Nonlinear Bending of Strait Beams

CONTENTS

- **The Euler-Bernoulli beam theory**
- **The Timoshenko beam theory**
  - Governing Equations
  - Weak Forms
  - Finite element models
  - Computer Implementation: calculation of element matrices
  - Numerical examples

Von Kármán NONLINEAR STRAINS

- **Green-Lagrange Strain Tensor Components**

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \tfrac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$$

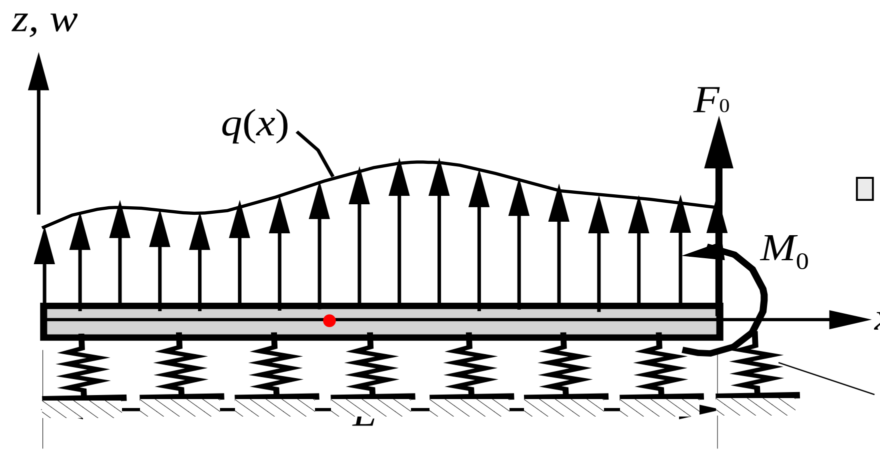
$$E_{xx} = \frac{\partial u_1}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

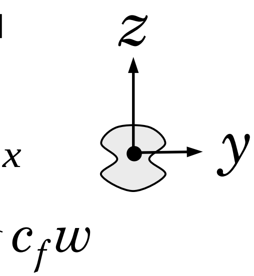
- **Order-of-magnitude assumption**

$$\frac{\partial u_1}{\partial x_1} \approx O(\varepsilon), \qquad \frac{\partial u_3}{\partial x_1} \approx O(\sqrt{\varepsilon})$$

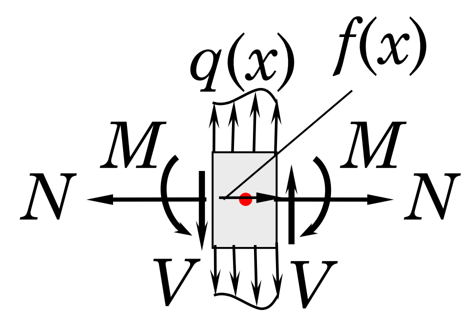
$$E_{xx} \approx \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS





Beam cross section



- Displacements and strain-displacement relations

$$\mathbf{u} = (u + z\,\boldsymbol{\theta}_x)\,\hat{\mathbf{e}}_1 + w\,\hat{\mathbf{e}}_3, \quad \boldsymbol{\theta}_x = -\frac{dw}{dx}$$

$$u_1(x,z) = u - z\frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{du}{dx} + \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2 w}{dx^2},$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS
for the Euler-Bernoulli beams

$$\delta W_I^e = \int_{V^e} \delta\varepsilon_{xx}\sigma_{xx}\, dV$$

$$= \int_{x_a}^{x_b}\int_{A^e}\Big[\Big(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\Big) - z\Big(\frac{d^2\delta w}{dx^2}\Big)\Big]\sigma_{xx}\, dA dx$$

$$= \int_{x_a}^{x_b}\Big[\Big(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\Big)N_{xx} - \frac{d^2\delta w}{dx^2}M_{xx}\Big]dx$$

$$\delta W_E^e = -\left[\int_{x_a}^{x_b} q\delta w dx + \int_{x_a}^{x_b} f\delta u dx + \sum_{i=1}^{6} Q_i^e\delta\Delta_i^e\right]$$

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0$$

$$\frac{dN}{dx} + f = 0, \qquad \frac{d^2M}{dx^2} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

Equilibrium equations

$$\frac{dN}{dx}+f=0,\quad \frac{dM}{dx}-V=0,\quad \frac{dV}{dx}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

$$\frac{dN}{dx}+f=0,\qquad \frac{d^2M}{dx^2}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

Stress resultants in terms of deflection

$$N=\int_A \sigma_{xx}\ dA=\int_A\left[E\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]dA=EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M=\int_A \sigma_{xx}\times z\ dA=\int_A\left[E\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]z\ dA=-EI\frac{d^2w}{dx^2}$$

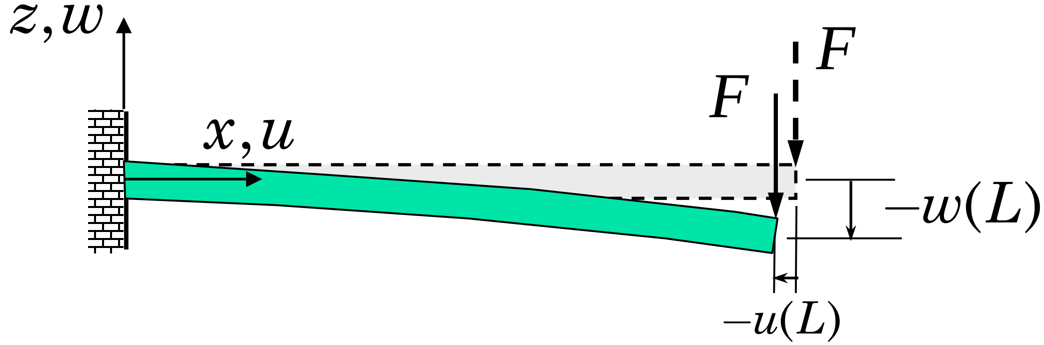
$$V=\frac{dM}{dx}=\frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right)$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

- **Equilibrium equations in terms of displacements (*u*,*w*)**

$$\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}-f=0$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left(\frac{dw}{dx}EA\left[\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right)-q=0$$



- Clearly, transverse load induces both axial displacement u and transverse displacement w.

