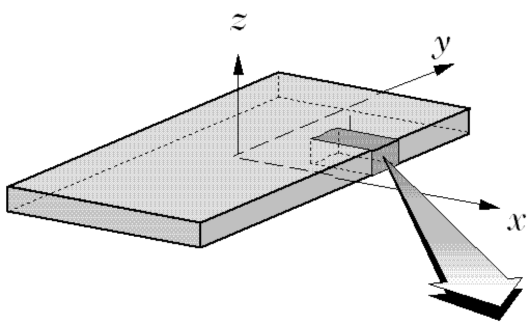
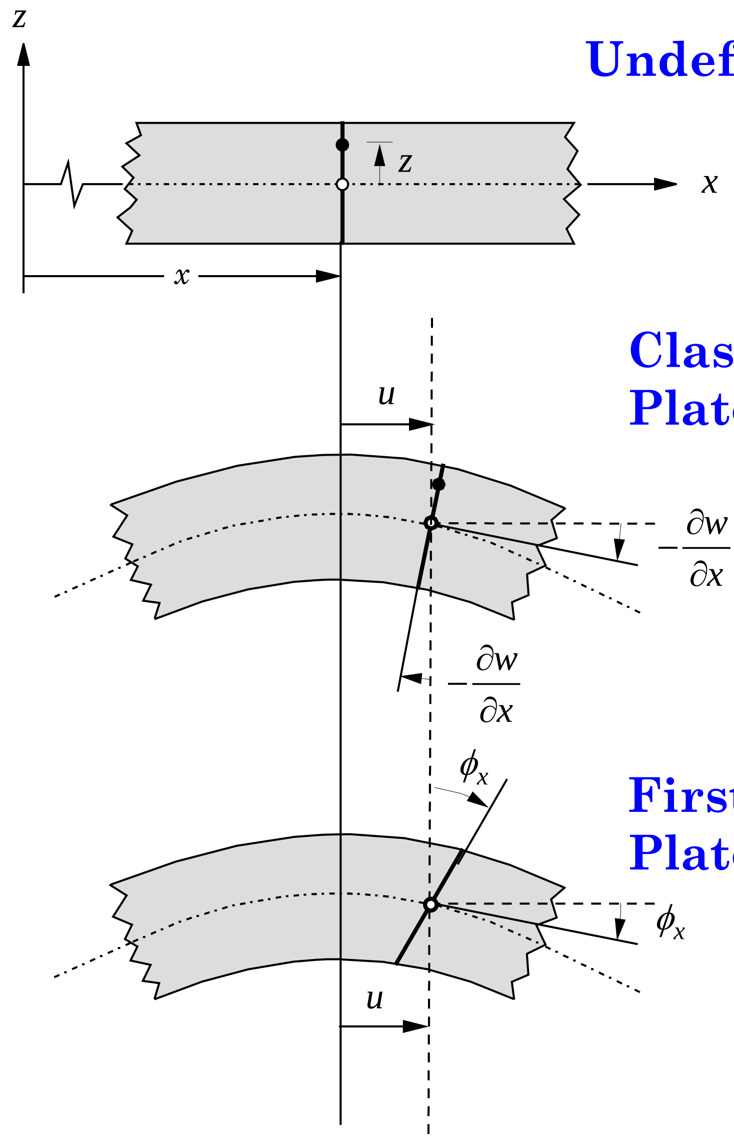
LAMINATE PLATE THEORIES AND ANALYTICAL SOLUTIONS OF RECTANGULAR PLATES

Bending, Vibration, Buckling

Kinematics of the Classical and Shear Deformation Plate Theories





Undeformed Edge

Classical (Kirchhoff) Plate Theory (CPT)

First-Order (Mindlin) Plate Theory (FST)

Classical Laminate Plate Theory (CLPT)

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z,t) = w_0(x,y,t)$$

First-order Shear Deformation Theory (FSDT)

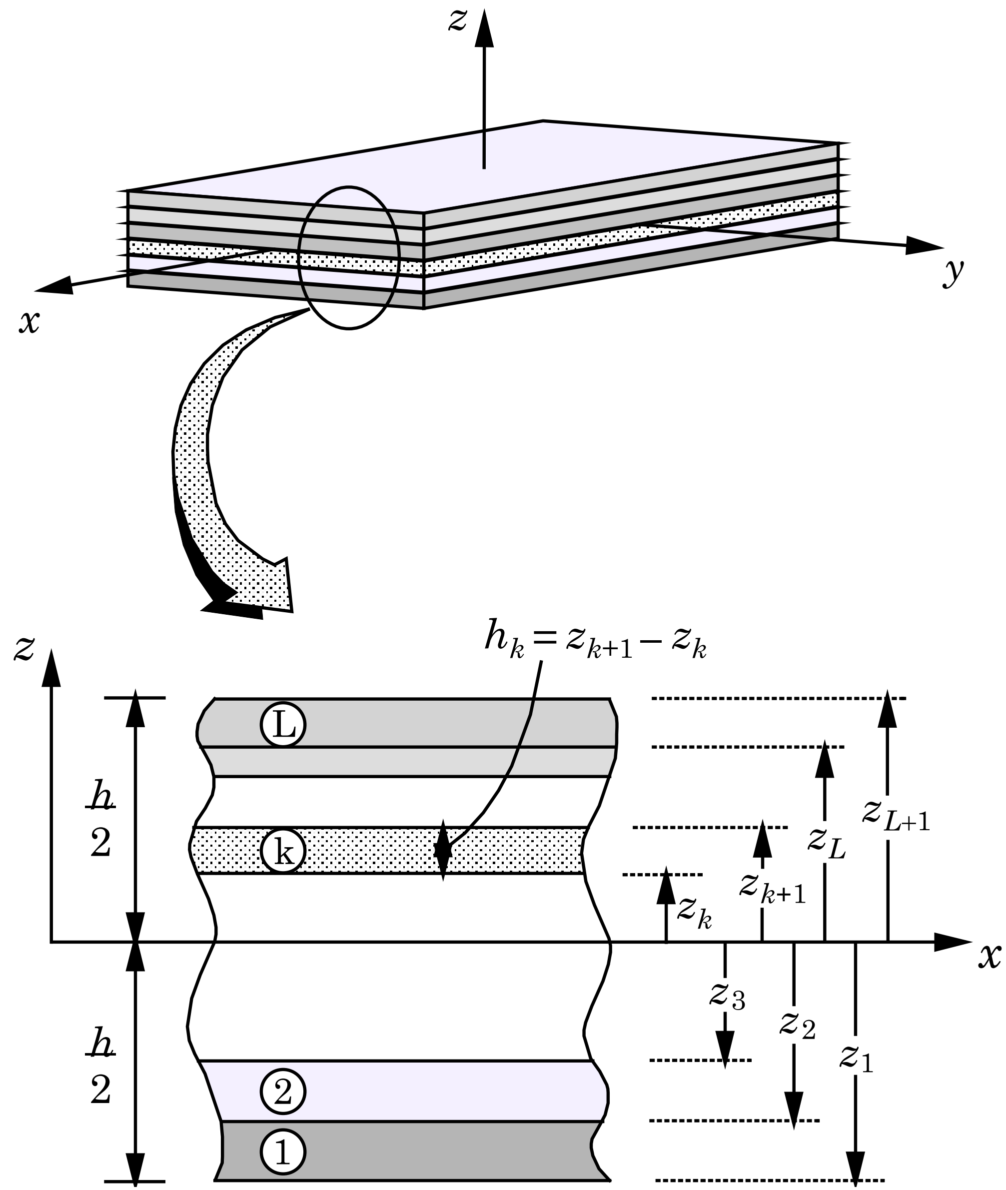
$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t)$$
$$w(x,y,z,t) = w_0(x,y,t)$$

Third-order Shear Deformation Theory (TSDT)

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right)$$
$$w(x,y,z,t) = w_0(x,y,t)$$

Coordinate System and Lamination Scheme

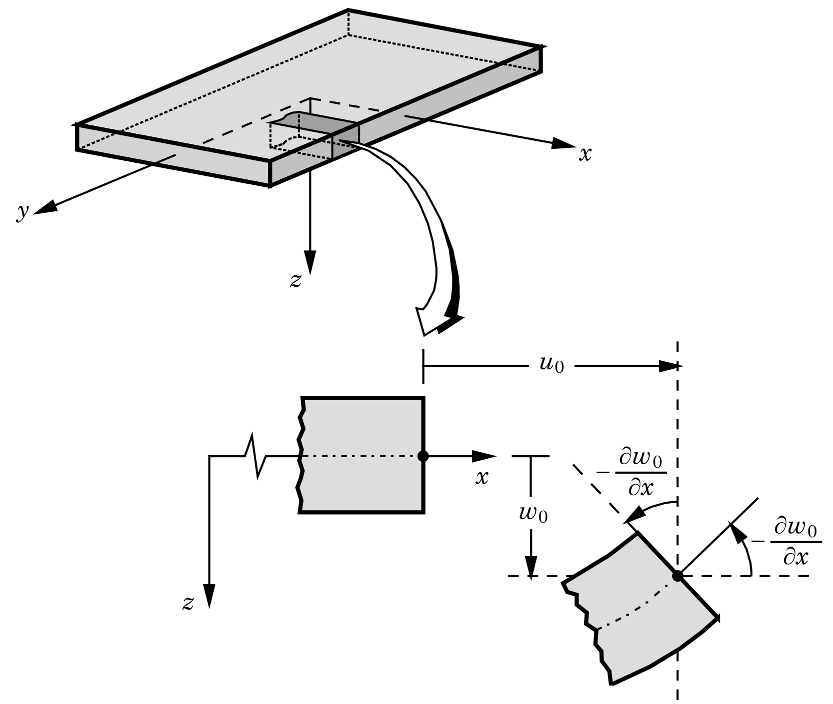
<u>Notation and Coordinate System Used in a Laminate Analysis</u>



