EULER-BERNOULLI AND TIMOSHENKO BEAM THEORIES

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Euler-Bernoulli beam theory

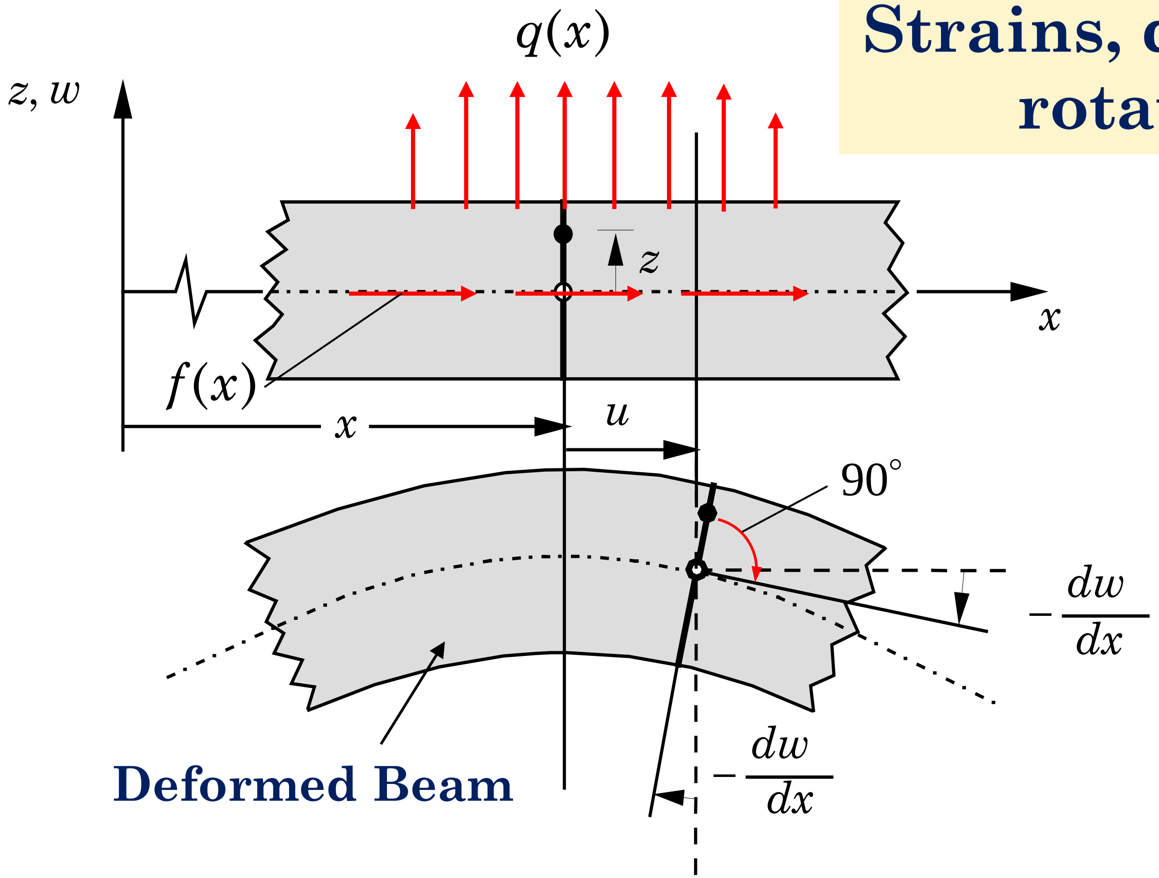
- Kinematics
- Equilibrium equations
- Governing equations in terms of the displacements

Timoshenko beam theory

- Kinematics
- Equilibrium equations
- Governing equations in terms of the displacements