EULER-BERNOULLI BEAM THEORY
(continued)

➢ **Weak forms**

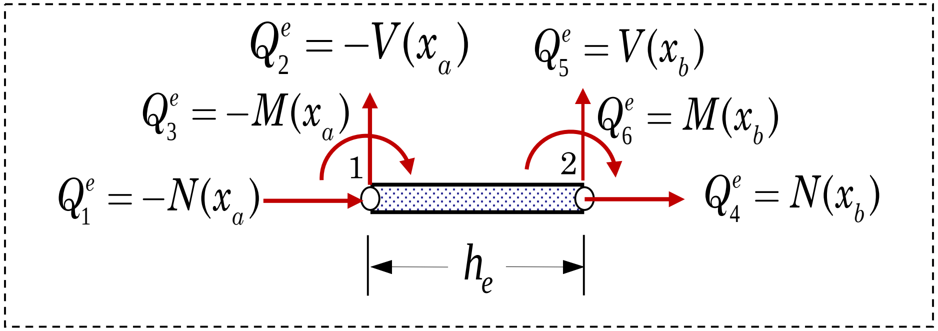
$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$v_1 \approx \delta u,\ \ v_2 \approx \delta w$$

$$0 = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N - v_1 f\right) dx - v_1(x_a) Q_1 - v_1(x_b) Q_4$$

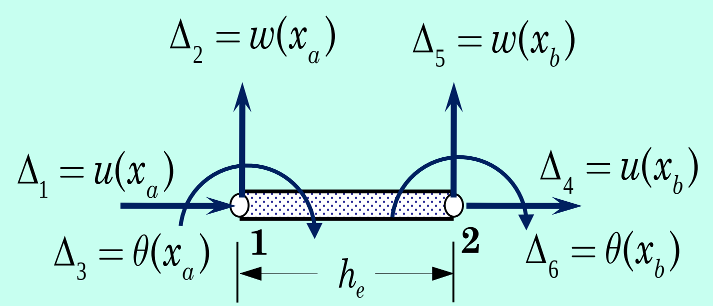
$$0 = \int_{x_a}^{x_b} \left[EI \frac{d^2 v_2}{dx^2}\frac{d^2 w}{dx^2} + \frac{dv_2}{dx}\left(N \frac{dw}{dx}\right) - v_2 q\right] dx$$

$$- v_2(x_a) Q_2 - \left(-\frac{dv_2}{dx}\right)_{x_a} Q_3 - v_2(x_b) Q_5 - \left(-\frac{dv_2}{dx}\right)_{x_b} Q_6$$

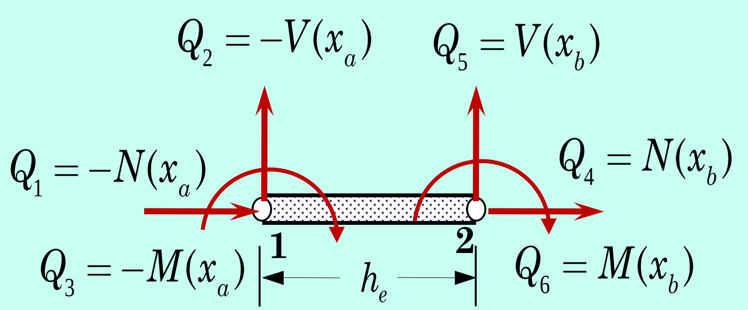


BEAM ELEMENT DEGREES OF FREEDOM

Generalized displacements



Generalized forces



FINITE ELEMENT APPROXIMATION

Primary variables (serve as the nodal variables that must be continuous across elements) $u,\, w,\; \theta = -\dfrac{dw}{dx}$

$$w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x), \quad u(x) \approx \sum_{j=1}^{n} u_j \psi_j(x),$$

Hermite cubic polynomials

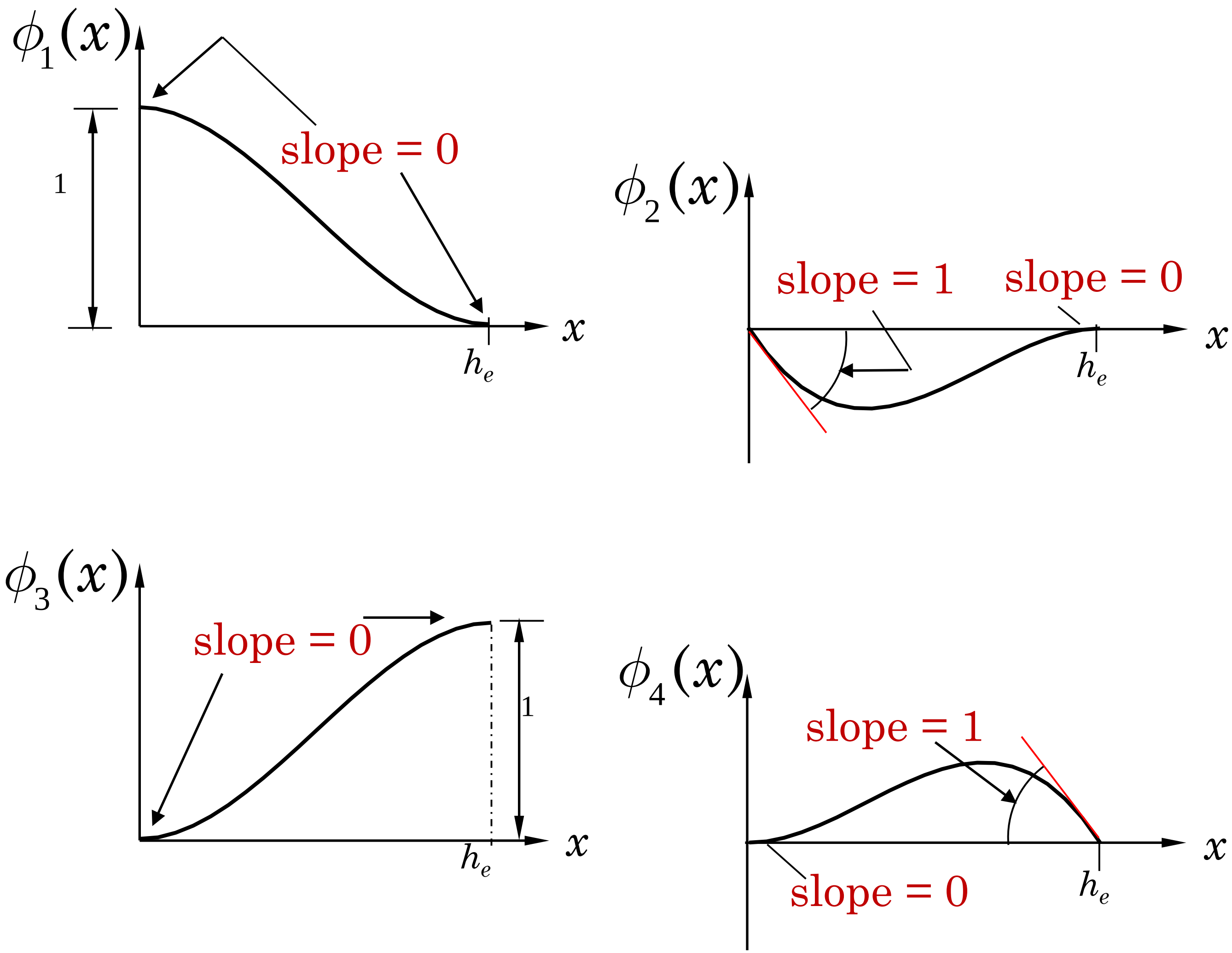
$$\phi_1^e = 1 - 3\left(\frac{x - x_a}{h_e}\right)^2 + 2\left(\frac{x - x_a}{h_e}\right)^3$$

