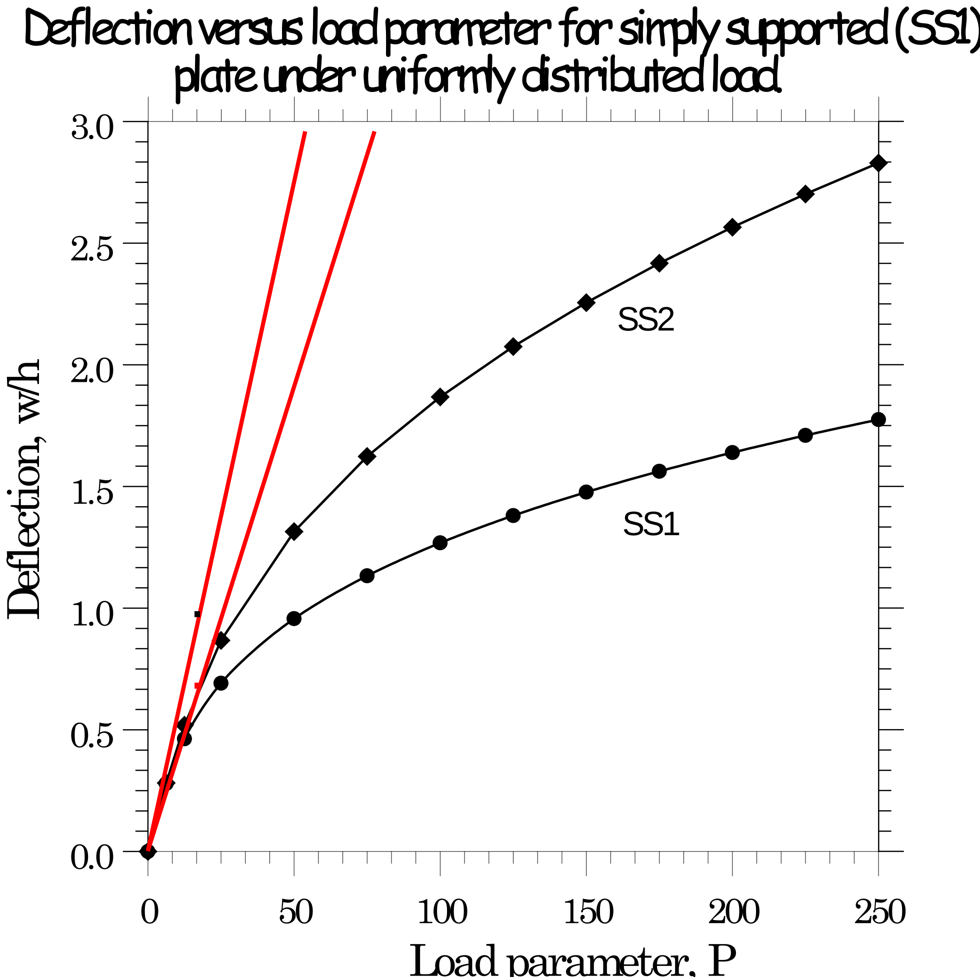
Full integration gives less accurate results than mixed integration, and the error increases with an increase in side-to-thickness ratio (a/h). This implies that mixed integration is essential for thin plates, especially when modeled by lower-order elements.

Full integration results in smaller errors for quadratic elements and refined meshes than for linear elements and/or coarser meshes.