Layers are numbered in the +ve *z* direction

Displacement Field of CLPT

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z,t) = w_0(x,y,t)$$



Strains

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

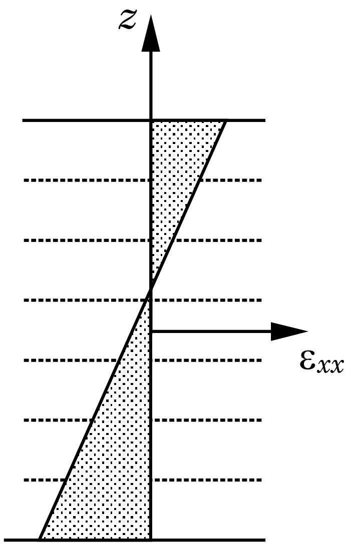
$$\{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix}$$

$$\{\varepsilon^1\} = \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

Lamina Constitutive Relations

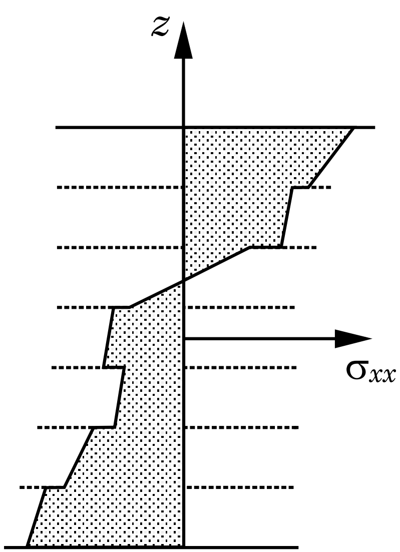
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \Delta T \right)$$

Strain and Stress Distributions Through the Thickness of the Laminate



Strain distribution is ***continuous*** through laminate thickness

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$



Stress distribution is ***discontinuous*** through laminate thickness

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

CLASSICAL LAMINATE THEORY (continued)

Principle of Virtual Displacements

$$0 = \int_{V^e} \sigma_{ij}\delta\varepsilon_{ij}\, dV - \int_{\Omega^e} q\,\delta w\, dxdy - \int_{\Gamma^e} \left(V_n\delta w + M_n\,\delta\theta_n\right)ds$$

$$= \int_{\Omega^e}\int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\,\delta\varepsilon_{yy} + \sigma_{xy}\,2\delta\varepsilon_{xy}\right)dzdxdy$$

$$- \int_{\Omega^e} q\,\delta w_0\, dxdy - \int_{\Gamma^e} \left(N_n\delta u_n + N_{ns}\delta u_{ns} + V_n\delta w_0 + M_n\,\delta\theta_n\right)ds$$

$$= \int_{\Omega^e} \left\{ \int_{-h/2}^{h/2} \left[\sigma_{xx}\left(\frac{\partial\delta u_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial x} - z\frac{\partial^2\delta w_0}{\partial x^2}\right) + \sigma_{yy}\left(\frac{\partial\delta v_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial\delta w_0}{\partial y} - z\frac{\partial^2\delta w_0}{\partial y^2}\right)\right.\right.$$

$$\left.\left. + \sigma_{xy}\left(\frac{\partial\delta u_0}{\partial y} + \frac{\partial\delta v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial\delta w_0}{\partial x} - 2z\frac{\partial^2\delta w_0}{\partial x\partial y}\right)\right]dz \right\} dxdy$$

$$- \int_{\Omega^e} q\,\delta w_0\, dxdy - \int_{\Gamma^e} \left(N_n\delta u_n + N_{ns}\delta u_{ns} + V_n\delta w_0 + M_n\,\delta\theta_n\right)ds$$

Principle of Virtual Displacements (continued)

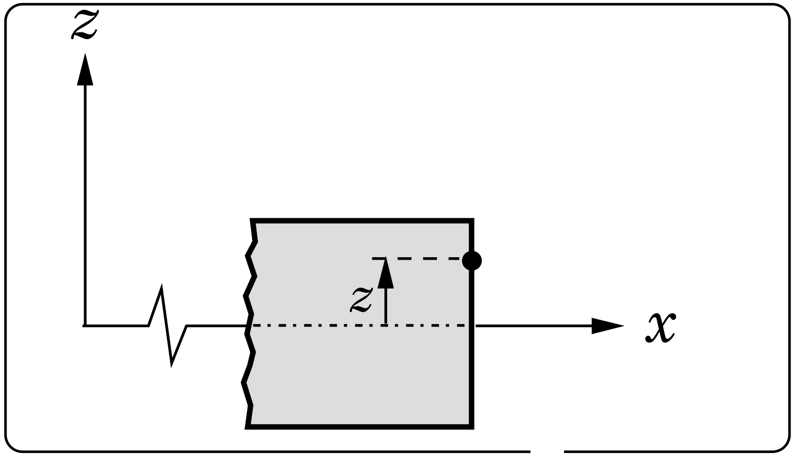
$$0 = \int_{\Omega^e} \left\{ N_{xx} \left(\frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} \right) + N_{yy} \left(\frac{\partial \delta v_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial y} \right) \right.$$

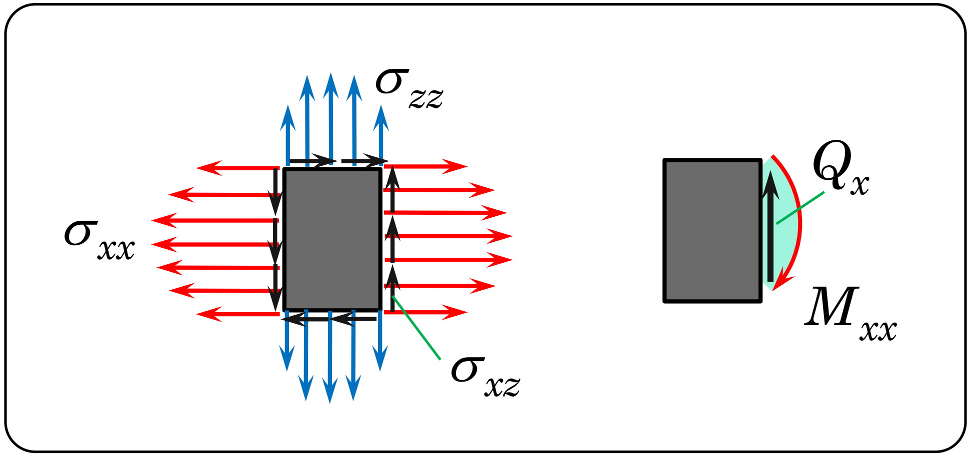
$$+ N_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial x} \right)$$

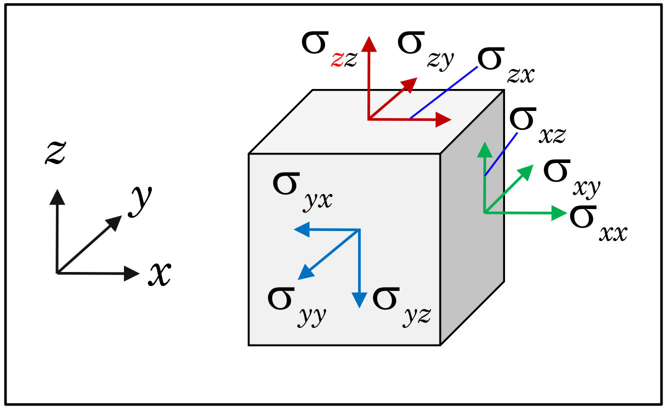
$$\left. - M_{xx} \frac{\partial^2 \delta w_0}{\partial x^2} - M_{yy} \frac{\partial^2 \delta w_0}{\partial y^2} - 2 M_{xy} \frac{\partial^2 \delta w_0}{\partial x \partial y} \right\} dxdy$$

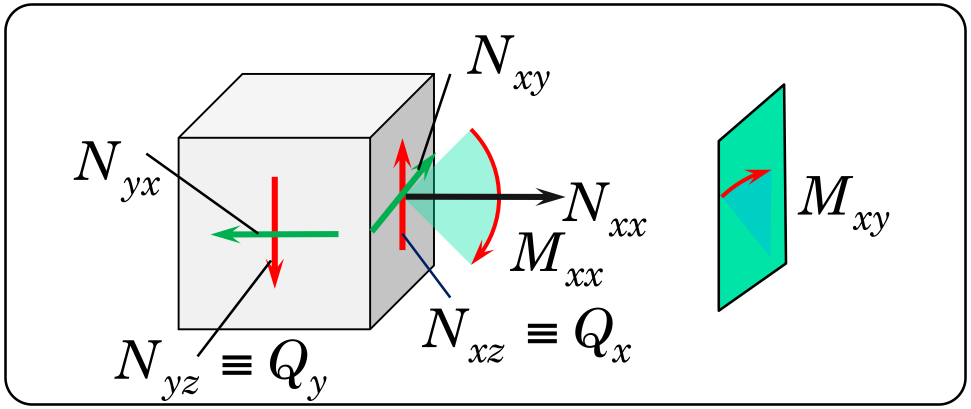
$$- \int_{\Omega^e} q \, \delta w_0 \, dxdy - \int_{\Gamma^e} \left(N_n \delta u_n + N_{ns} \delta u_{ns} + V_n \delta w_0 + M_n \, \delta \theta_n \right) ds$$