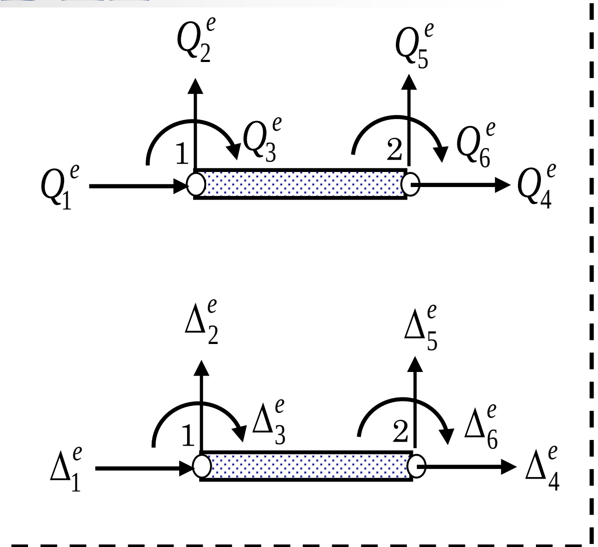
$$\phi_2^e = -(x - x_a)\left(1 - \frac{x - x_a}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_a}{h_e}\right)^2 - 2\left(\frac{x - x_a}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_a)\left[\left(\frac{x - x_a}{h_e}\right)^2 - \frac{x - x_a}{h_e}\right]$$

HERMITE CUBIC INTERPOLATION FUNCTIONS $\phi_i(x)$





➢ Finite Element Equations

$$u(x) \approx \sum_{j=1}^{2} u_j \psi_j(x), \quad w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x)$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{\Delta\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} EA \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx, \quad K_{ij}^{12} = \frac{1}{2} \int_{x_a}^{x_b} EA \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_j}{dx} dx,$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} EA \frac{dw}{dx} \frac{d\phi_i}{dx} \frac{d\psi_j}{dx} dx, \quad F_i^1 = \int_{x_a}^{x_b} f \psi_i \, dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_4$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} EI \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} dx + \int_{x_a}^{x_b} EA \left(\frac{dw}{dx}\right)^2 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx,$$

$$F_i^2 = \int_{x_a}^{x_b} q \psi_i \, dx + \phi_i(x_a) Q_2 + \phi_i(x_b) Q_5 + \left(-\frac{d\phi_i}{dx}\right)_{x_a} Q_3 + \left(-\frac{d\phi_i}{dx}\right)_{x_b} Q_6$$

MEMBRANE LOCKING

Membrane strain

$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2$$

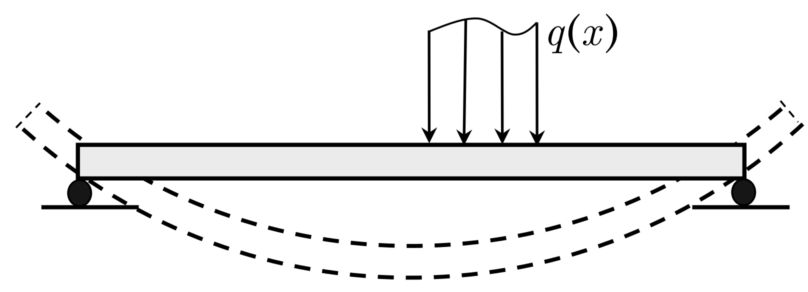
$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{du}{dx} = -\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2$$

Remedy

$\Rightarrow$ make $\left(\dfrac{dw}{dx}\right)^2$ to behave like a constant

Beam on roller supports



Direct Iteration

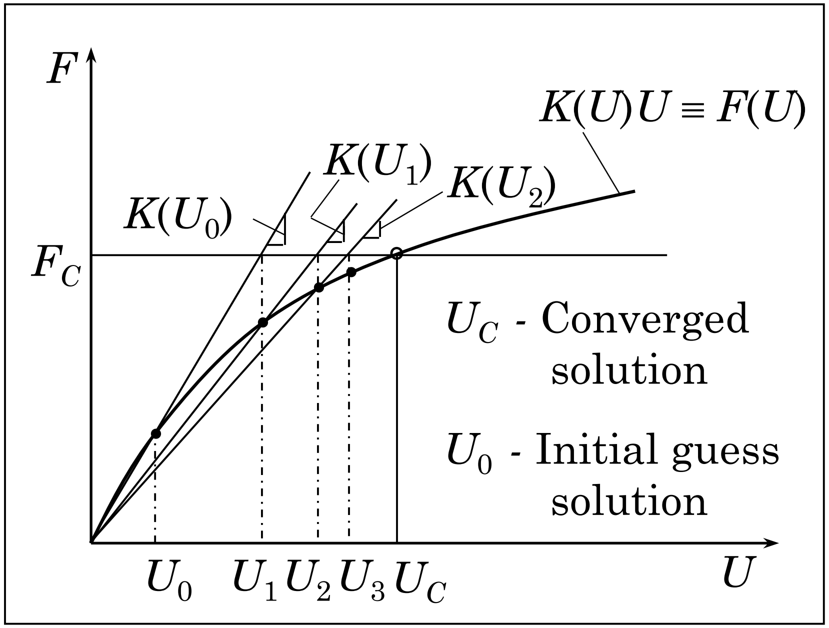
Non-Linear Finite Element Model

$$[K^e(\Delta^e)]\{\Delta^e\} = \{F^e\} \Rightarrow \text{assembled } [K(U)]\{U\} = \{F\}$$

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$[K(\{U\}^r)]\{U\}^{r+1} = \{F\}$$



SOLUTION OF NONLINEAR EQUATIONS (continued)

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$[K(\{U\}^r)]\{U\}^{r+1} = \{F\}$$

Convergence Criterion

$$\varepsilon = \sqrt{\frac{\sum_{I=1}^{NEQ} \left(U_I^r - U_I^{r+1}\right)^2}{\sum_{I=1}^{NEQ} \left(U_I^{r+1}\right)^2}} \leq \text{specified tolerance}$$

SOLUTION OF NONLINEAR EQUATIONS

Newton's Iteration Method

Taylor's series

$$\text{Residual,}\quad \{R\} \equiv [K(\{U\}^r)]\{U\}^{r+1} - \{F\}^r$$

$$\{R(U^{r+1})\} = \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + \frac{1}{2!}(U^{r+1} - U^r)^2\left[\frac{\partial^2 R}{\partial U^2}\right]^r + \cdots$$

$$\approx \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + O(\delta U)^2, \qquad \boxed{\delta U = U^{r+1} - U^r}$$