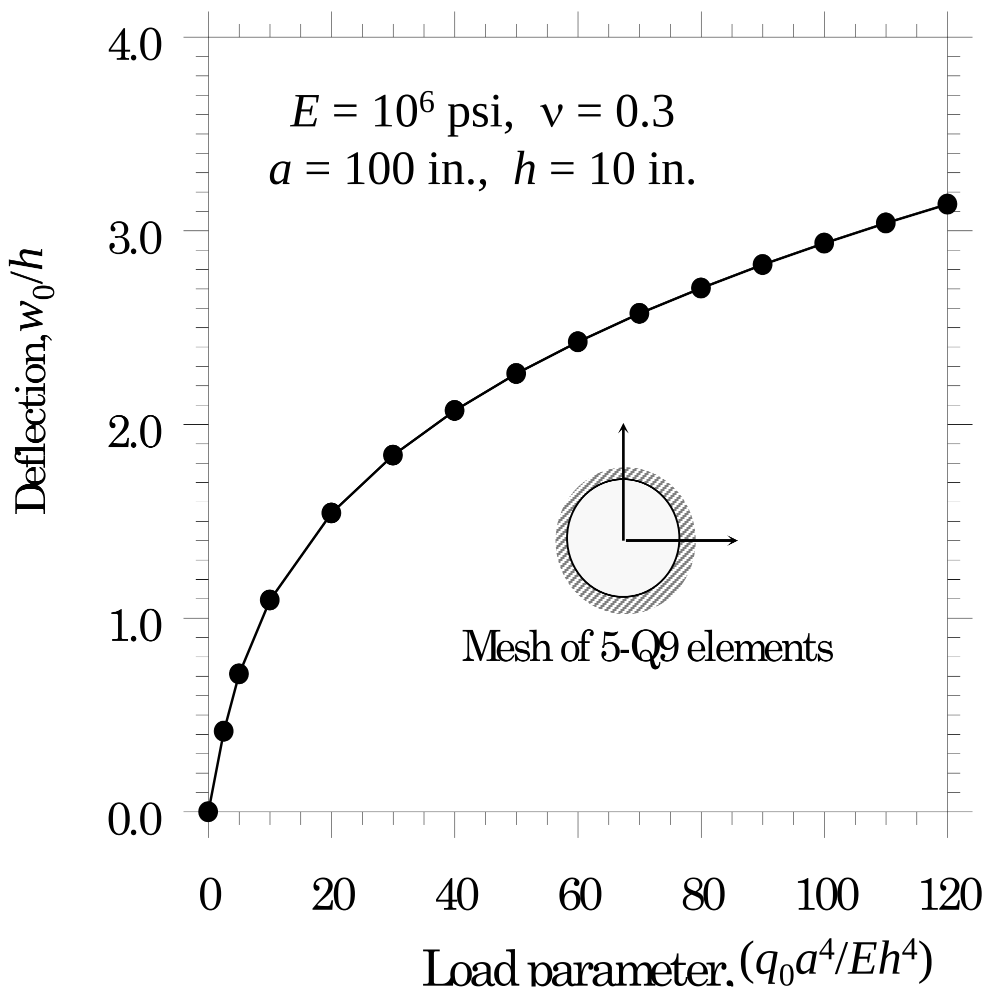
Nonlinear Analysis of Simply Supported Plate (SS-1)