SIGN CONVENTION FOR STRESS RESULTANTS

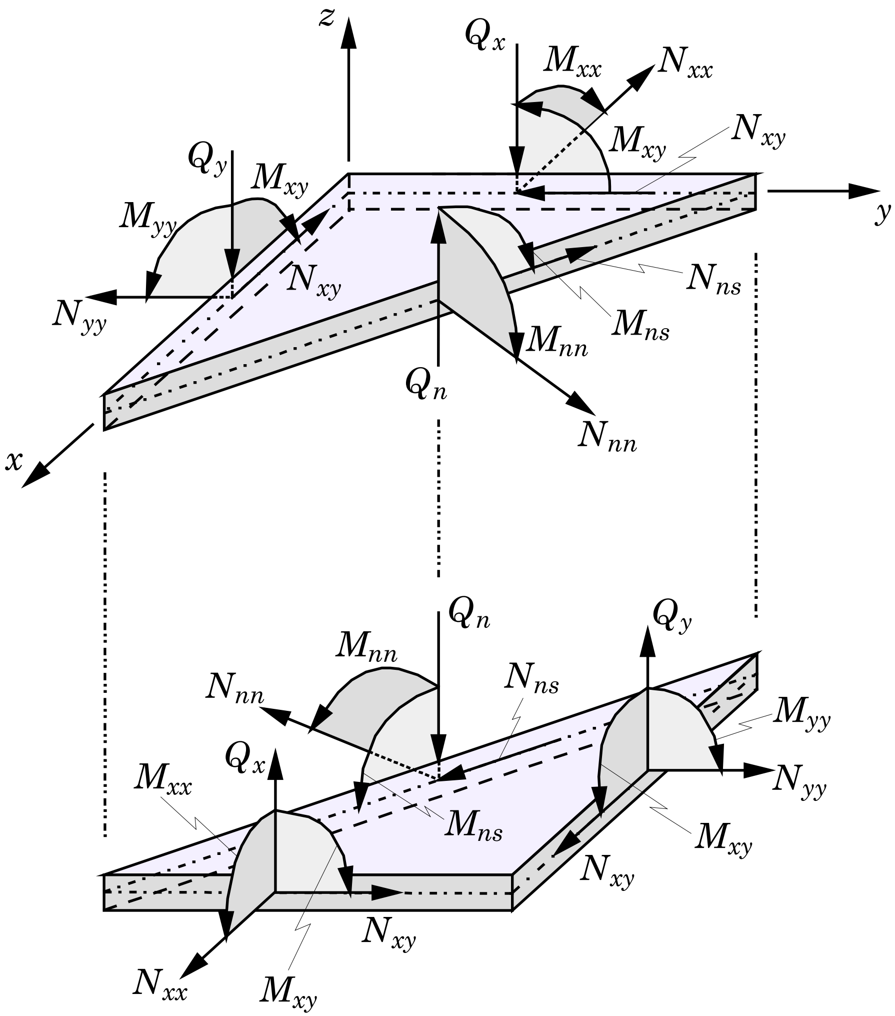








STRESS RESULTANTS



$$N_{\xi\eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi\eta}\, dz \quad (\xi, \eta = x, y, z)$$

$$M_{\xi\eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi\eta} z\, dz \quad (\xi, \eta = x, y)$$

$$Q_x = N_{xz}, \quad Q_y = N_{yz}$$

Equations of Motion of the Classical Plate Theory

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} z\, dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

Laminate Stiffnesses

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}(1, z, z^2) dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2) dz$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k)\ , \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3)$$

Equations of Motion of the Classical Plate Theory

$$\delta u_0: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right)$$

$$\delta v_0: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right)$$

$$\delta w_0: \quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + \mathcal{N}(w_0) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) + I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} dz$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

Stress Resultant-Displacement Relations

$$N_{xx}=\int_{-h/2}^{h/2}\sigma_{xx}\,dz=\int_{-h/2}^{h/2}\left\{\bar{Q}_{11}\left(\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2-z\frac{\partial^2 w_0}{\partial x^2}\right)+\bar{Q}_{12}\left(\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2-z\frac{\partial^2 w_0}{\partial y^2}\right)\right\}dz$$

$$+\int_{-h/2}^{h/2}\bar{Q}_{16}\left(\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}-2z\frac{\partial^2 w_0}{\partial x\partial y}\right)dz$$

$$=A_{11}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{12}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]+A_{16}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]$$

$$-B_{11}\frac{\partial^2 w_0}{\partial x^2}-B_{12}\frac{\partial^2 w_0}{\partial y^2}-2B_{16}\frac{\partial^2 w_0}{\partial x\partial y}$$

$$M_{xx}=\int_{-h/2}^{h/2}\sigma_{xx}\,z\,dz=\int_{-h/2}^{h/2}z\left[\bar{Q}_{11}\left(\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2-z\frac{\partial^2 w_0}{\partial x^2}\right)+\bar{Q}_{12}\left(\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2-z\frac{\partial^2 w_0}{\partial y^2}\right)\right]dz$$

$$+\int_{-h/2}^{h/2}z\,\bar{Q}_{16}\left(\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}-2z\frac{\partial^2 w_0}{\partial x\partial y}\right)dz$$

$$=B_{11}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+B_{12}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]+B_{16}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]$$

$$-D_{11}\frac{\partial^2 w_0}{\partial x^2}-D_{12}\frac{\partial^2 w_0}{\partial y^2}-2D_{16}\frac{\partial^2 w_0}{\partial x\partial y}$$

Laminated Plate Constitutive Equations (Linear CLPT)

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y} - N_{xx}^T$$

$$N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - B_{12}\frac{\partial^2 w}{\partial x^2} - B_{22}\frac{\partial^2 w}{\partial y^2} - 2B_{26}\frac{\partial^2 w}{\partial x \partial y} - N_{yy}^T$$

$$N_{xy} = A_{16}\frac{\partial u}{\partial x} + A_{26}\frac{\partial v}{\partial y} + A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} - 2B_{66}\frac{\partial^2 w}{\partial x \partial y} - N_{xy}^T$$

$$M_{xx} = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y} + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T$$

$$M_{yy} = B_{12}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T$$

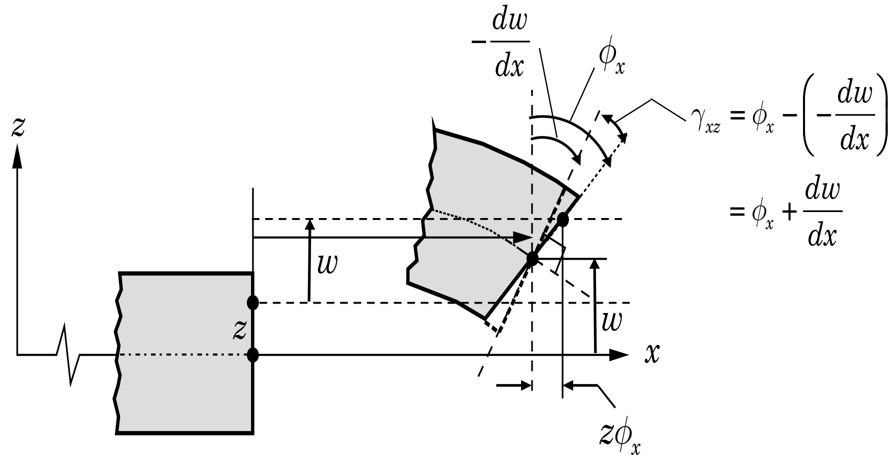
$$M_{xy} = B_{16}\frac{\partial u}{\partial x} + B_{26}\frac{\partial v}{\partial y} + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \quad Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$



Linear strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} + z\frac{\partial \phi_y}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$\gamma_{xz} = \left(\phi_x + \frac{\partial w}{\partial x}\right), \qquad \gamma_{yz} = \left(\phi_y + \frac{\partial w}{\partial y}\right)$$

EQUATIONS OF MOTION OF FSDT

Equations of motion (FSDT)

$$\sum F_x = 0: \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\sum F_y = 0: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\sum F_z = 0: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\sum M_y = 0: \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\sum M_x = 0: \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2}$$

Definitions of stress resultants

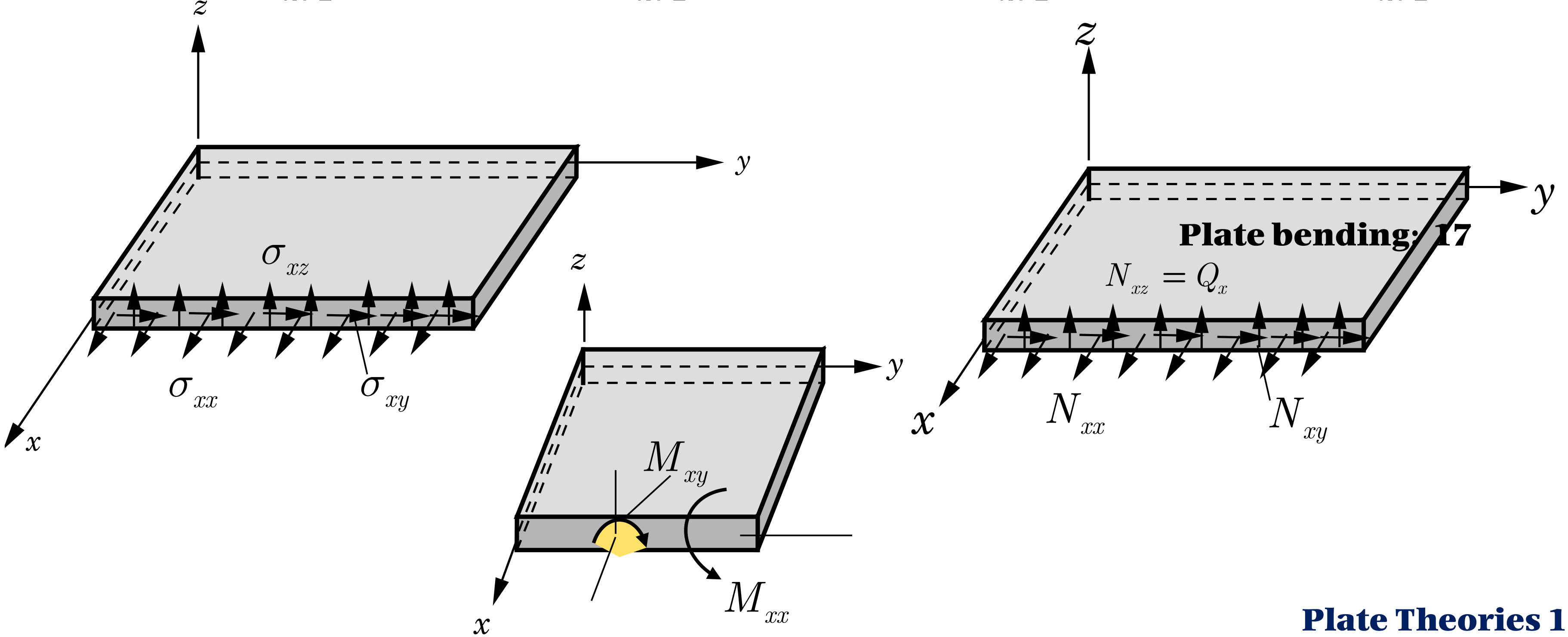
$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz, \quad Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz}\, dz$$