Requiring the residual $\{R\}^{r+1}$ to be zero at the $r+1^{\text{st}}$ iteration, we have

$$\boxed{[K^{\tan}(\{U\}^r)]\{\delta U\} = -\{R\}^r = \{F\}^r - [K(U^r)]^r\{U\}^r}$$

The tangent matrix at the element level is

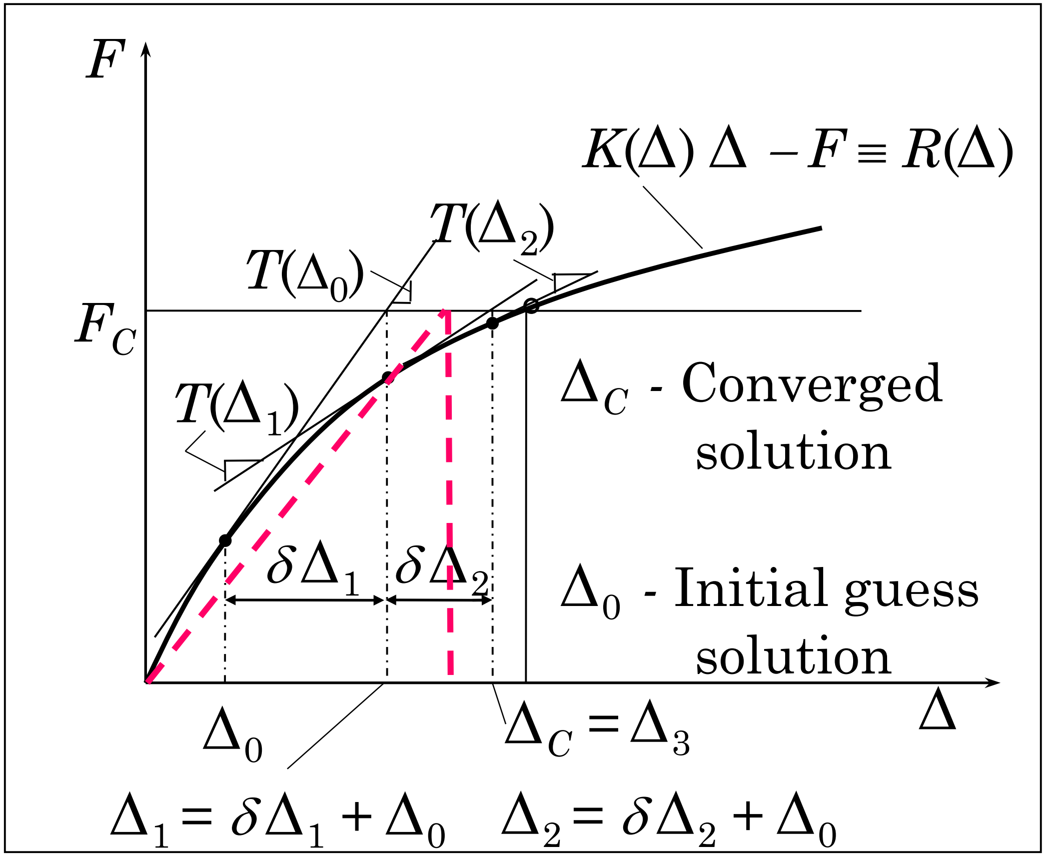
$$\left(K_{ij}^{\alpha\beta}\right)^{\tan} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n} K_{ip}^{\alpha\lambda}\Delta_p^\gamma - F_i^\alpha\right)$$

SOLUTION OF NONLINEAR EQUATIONS

Newton's Iteration (continued)

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n_\beta} K_{ip}^{\alpha\gamma}\Delta_p^\gamma - F_i^\alpha\right) = K_{ij}^{\alpha\beta} + \sum_{\gamma=1}^{2}\sum_{p=1}^{n}\frac{\partial K_{ip}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_p^\gamma \equiv T_{ij}^{\alpha\beta}$$

$$[T(\{\Delta\}^r)]\{\delta\Delta\} = \{F\}^r - [K(\Delta^r)]^r\{\Delta\}^r, \quad \{\Delta\}^{r+1} = \{\Delta\}^r + \{\delta\Delta\}$$