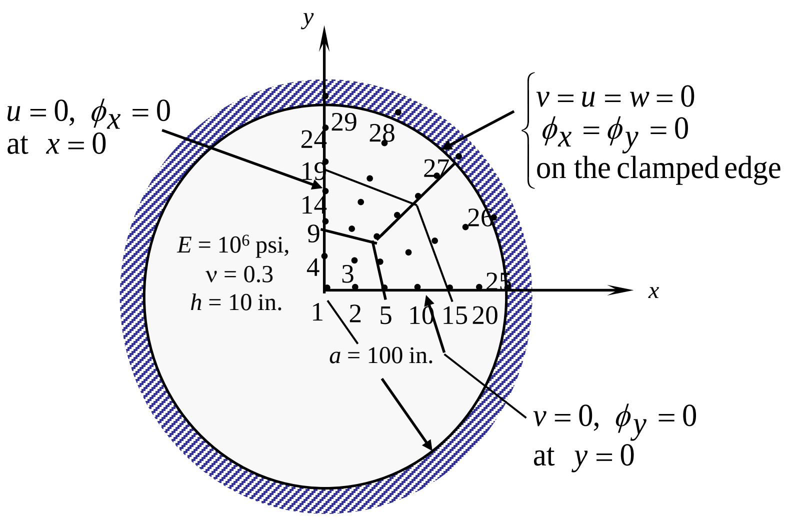




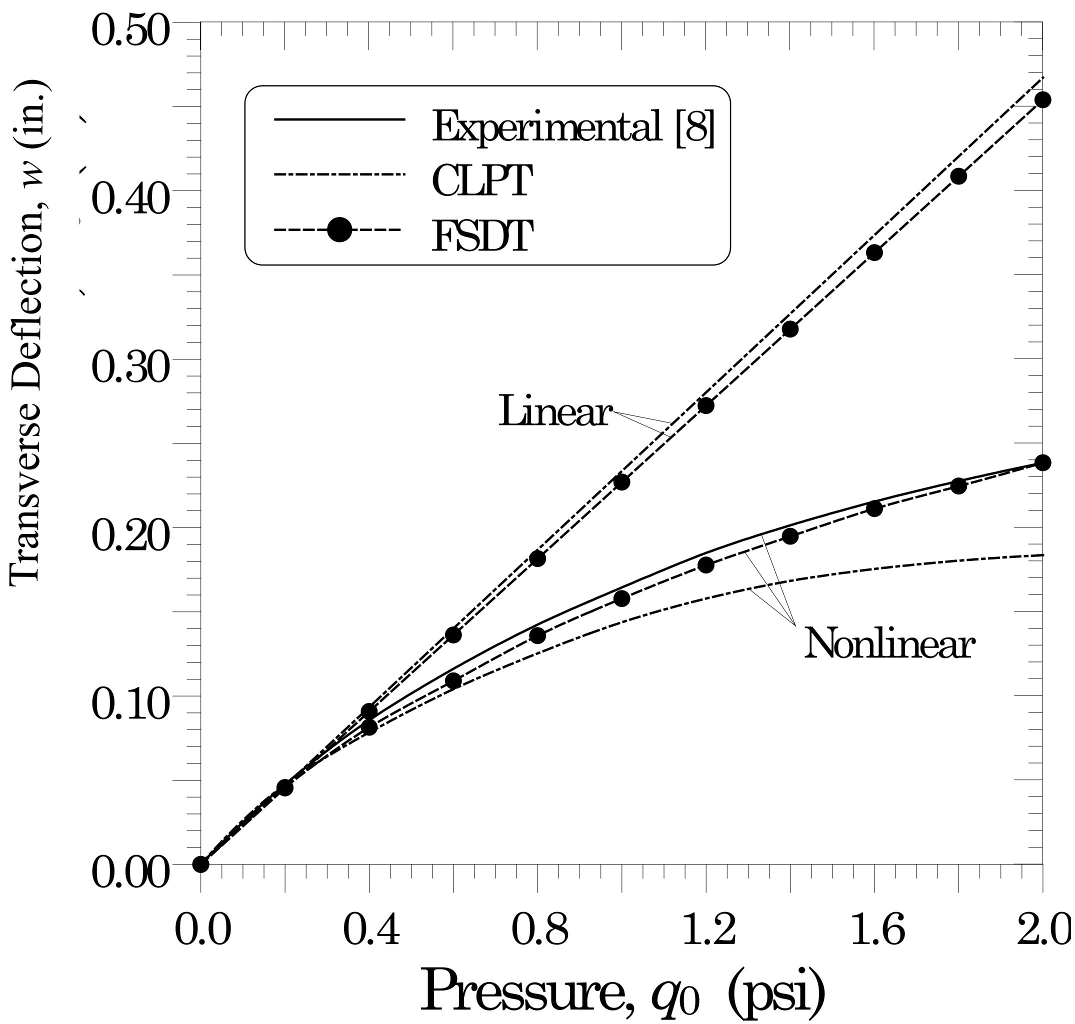


Clamped Circular Plate under UDL





Simply Supported (SS2) Orthotropic* Plate



Geometry and Material Properties