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz, \quad M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\, dz, \quad M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz, \quad Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz}\, dz$$

Stresses and Stress Resultants on an edge of a Plate

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz, \quad Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz}\, dz$$

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz, \quad M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\, dz, \quad M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz, \quad Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz}\, dz$$



Navier Solution of Simply Supported Orthotropic Plates (CLPT)

$$-\left[D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4}\right] + q = 0$$

$$w_0(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}\ \sin\alpha x\ \sin\beta y$$

$$q(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{mn} \sin\alpha x\ \sin\beta y$$

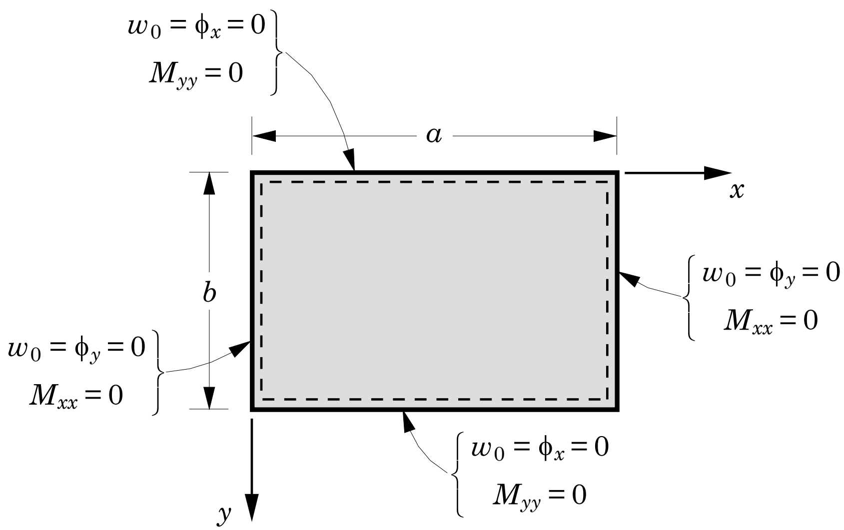
$$Q_{mn} = \frac{4}{ab}\int_0^b\int_0^a\ q(x,y)\ \sin\alpha x\ \sin\beta y\ dxdy$$

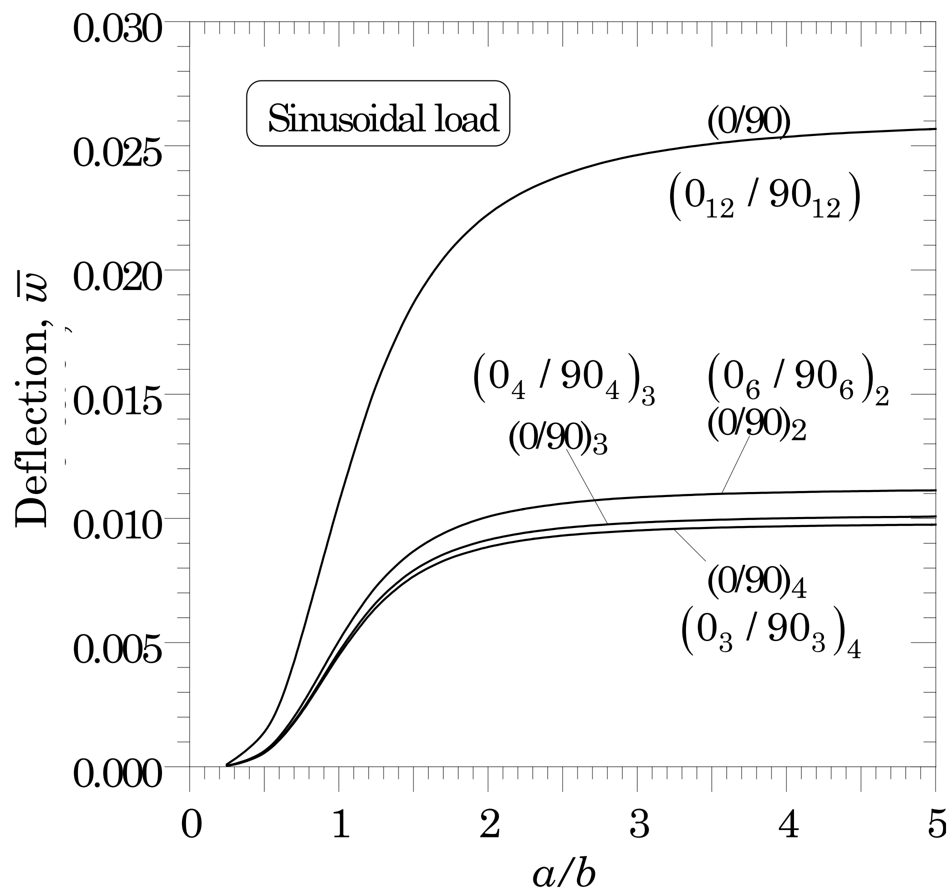
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big\{-W_{mn}\left[D_{11}\alpha^4 + 2(D_{12}+2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right]$$
$$+ Q_{mn}\Big\}\sin\alpha x\ \sin\beta y = 0$$

$$W_{mn} = \frac{Q_{mn}}{d_{mn}},\ \ d_{mn} = \frac{\pi^4}{b^4}\left[D_{11}m^4 s^4 + 2(D_{12}+2D_{66})m^2 n^2 s^2 + D_{22}n^4\right]$$