Summary of the N-R Method

$$[T(\{\Delta\}^{(r-1)}][\delta\Delta\}^r = -\{R(\{\Delta\}^{(r-1)})\}$$

$$\{\Delta\}^r = \{\Delta\}^{(r-1)} + \{\delta\Delta\}$$

Computation of tangent stiffness matrix

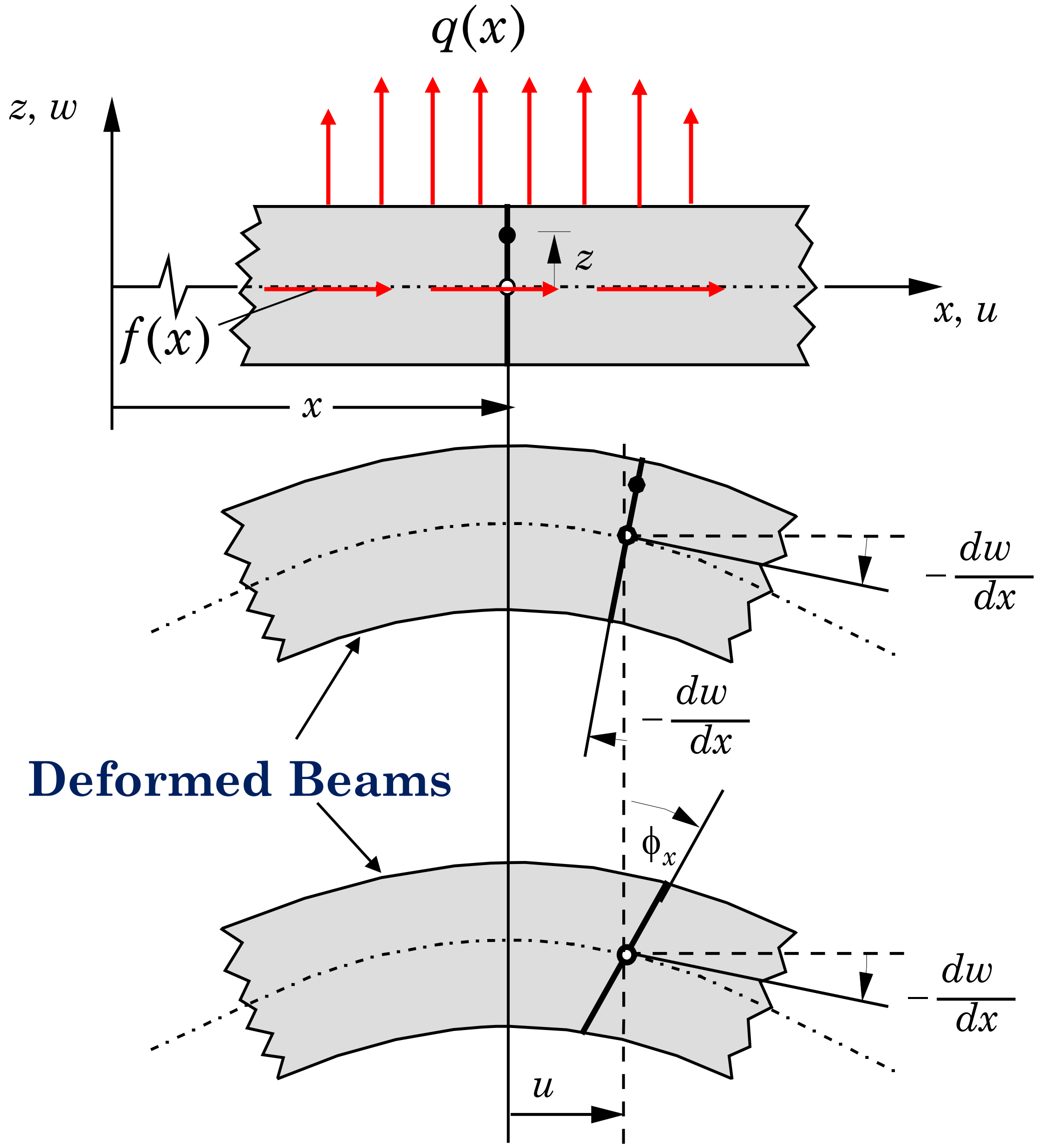
$$R_i^\alpha = \sum_{\gamma=1}^{2}\sum_{p=1} K_{ip}^{\alpha\gamma}\Delta_p^\gamma - F_i^\alpha = \sum_{p=1}^{n} K_{ip}^{\alpha 1}u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2}\overline{\Delta}_P - F_i^\alpha$$

$$T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}\right) = K_{ij}^{\alpha\beta} + \sum_{p=1}^{n} \frac{\partial}{\partial \Delta_j^\beta}\left(K_{ip}^{\alpha 1}\right)u_p + \sum_{P=1}^{4}\frac{\partial}{\partial \Delta_j^\beta}\left(K_{iP}^{\alpha 2}\right)\overline{\Delta}_P$$

$$T_{ij}^{11} = K_{ij}^{11} + \sum_{p=1}^{n}\frac{\partial K_{ip}^{11}}{\partial u_j}u_p + \sum_{P=1}^{4}\frac{\partial K_{iP}^{12}}{\partial u_j}\overline{\Delta}_P$$

$$= K_{ij}^{11} + \sum_{p=1}^{n} 0\cdot u_p + \sum_{P=1}^{4} 0\cdot\overline{\Delta}_P$$

THE TIMOSHENKO BEAM THEORY



Undeformed Beam

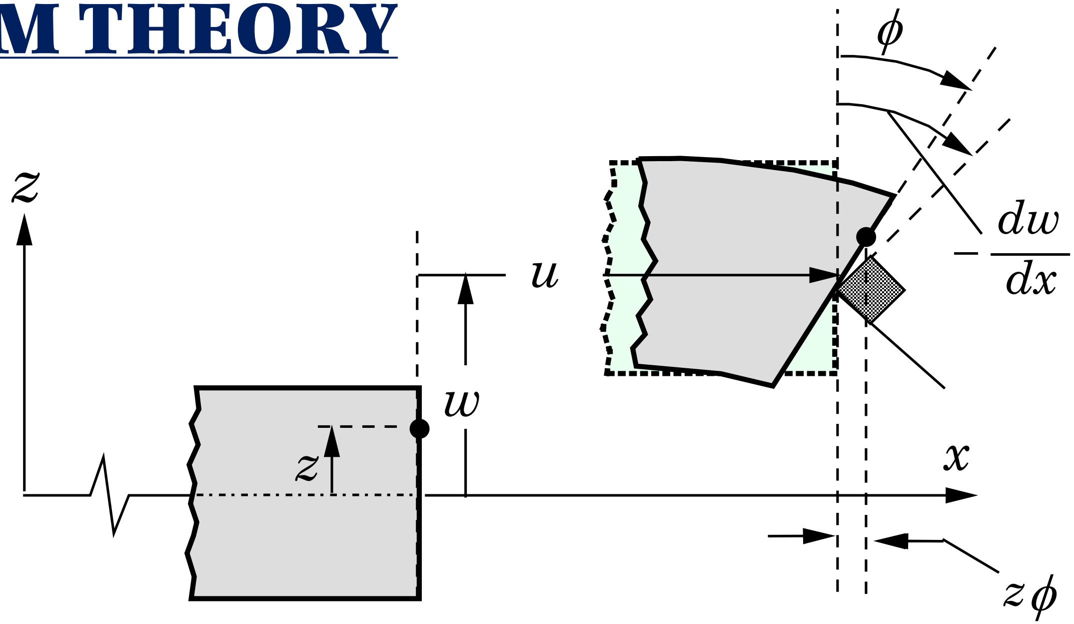
Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)
Straightness and inextensibility

KINEMATICS OF THE TIMOSHENKO BEAM THEORY

Displacement field

$$\mathbf{u} = (u + z\,\phi_x)\,\hat{\mathbf{e}}_1 + w\,\hat{\mathbf{e}}_3$$



$$u_1(x,z) = u(x) + z\phi(x),$$

$$u_2 = 0, \quad u_3(x,z) = w(x)$$

$$E_{xx} \approx \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2$$

$$= \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}$$

$$2E_{xz} \approx 2\varepsilon_{xz} = \gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}$$

$$= \phi_x + \frac{dw}{dx}$$

Constitutive Equations

$$\sigma_{xx} = E\,\varepsilon_{xx}, \quad \sigma_{xz} = G\,\gamma_{xz}$$

TIMOSHENKO BEAM THEORY (continued)

Equilibrium Equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0$$

Beam Constitutive Equations

$$N = \int_A \sigma_{xx}\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] dA = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M = \int_A \sigma_{xx} z\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] z\, dA = EI\frac{d\phi_x}{dx}$$

$$V = K_s \int_A \sigma_{xz}\, dA = GK_s\left(\phi_x + \frac{dw}{dx}\right)\int_A dA = GAK_s\left(\phi_x + \frac{dw}{dx}\right)$$