KINEMATICS OF THE LINEARIZED EULER-BERNOULLI BEAM THEORY

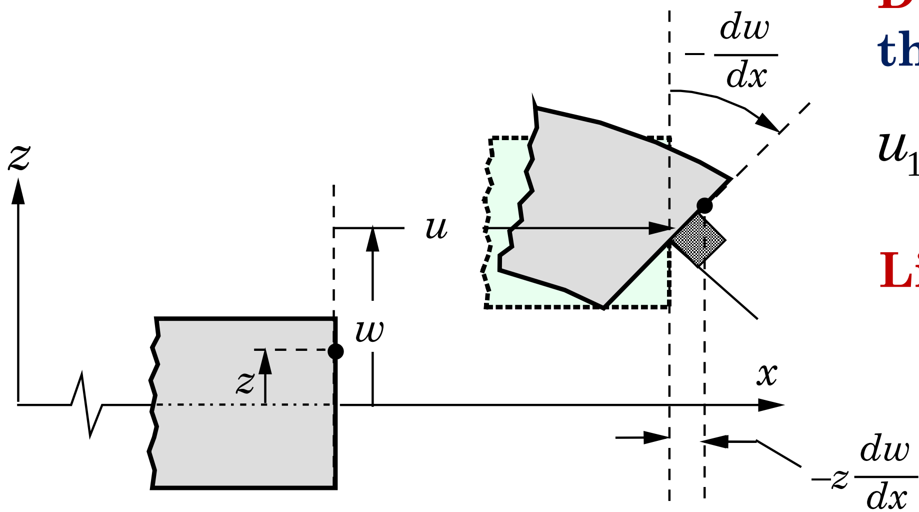
Strains, displacements, and rotations are small



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT) is based on the assumptions of
(1) straightness,
***(2) inextensibility,* and**
(3) normality

Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



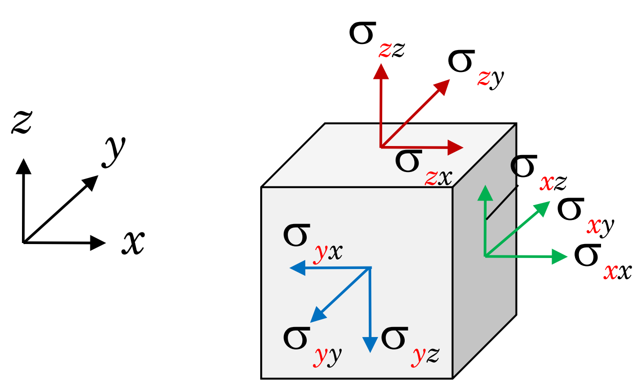
Displacement field (constructed using the hypothesis)

$$u_1(x,z) = u - z\frac{dw}{dx}, \qquad u_2 = 0, \quad u_3 = w(x)$$

Linear strains

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} - z\frac{d^2w}{dx^2},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\frac{dw}{dx} + \frac{dw}{dx} = 0.$$