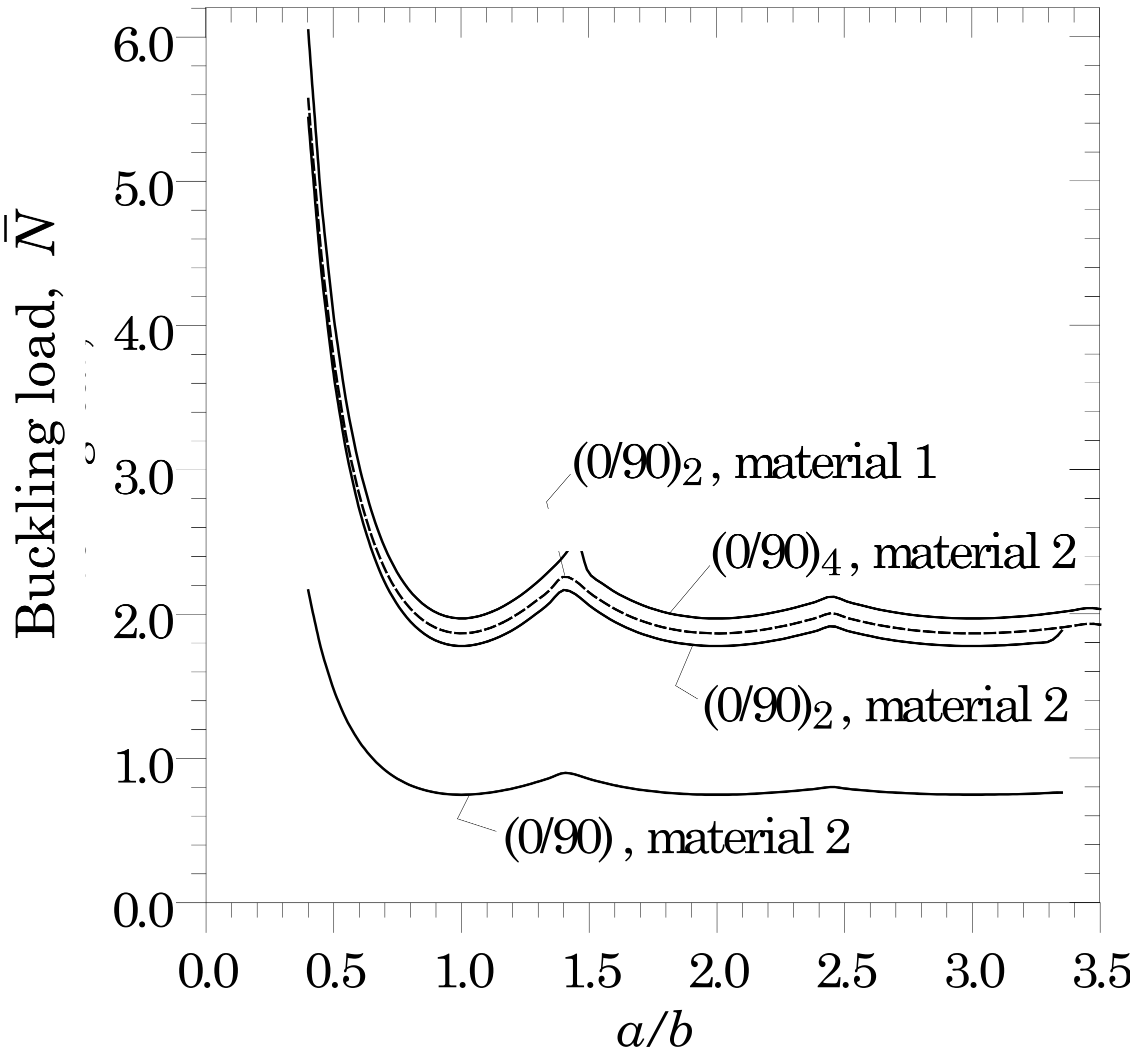
ANALYTICAL SOLUTIONS

Bending of Simply Supported Plates (SS-1)-CLPT





Buckling of Plates (SS-1) - CLPT



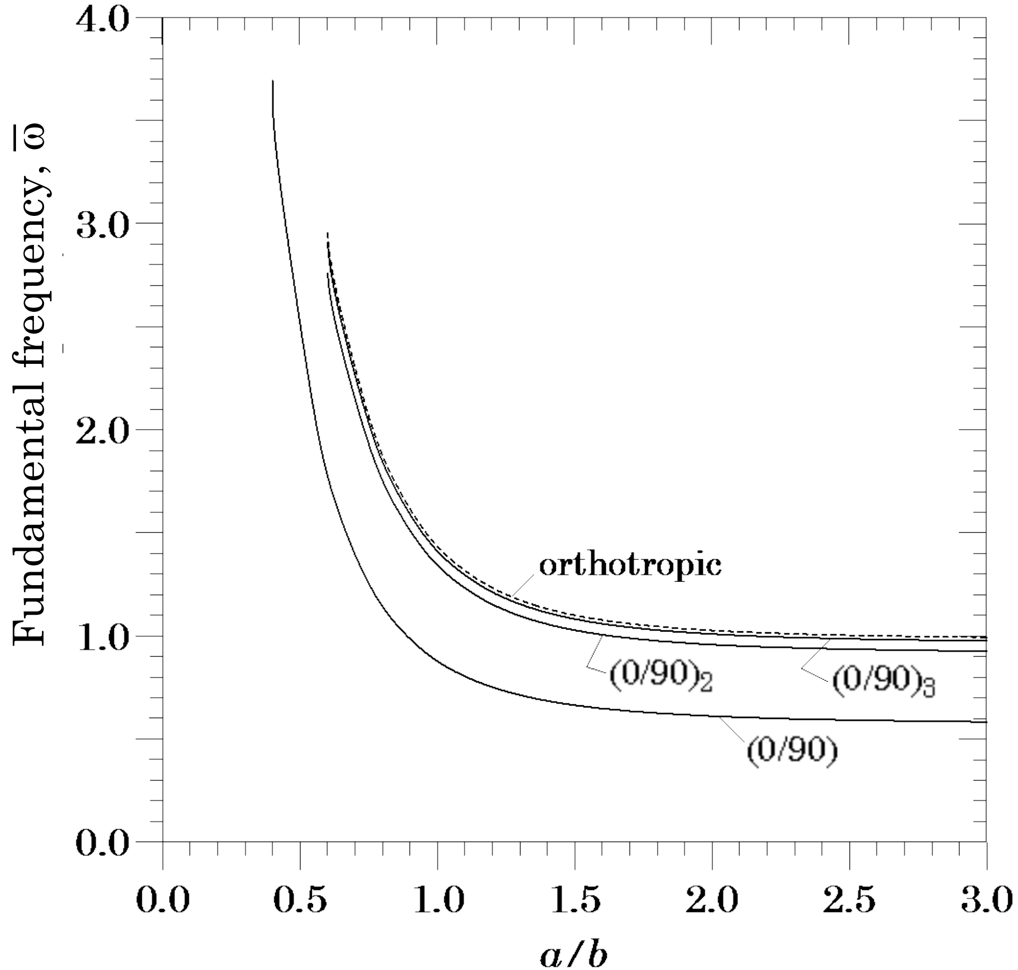
$(0_{12}/90_{12})$

$(0_4/90_4)_3$

$(0_3/90_3)_4$

$(0_6/90_6)_2$

Vibration of Plates (SS-1) - CLPT



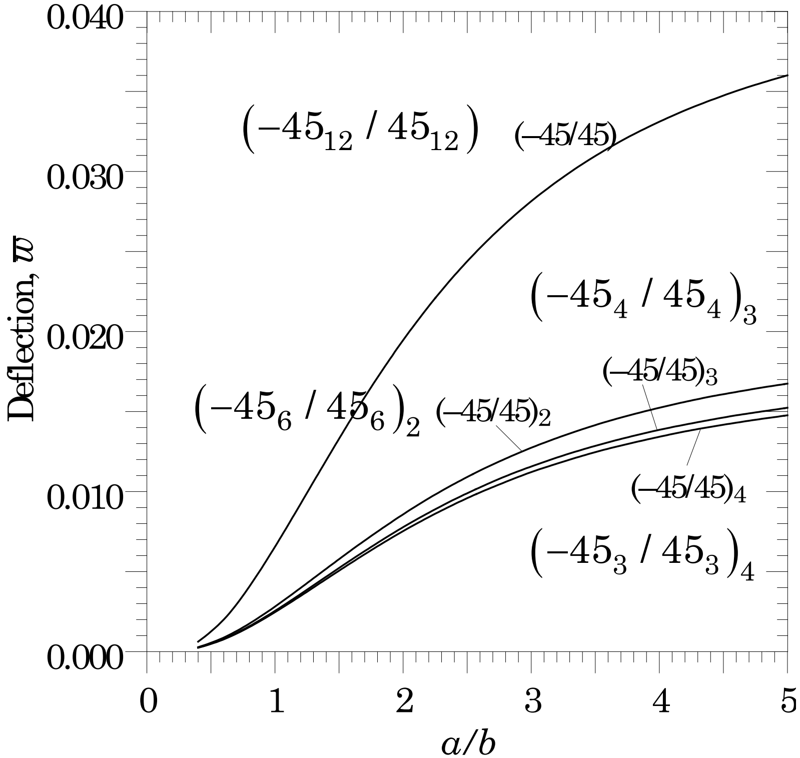
$$(0/90) = (0_{12}/90_{12})$$

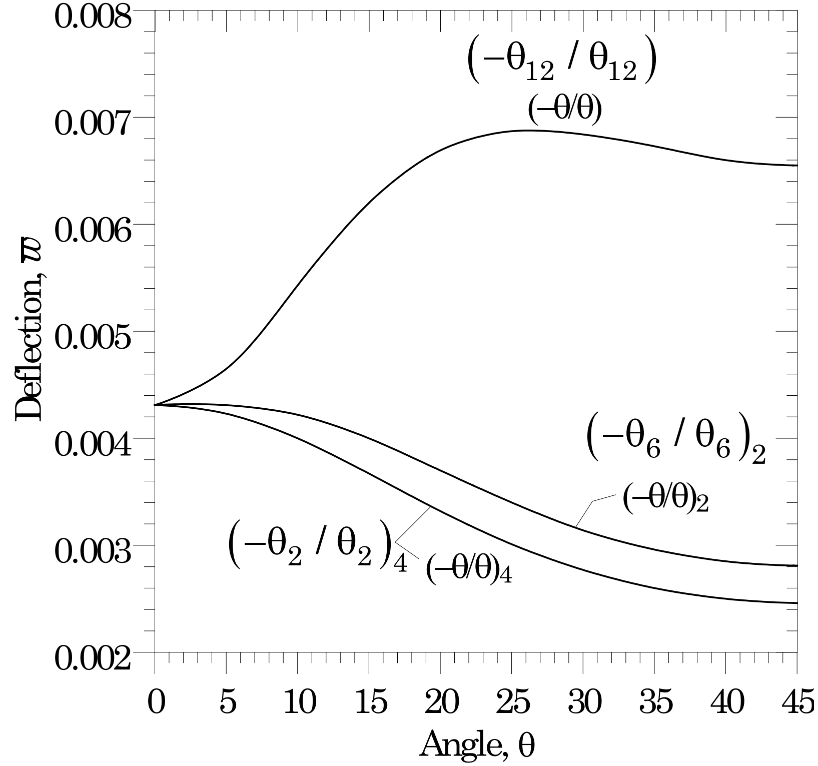
$$(0/90)_2 = (0_6/90_6)_2$$

$$(0/90)_4 = (0_3/90_3)_4$$

$$(0/90)_3 = (0_4/90_4)_3$$

Bending of Angle-ply Plates (SS-2)-CLPT





BUCKLING OF LAMINATED COMPOSITE PLATES USING FSDT

$$\frac{\partial}{\partial x}\left[A_{11}\frac{\partial u_0}{\partial x}+A_{12}\frac{\partial v_0}{\partial y}+A_{16}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{11}\frac{\partial \phi_x}{\partial x}+B_{12}\frac{\partial \phi_y}{\partial y}\right.$$
$$\left.+B_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[A_{16}\frac{\partial u_0}{\partial x}+A_{26}\frac{\partial v_0}{\partial y}\right.$$
$$\left.+A_{66}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{16}\frac{\partial \phi_x}{\partial x}+B_{26}\frac{\partial \phi_y}{\partial y}+B_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]=0$$

$$\frac{\partial}{\partial x}\left[A_{16}\frac{\partial u_0}{\partial x}+A_{26}\frac{\partial v_0}{\partial y}+A_{66}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{16}\frac{\partial \phi_x}{\partial x}+B_{26}\frac{\partial \phi_y}{\partial y}\right.$$
$$\left.+B_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[A_{12}\frac{\partial u_0}{\partial x}+A_{22}\frac{\partial v_0}{\partial y}\right.$$
$$\left.+A_{26}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{12}\frac{\partial \phi_x}{\partial x}+B_{22}\frac{\partial \phi_y}{\partial y}+B_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]=0$$

$$\frac{\partial}{\partial x}\left[KA_{45}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)+KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)\right]$$
$$+\frac{\partial}{\partial y}\left[KA_{44}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)+KA_{45}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)\right]$$
$$+\hat{N}_{xx}\frac{\partial^2 w_0}{\partial x^2}+2\hat{N}_{xy}\frac{\partial^2 w_0}{\partial x \partial y}+\hat{N}_{yy}\frac{\partial^2 w_0}{\partial y^2}=0$$

BUCKLING OF LAMINATED COMPOSITE PLATES USING FSDT

Governing Equations (continued)

$$\frac{\partial}{\partial x}\left[B_{11}\frac{\partial u_0}{\partial x}+B_{12}\frac{\partial v_0}{\partial y}+B_{16}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{11}\frac{\partial \phi_x}{\partial x}+D_{12}\frac{\partial \phi_y}{\partial y}\right.$$
$$\left.+D_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[B_{16}\frac{\partial u_0}{\partial x}+B_{26}\frac{\partial v_0}{\partial y}\right.$$
$$\left.+B_{66}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{16}\frac{\partial \phi_x}{\partial x}+D_{26}\frac{\partial \phi_y}{\partial y}+D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]$$
$$-\left[KA_{45}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)+KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)\right]=0$$

$$\frac{\partial}{\partial x}\left[B_{16}\frac{\partial u_0}{\partial x}+B_{26}\frac{\partial v_0}{\partial y}+B_{66}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{16}\frac{\partial \phi_x}{\partial x}\right.$$
$$\left.+D_{26}\frac{\partial \phi_y}{\partial y}+D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[B_{12}\frac{\partial u_0}{\partial x}+B_{22}\frac{\partial v_0}{\partial y}\right.$$
$$\left.+B_{26}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{12}\frac{\partial \phi_x}{\partial y}+D_{22}\frac{\partial \phi_y}{\partial y}+D_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]$$
$$-\left[KA_{44}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)+KA_{45}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)\right]=0$$