WEAK FORMS OF TBT

$$0 = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N - v_1 f \right) dx - v_1(x_a) Q_1 - v_1(x_b) Q_4$$

$$= \int_{x_a}^{x_b} \left\{ EA \frac{dv_1}{dx} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] - v_1 f \right\} dx - v_1(x_a) Q_1 - v_1(x_b) Q_4$$

$$0 = \int_{x_a}^{x_b} \left\{ \frac{dv_2}{dx} \left[GAK_s \left(\phi_x + \frac{dw}{dx} \right) \right] + \frac{dv_2}{dx} N \frac{dw}{dx} - v_2 q \right\} dx$$
$$- v_2(x_a) \cdot Q_2 - v_2(x_b) \cdot Q_5$$

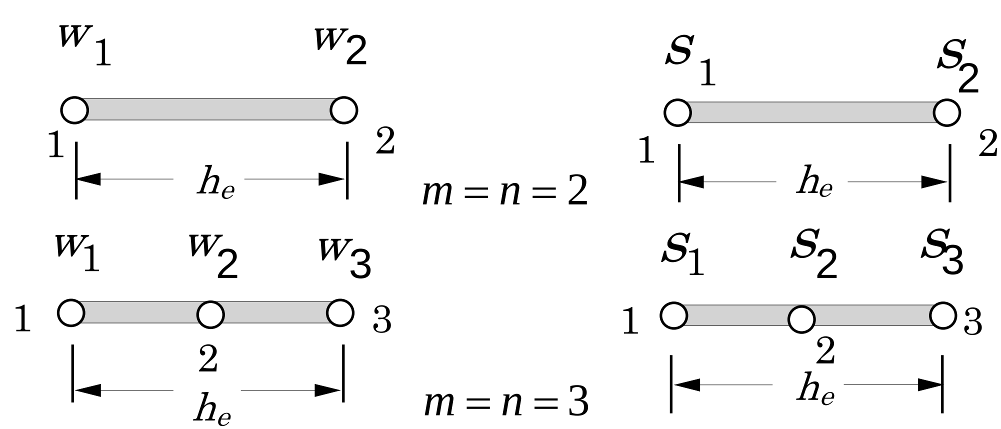
$$0 = \int_{x_a}^{x_b} \left[\frac{dv_3}{dx} \left(EI \frac{d\phi_x}{dx} \right) + GAK_s v_3 \left(\phi_x + \frac{dw}{dx} \right) \right] dx$$
$$- v_3(x_a) \cdot Q_3 - v_3(x_b) \cdot Q_6$$

FINITE ELEMENT MODELS OF TIMOSHENKO BEAMS

Finite Element Approximation

$$u \approx \sum_{j=1}^{m} u_j \psi_j^{(1)}(x), \quad w \approx \sum_{j=1}^{n} w_j \psi_j^{(2)}(x), \quad \phi \approx \sum_{j=1}^{p} S_j \psi_j^{(3)}(x)$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix}$$



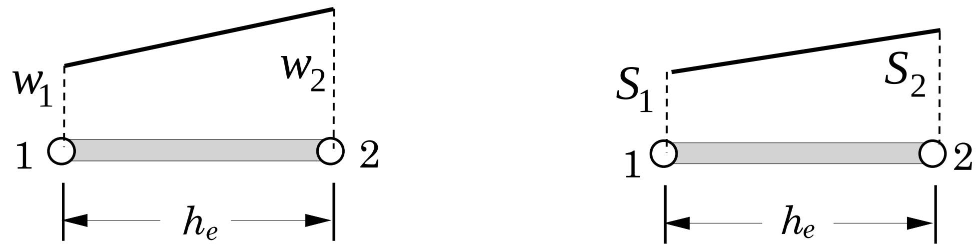
SHEAR LOCKING IN TIMOSHENKO BEAMS

(1) Thick beam experiences shear deformation, $\phi_x \neq -\dfrac{dw}{dx}$

(2) Shear deformation is negligible in thin beams, $\phi_x = -\dfrac{dw}{dx}$

Linear interpolation of both w, ϕ_x

$$w(x) \approx w_1\psi_1(x) + w_2\psi_2(x), \qquad \phi_x(x) \approx S_1\psi_1(x) + S_2\psi_2(x)$$



In the thin beam limit it is not possible for the element to realize the requirement

$$\phi_x = -\frac{dw}{dx}$$

SHEAR LOCKING - REMEDY

In the thin beam limit, ϕ should become constant so that it matches dw/dx. However, if ϕ is a constant then the bending energy becomes zero. If we can mimic the two states (constant and linear) in the formulation, we can overcome the problem. Numerical integration of the coefficients allows us to evaluate both ϕ and $d\phi/dx$ as constants. The terms **highlighted** should be evaluated using "reduced integration".

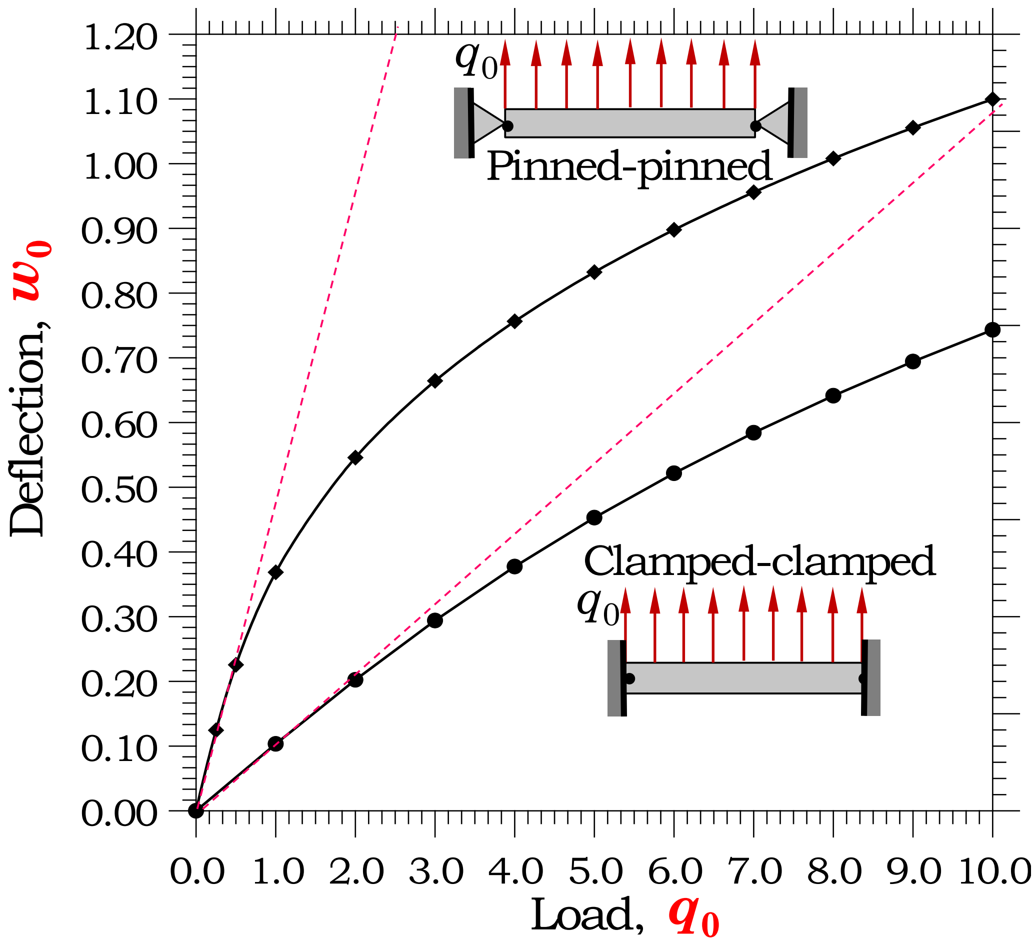
$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + \ldots \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} \, dx = K_{ji}^{32}$$