$a = b = 12$ in, $h = 0.138$ in

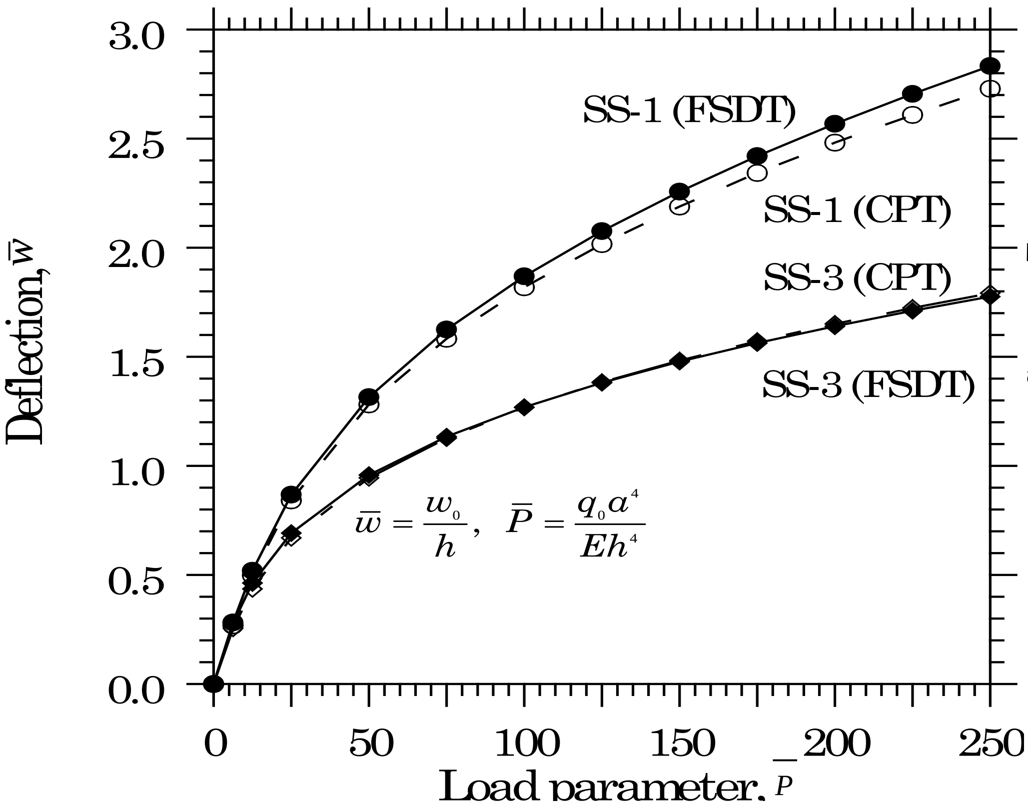
$E_1 = 3\times10^6$ psi, $E_2 = 1.28\times10^6$ psi

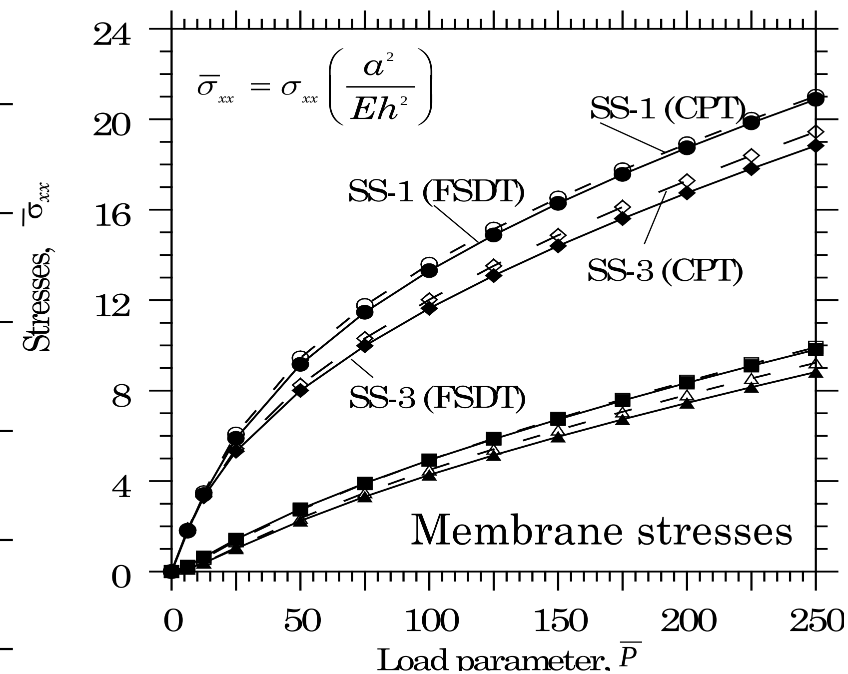
$G_{12} = G_{23} = G_{13} = 0.37\times10^6$ psi