BUCKLING OF LAMINATED COMPOSITE PLATES USING FSDT
The Navier Solution

SS-1 Boundary Conditions

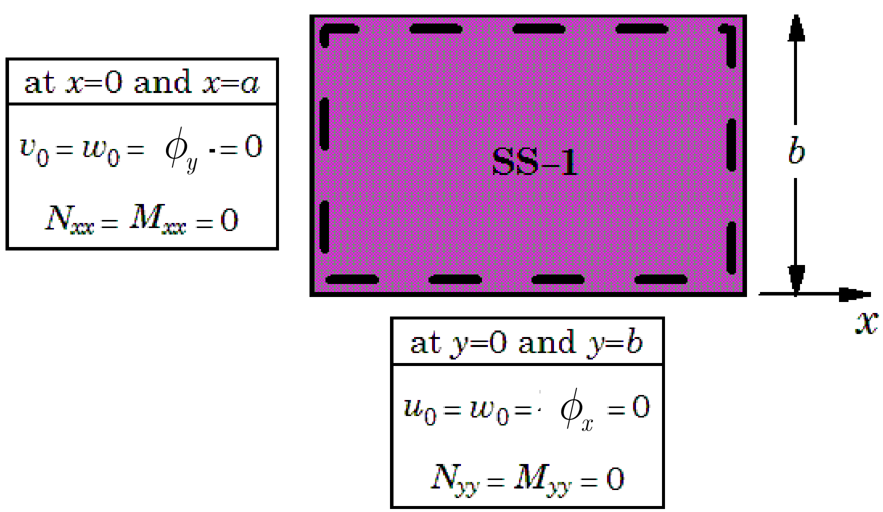
$$u_0(x,0)=0,\quad u_0(x,b)=0,\quad v_0(0,y)=0,\quad v_0(a,y)=0$$

$$w_0(x,0)=0,\quad w_{)}(x,b)=0,\quad w_0(0,y)=0,\quad w_0(a,y)=0$$

$$\phi_x(x,0)=0,\quad \phi_x(x,b)=0,\quad \phi_y(0,y)=0,\quad \phi_y(a,y)=0$$

$$N_{xx}(0,y)=0,\quad N_{xx}(a,y)=0,\quad N_{yy}(x,0)=0,\quad N_{yy}(x,b)=0$$

$$M_{xx}(0,y)=0,\quad M_{xx}(a,y)=0,\quad M_{yy}(x,0)=0,\quad M_{yy}(x,b)=0$$



BUCKLING OF *ANTISYMMETRIC CROSS-PLY* LAMINATES USING FSDT
The Navier Solution

$$u_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn} \cos\alpha x \sin\beta y$$

$$v_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn} \sin\alpha x \cos\beta y$$

$$w_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn} \sin\alpha x \sin\beta y$$

$$\phi_x(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_{mn} \cos\alpha x \sin\beta y$$

$$\phi_y(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Y_{mn} \sin\alpha x \cos\beta y$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$.

$$A_{16} = 0, \quad A_{26} = 0, \quad A_{45} = 0, \quad B_{16} = 0, \quad B_{26} = 0, \quad D_{16} = 0, \quad D_{26} = 0$$

$$\hat{N}_{xx} = -N_0, \quad \hat{N}_{yy} = -kN_0, \quad k = \frac{\hat{N}_{yy}}{\hat{N}_{xx}}$$

BUCKLING OF *ANTISYMMETRIC CROSS-PLY* LAMINATES USING FSDT
The Navier Solution

$$\begin{bmatrix} s_{11} & s_{12} & 0 & s_{14} & s_{15} \\ s_{12} & s_{22} & 0 & s_{24} & s_{25} \\ 0 & 0 & s_{33} - N_0(\alpha^2 + k\beta^2) & s_{34} & s_{35} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$s_{11} = (A_{11}\alpha^2 + A_{66}\beta^2), \quad s_{12} = (A_{12} + A_{66})\alpha\beta$$

$$s_{14} = (B_{11}\alpha^2 + B_{66}\beta^2), \quad s_{15} = (B_{12} + B_{66})\alpha\beta$$

$$s_{22} = (A_{66}\alpha^2 + A_{22}\beta^2), \quad s_{24} = s_{15}$$

$$s_{25} = (B_{66}\alpha^2 + B_{22}\beta^2), \quad s_{33} = K(A_{55}\alpha^2 + A_{44}\beta^2)$$

$$s_{34} = KA_{55}\alpha, \quad s_{35} = KA_{44}\beta, \quad s_{44} = (D_{11}\alpha^2 + D_{66}\beta^2 + KA_{55})$$

$$s_{45} = (D_{12} + D_{66})\alpha\beta, \quad s_{55} = (D_{66}\alpha^2 + D_{22}\beta^2 + KA_{44})$$

BUCKLING OF *ANTISYMMETRIC CROSS-PLY* LAMINATES USING FSDT
The Navier Solution

$$N_0 = \left(\frac{1}{\alpha^2 + k\beta^2}\right)\left(\frac{K^2 A_{44} A_{55} c_{33} + \left(K A_{55}\alpha^2 + K A_{44}\beta^2\right) c_1}{c_1 + K A_{44} c_2 + K A_{55} c_3 + K^2 A_{44} A_{55}}\right)$$

$$= \left(\frac{1}{\alpha^2 + k\beta^2}\right)\left(\frac{c_{33} + \left(\frac{\alpha^2}{K A_{44}} + \frac{\beta^2}{K A_{55}}\right) c_1}{1 + \frac{c_1}{K^2 A_{44} A_{55}} + \frac{c_2}{K A_{55}} + \frac{c_3}{K A_{44}}}\right)$$

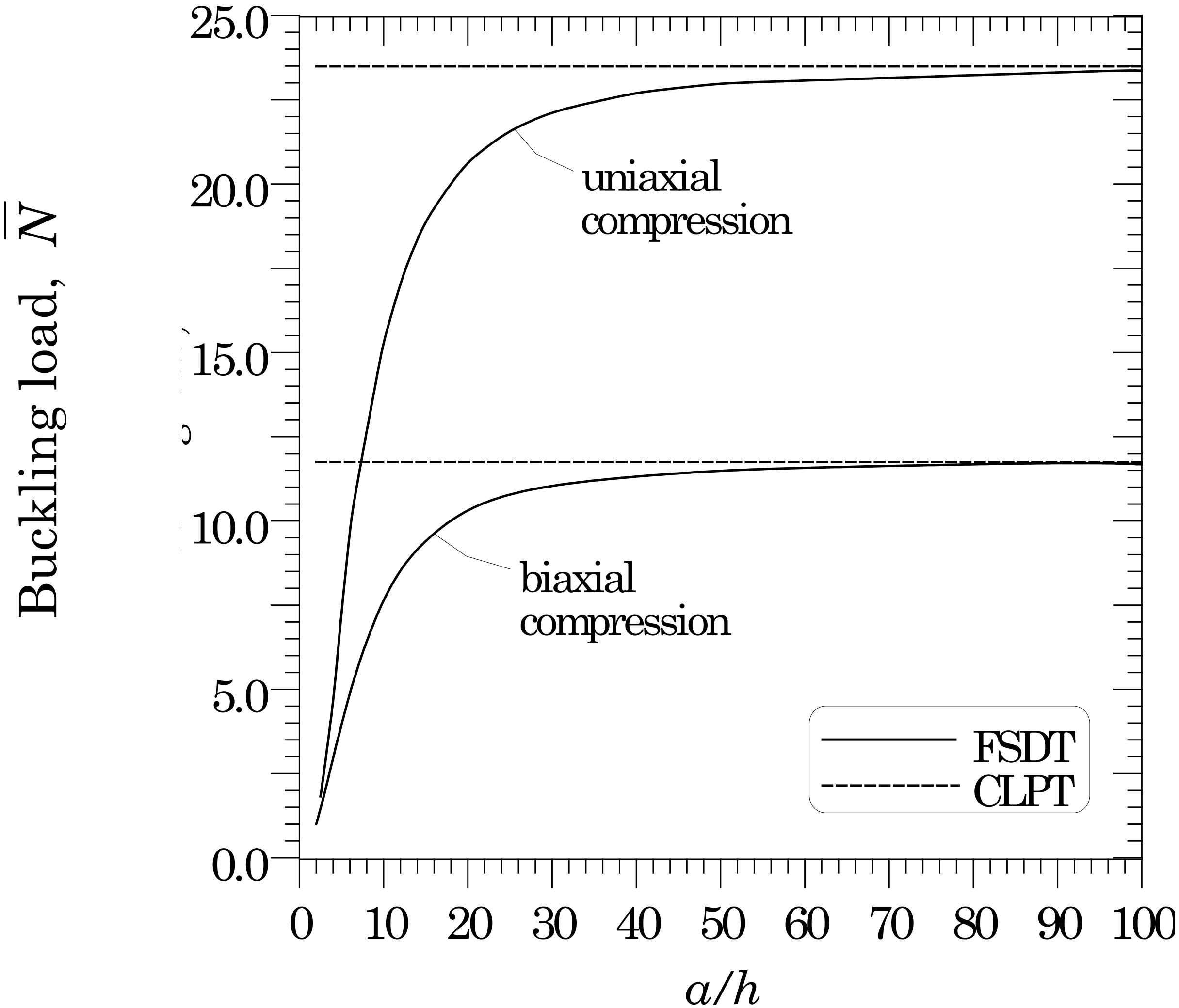
$$c_{33} = D_{11}\alpha^4 + 2\left(D_{12} + 2D_{66}\right)\alpha^2\beta^2 + D_{22}\beta^4, \quad c_1 = c_2 c_3 - (c_4)^2 > 0$$

$$c_2 = D_{11}\alpha^2 + D_{66}\beta^2, \; c_3 = D_{66}\alpha^2 + D_{22}\beta^2, \; c_4 = \left(D_{12} + D_{66}\right)\alpha\beta$$