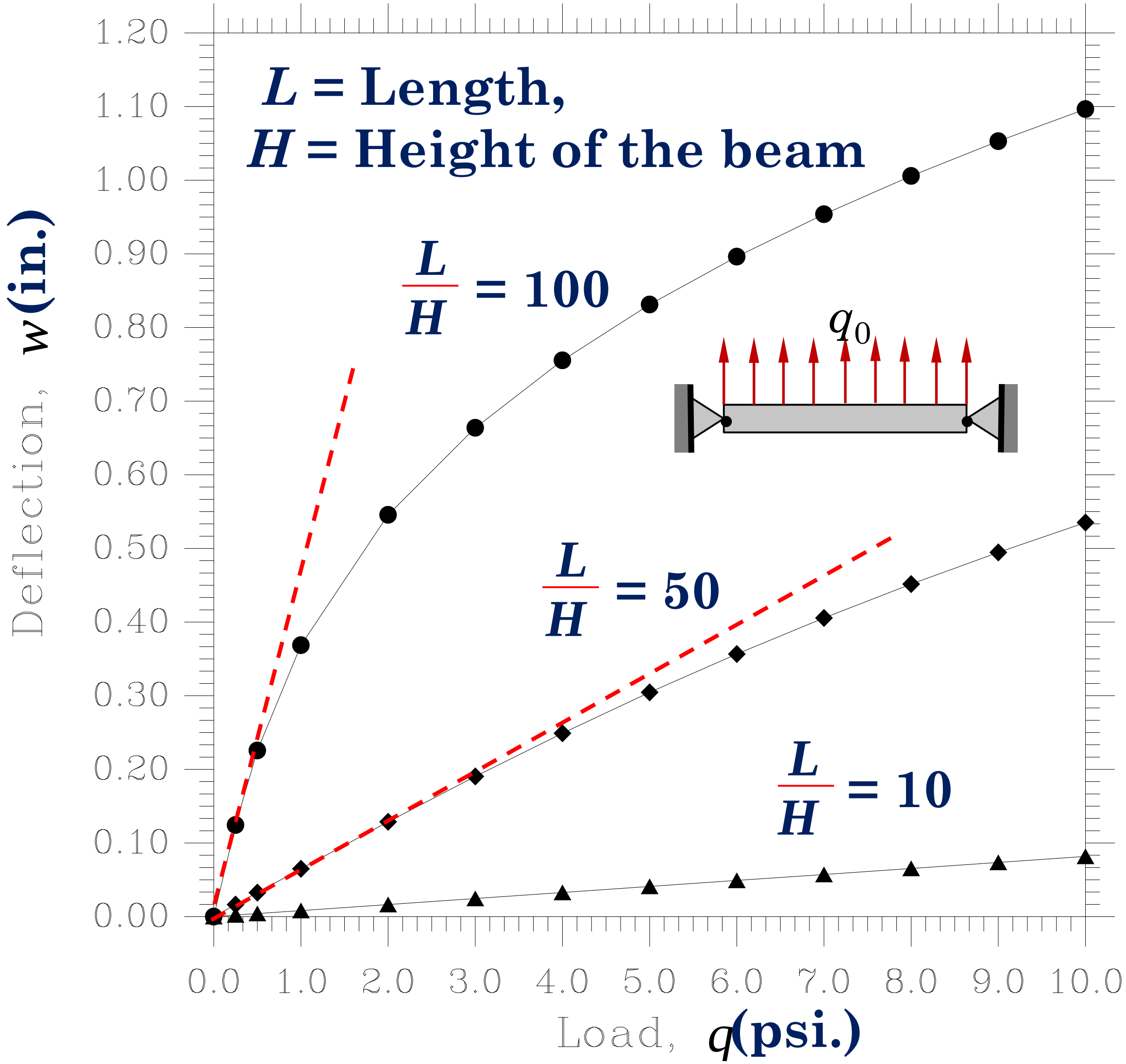
$$K_{ij}^{33} = \int_{x_a}^{x_b} \left[EI \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + GAK_s \psi_i^{(3)} \psi_j^{(3)} \right] dx$$