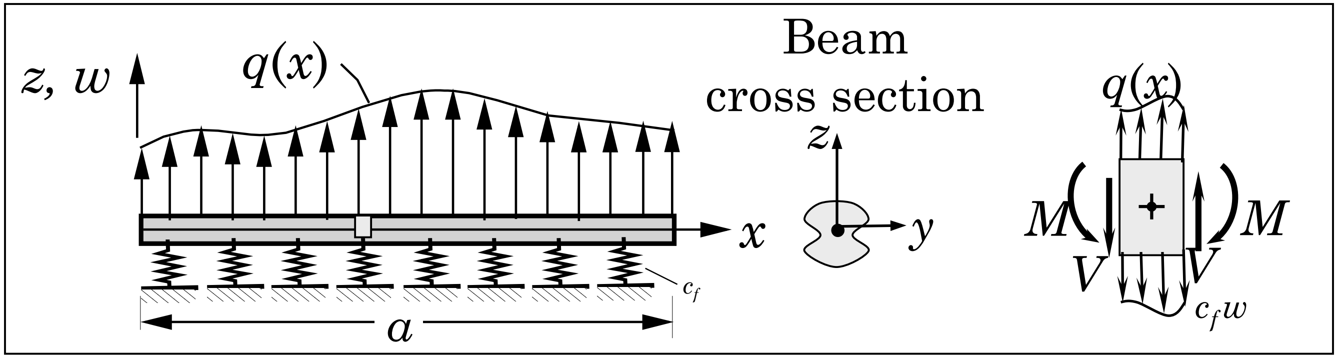
Notation for stress components

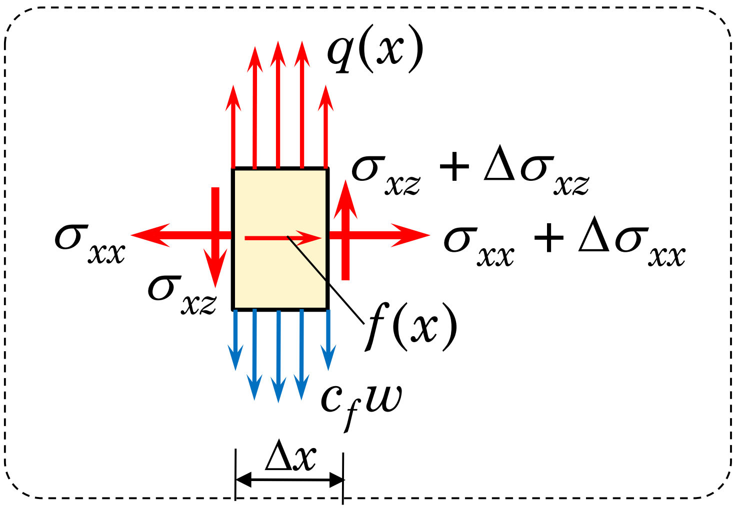
Constitutive relations

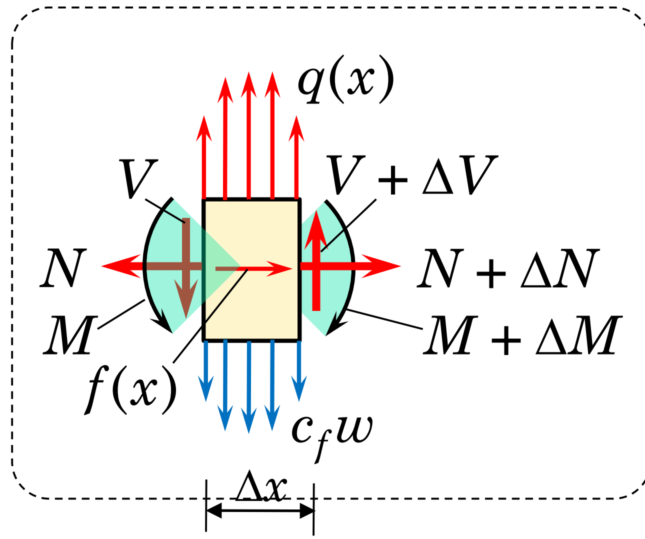
$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx} - Ez\frac{d^2w}{dx^2},$$