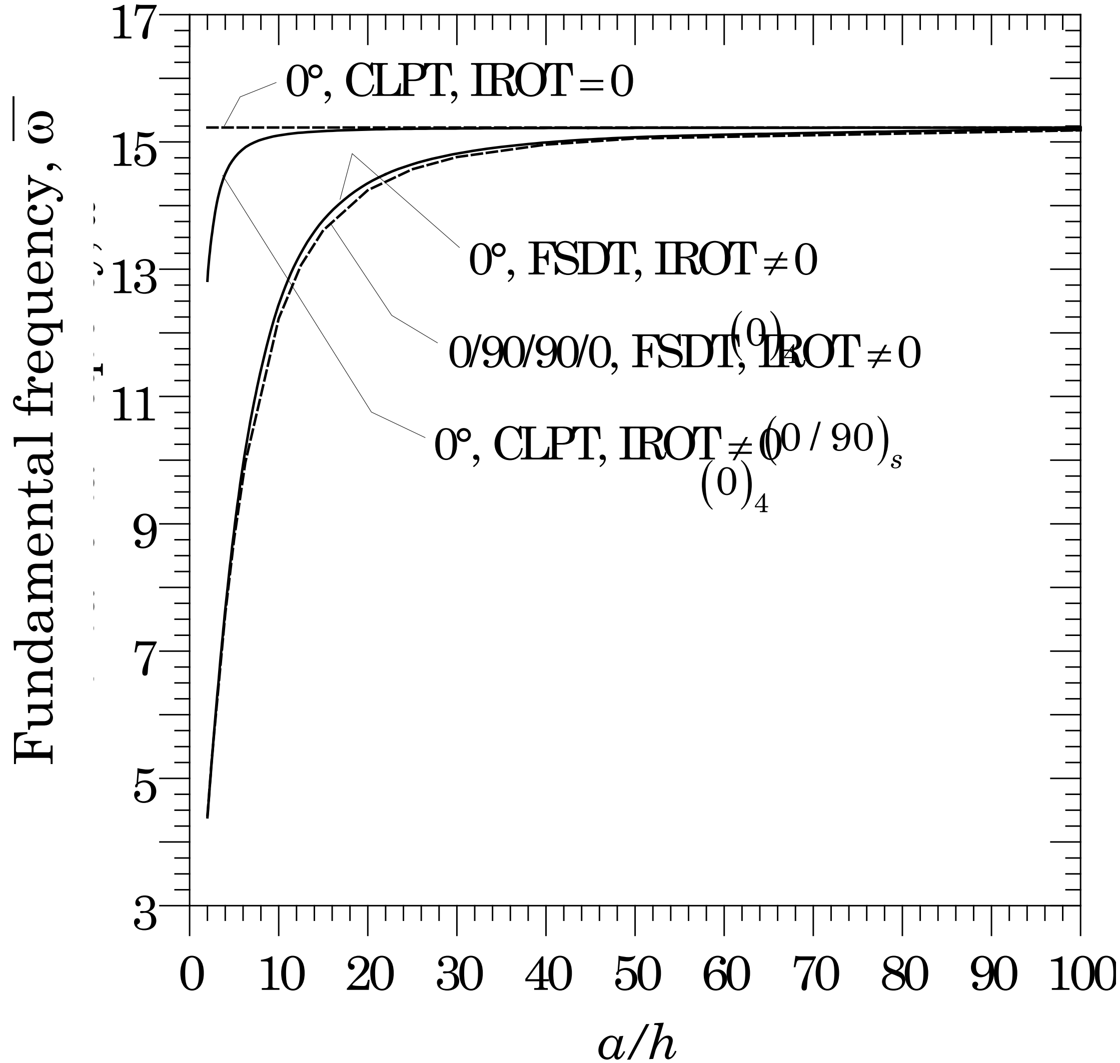
$$\frac{c_{33} + k_1}{1 + k_2} \quad \text{where} \quad k_1 < k_2, \quad c_{33} \geq \frac{c_{33} + k_1}{1 + k_2}$$

$$N_{cr} = 4D\left(\frac{\pi}{a}\right)^2 \frac{\left[1 + \frac{3(1-\nu^2)\pi^2 (h/a)^2}{K}\right]}{\left[1 + \frac{72(1+\nu)(1-\nu^2)\pi^4 (h/a)^4}{K^2} + \frac{6(1+\nu)(3-\nu)\pi^2 (h/a)^2}{K}\right]}$$

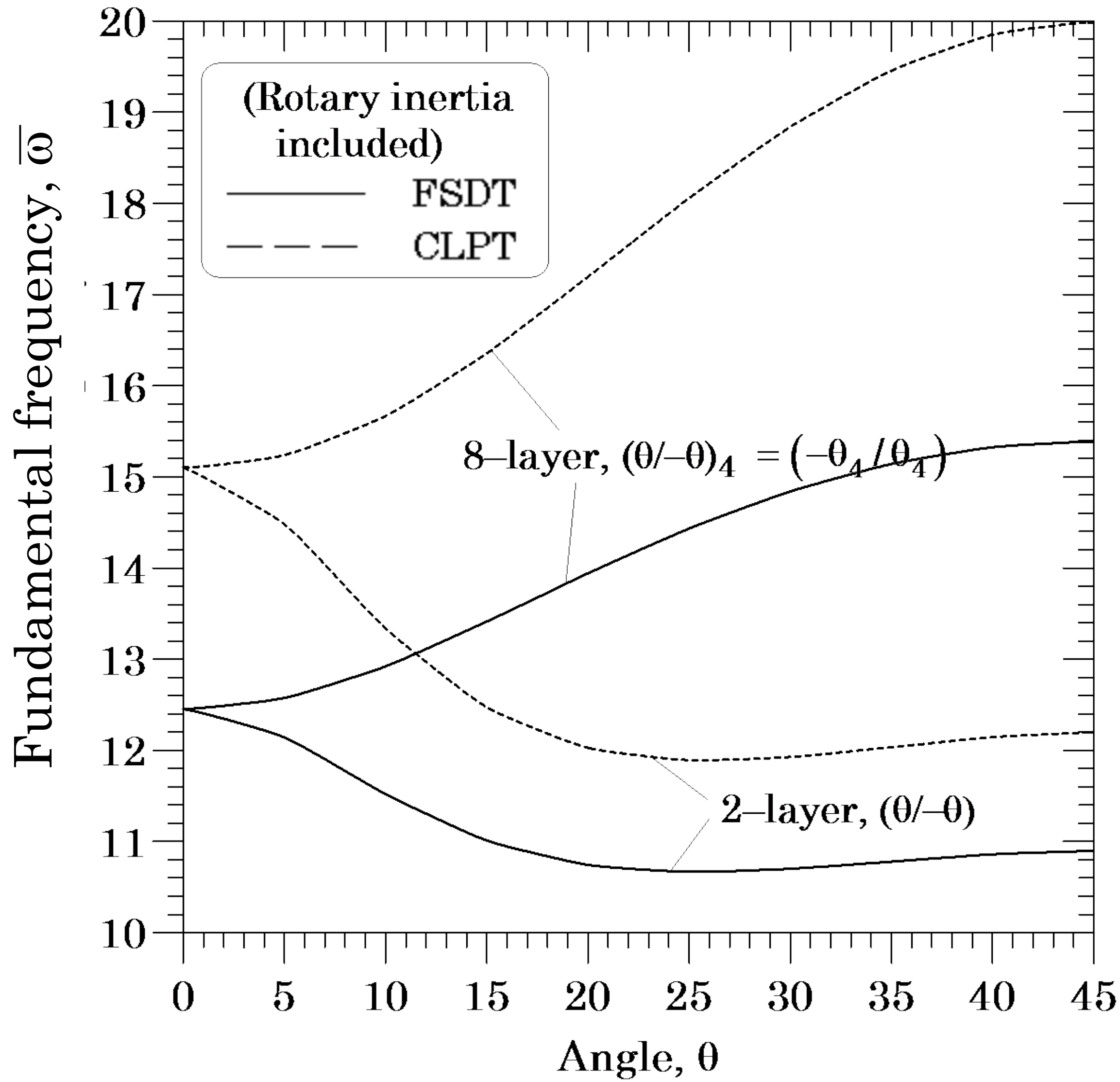
Effect of Shear Deformation on Buckling



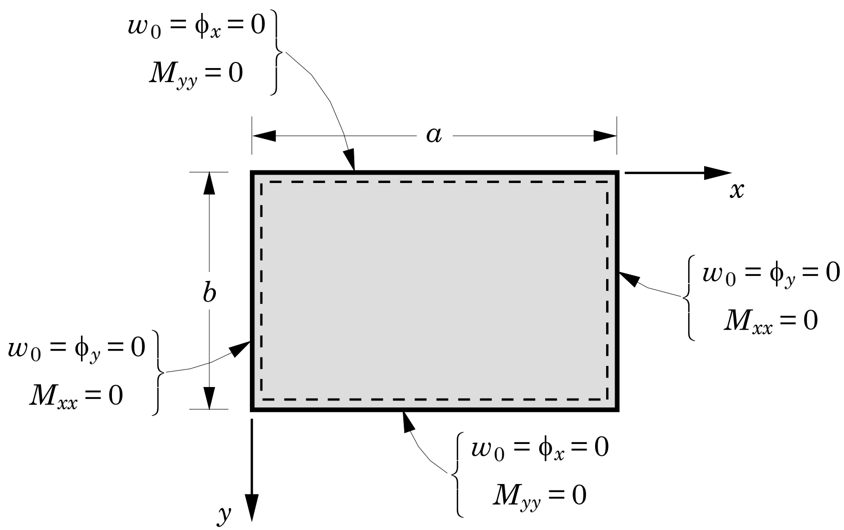
Effect of Shear Deformation On Vibration