NUMERICAL EXAMPLES

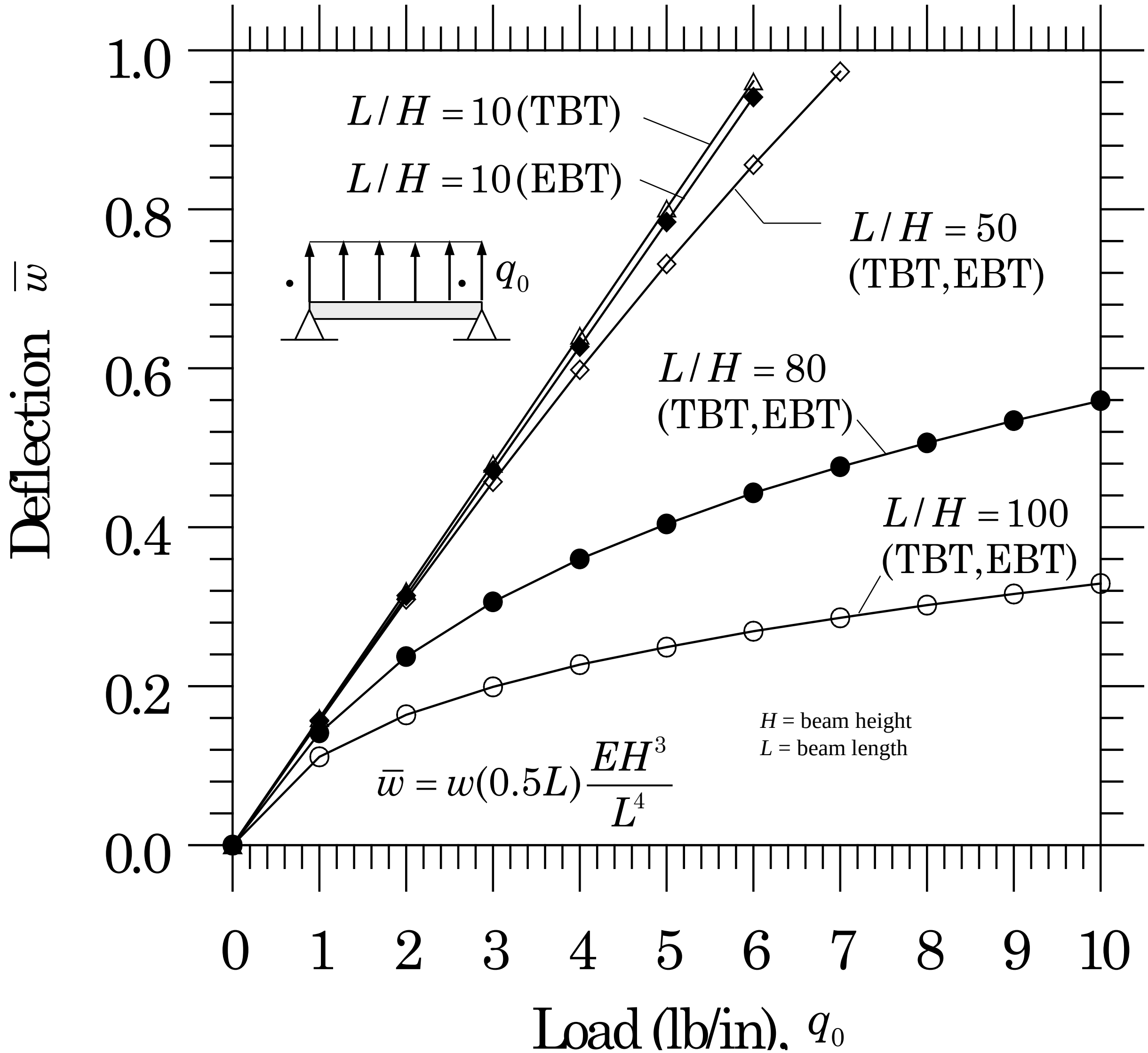
Pinned-pinned beam (EBT)



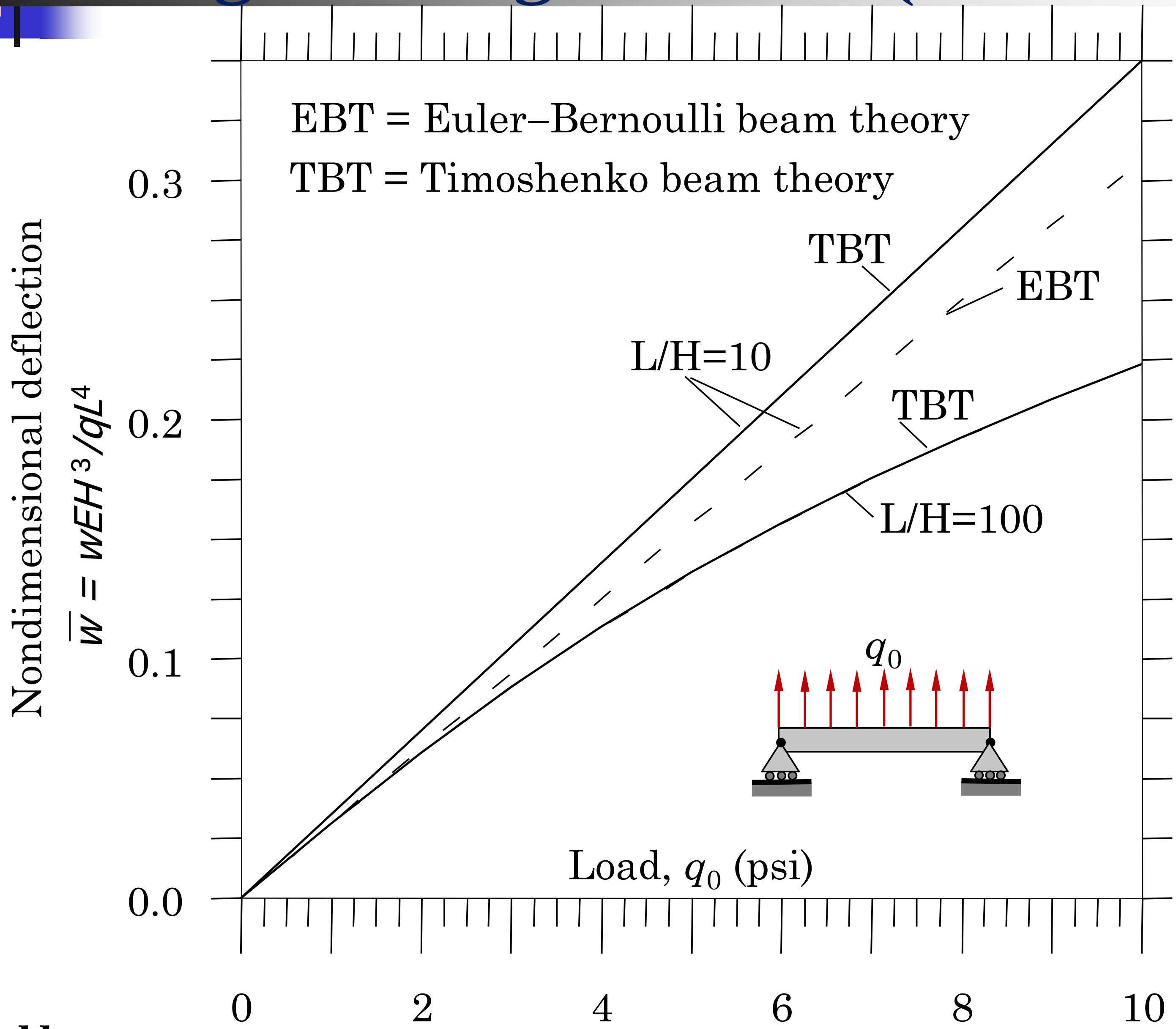
Pinned-pinned beam (TBT)



Pinned-pinned beam (EBT, TBT)



Hinged-Hinged beam (EBT and TBT)



SUMMARY

In this lecture we have covered the following topics:

- Derived the governing equations of the **Euler-Bernoulli beam theory**
- Derived the governing equations of the **Timoshenko beam theory**
- Developed Weak forms of EBT and TBT
- Developed Finite element models of EBT and TBT
- Discussed **membrane locking** (due to the geometric nonlinearity)
- Discussed **shear locking** in Timoshenko beam finite element
- Discussed examples