$\nu_{12} = \nu_{23} = \nu_{13} = 0.32$

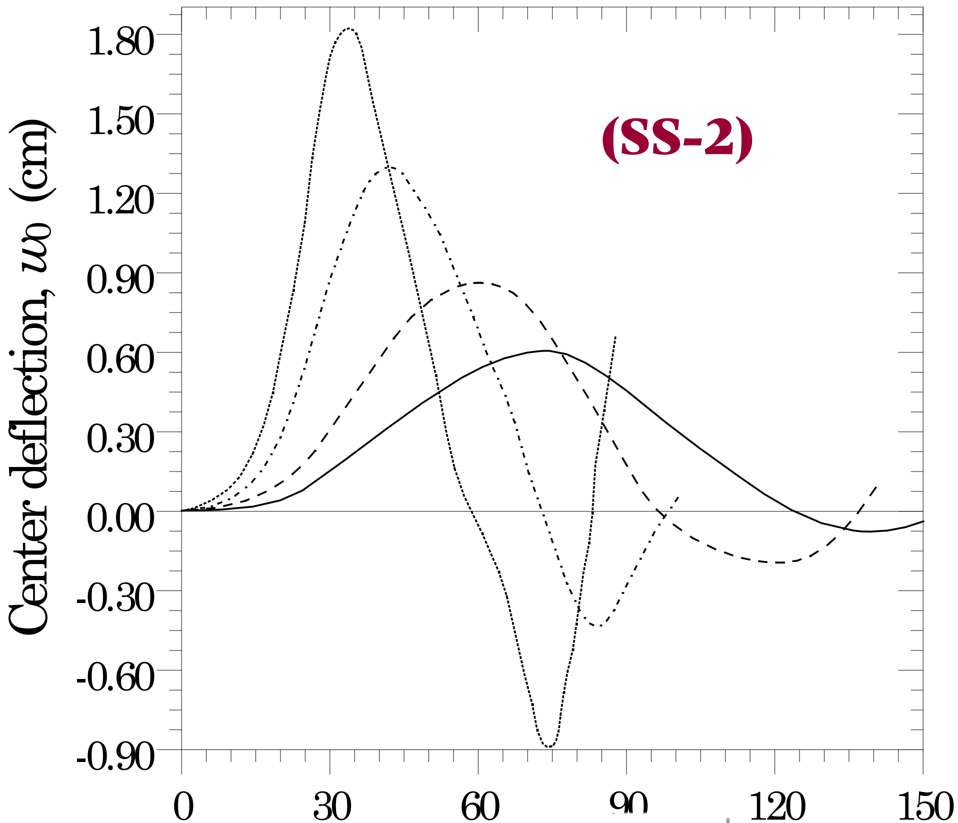
[8] Zaghloul, S. A. and Kennedy, J. B., ``Nonlinear Behavior of Symmetrically Laminated Plates," *Journal of Applied Mechanics*, 42, 234-236, 1975.

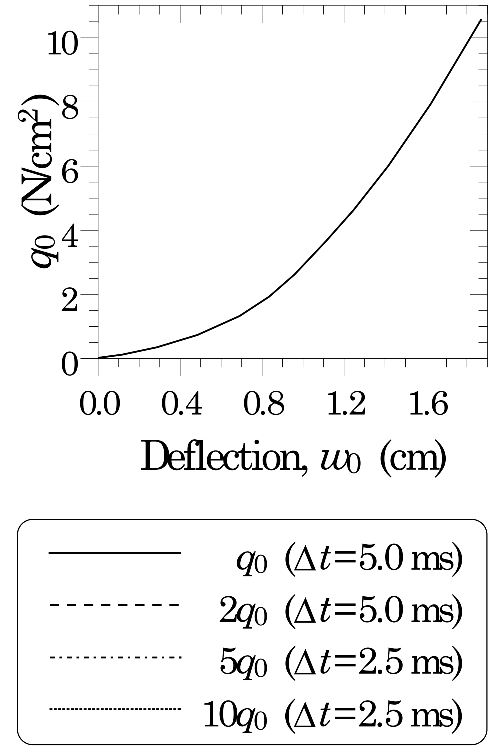
Deflection vs. load parameter for plates under uniformly distributed load





Center Deflection vs. Time for a Simply Supported Isotropic Plate Under Suddenly Applied Uniformly Distributed Pressure Load





$a = b = 243.8$ cm, $h = 0.635$ cm,

$\rho = 2.547\times 10^{-6}$ N-s^2/cm^4,

$E_1 = E_2 = 7.031\times 10^5$ N/cm^2, $\nu_{12} = 0.25$

$q_0 = 4.882\times 10^{-4}$ N/cm^2, $\Delta t = 0.005$ s = 5ms

SUMMARY

In this lecture we have covered the following topics:

- Governing Equations of FSDT
- Finite element models of FSDT
- Tangent stiffness coefficients
- Shear and membrane locking
- Programming aspects (including stress computation)
- Numerical examples