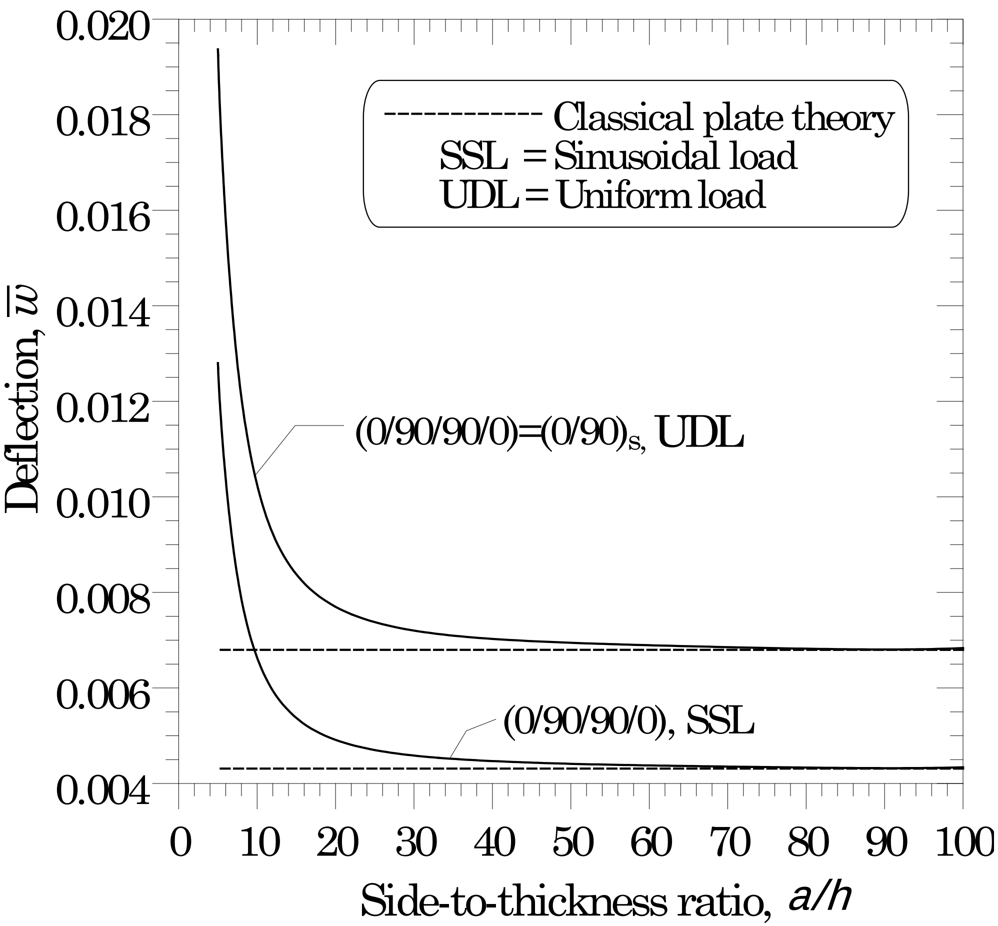


Shear Deformation in Angle-Ply Plates

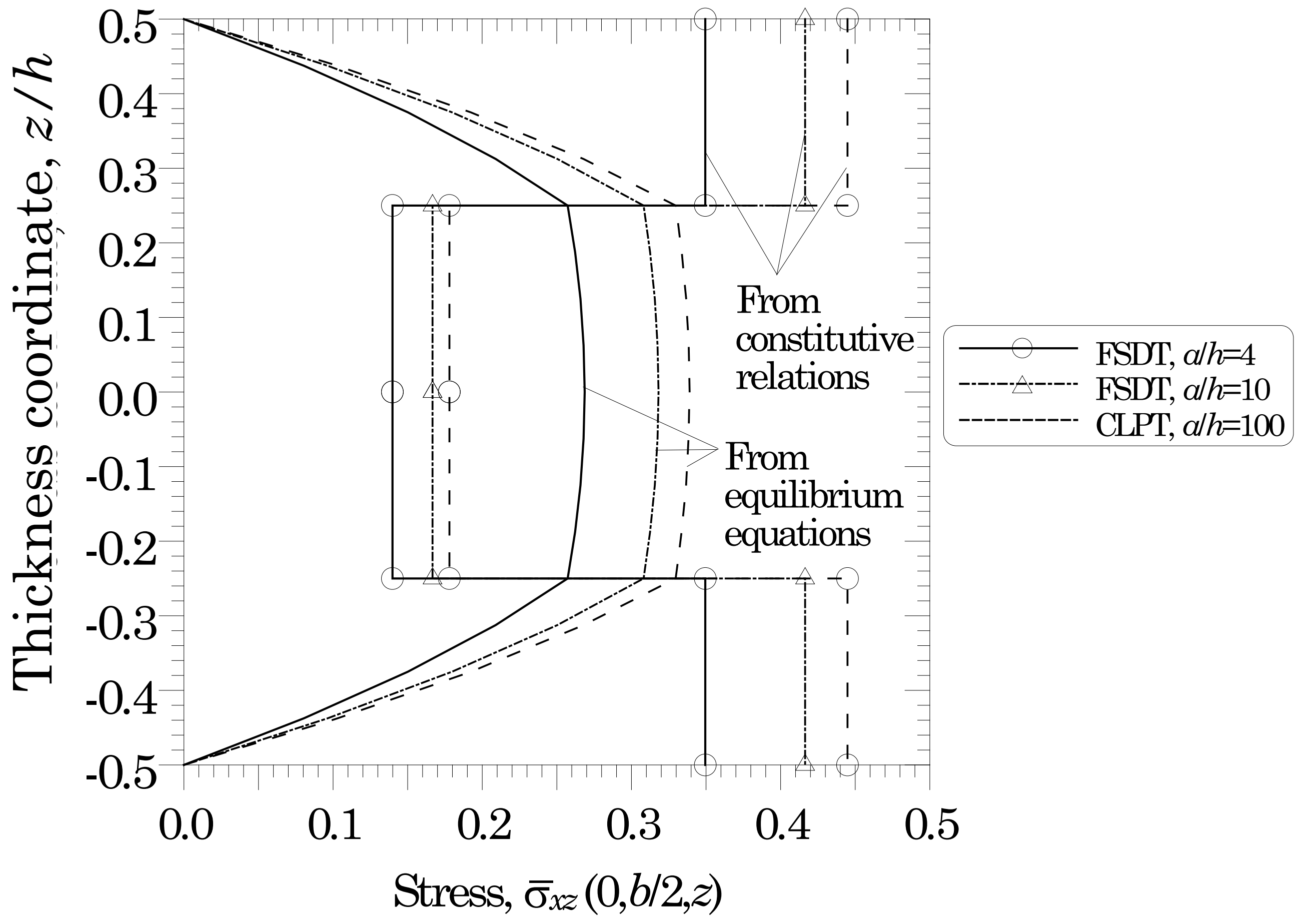


Effect of Shear Deformation on Bending Deformation

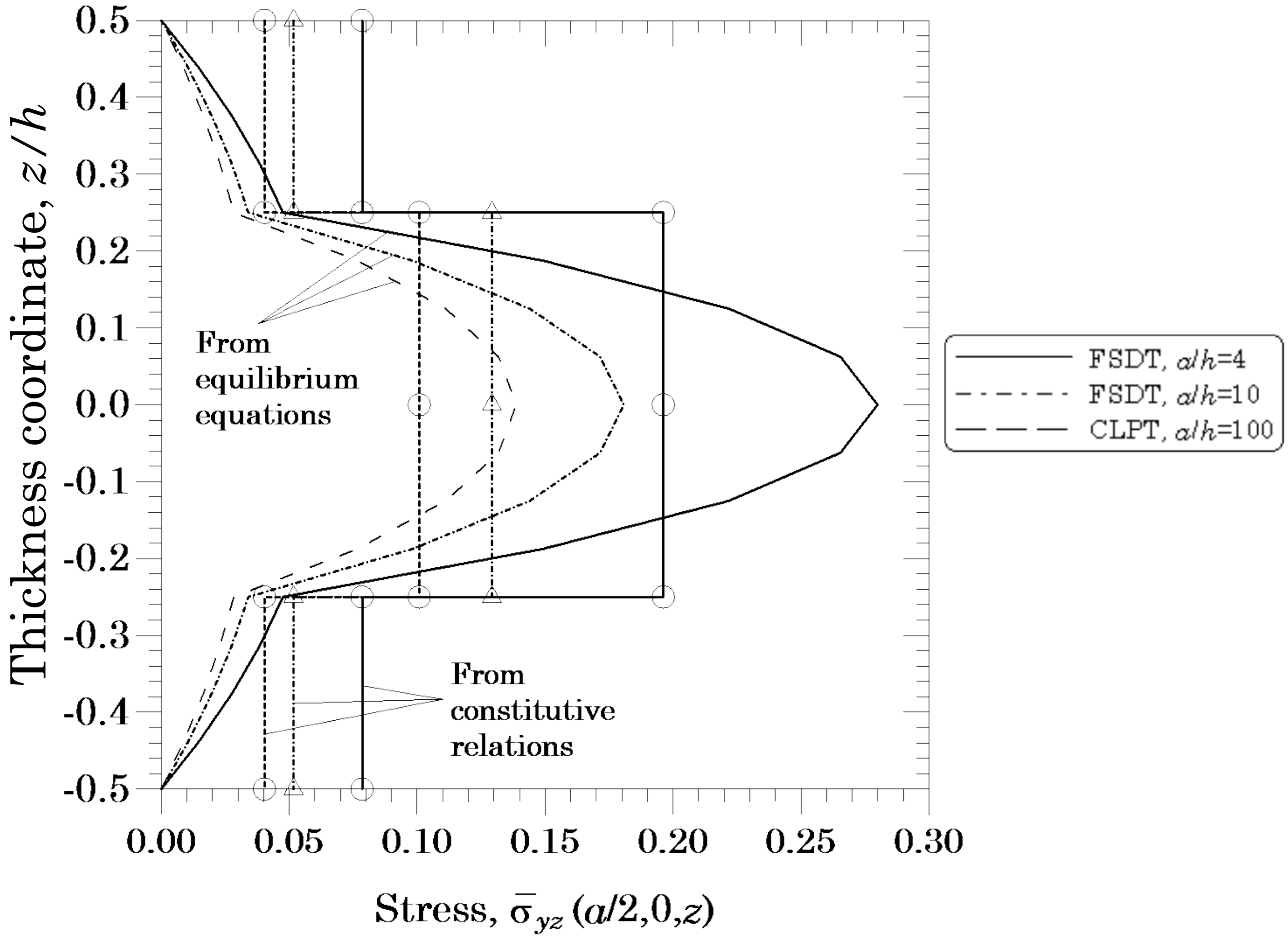




Transverse Shear Stresses



Transverse Shear Stresses



Shear Deformation in Angle-Ply Plates