$$\sigma_{xz} = G\gamma_{xz} = 0$$

Euler-Bernoulli Beam Theory: Vector Approach





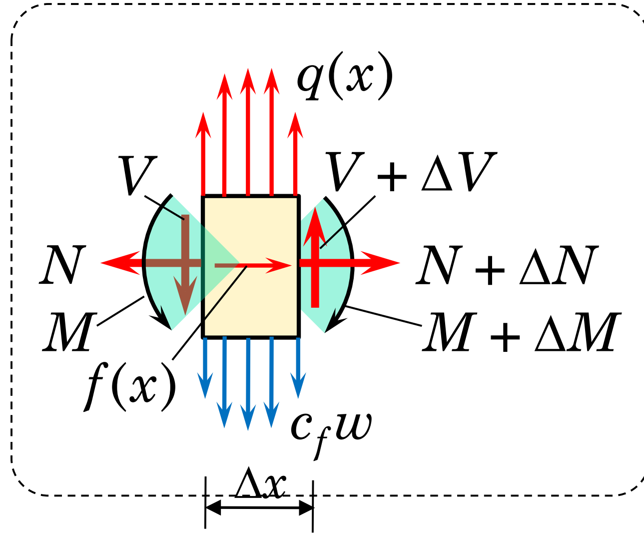


Definition of stress resultants

$$N = \int_A \sigma_{xx} \, dA, \quad M = \int_A \sigma_{xx} \cdot z \, dA, \quad V = \int_A \sigma_{xz} \, dA.$$

Euler-Bernoulli Beam Theory

Summation of forces in the x and z directions and moments about the y-axis.



$$\sum F_x = 0: \quad -N + (N+\Delta N) + f\Delta x = 0$$

$$\lim \Delta x \to 0 \quad \frac{\Delta N}{\Delta x} + f = 0 \quad \Rightarrow \boxed{\frac{dN}{dx} + f = 0}$$

$$\sum F_z = 0: \quad -V + (V+\Delta V) + q\Delta x - c_f w \Delta x = 0$$

$$\lim \Delta x \to 0 \quad \frac{\Delta V}{\Delta x} + q - c_f w = 0 \quad \Rightarrow \boxed{\frac{dV}{dx} + q - c_f w = 0}$$

$$\sum M_y = 0: \quad -V\Delta x - M + (M+\Delta M) + (q\Delta x)\alpha\Delta x - (c_f w\Delta x)\beta\Delta x = 0$$

$$\lim \Delta x \to 0 \quad -V + \frac{\Delta M}{\Delta x} + (q\Delta x)\alpha - (c_f w \Delta x)\beta = 0 \quad \Rightarrow \boxed{\frac{dM}{dx} - V = 0}$$

Euler-Bernoulli Beam Theory (Continued)

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + q - c_f w = 0$$

$$\int_A 1 \cdot dA = A, \qquad \int_A z \cdot dA = 0, \qquad \int_A z^2 \cdot dA = I$$

Stress resultants in terms of deflection

$$N = \int_A \sigma_{xx} \; dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) dA = EA\frac{du}{dx}$$

$$M = \int_A \sigma_{xx} \times z \; dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) z \; dA = -EI\frac{d^2w}{dx^2}$$

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2} \right)$$

Euler-Bernoulli Beam Theory (Continued)

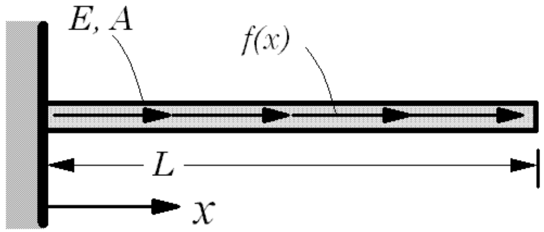
Governing equations in terms of the displacements

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-f=0,\quad 0<x<L$$

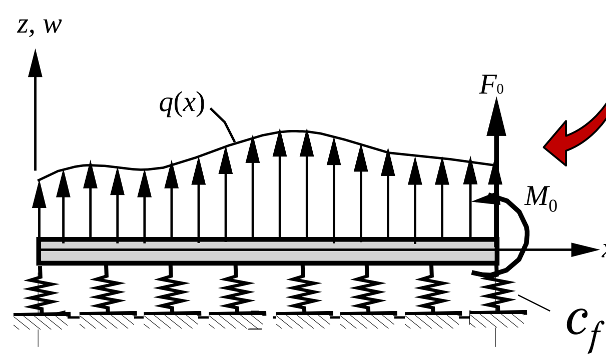
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q=0,\quad 0<x<L$$

Bars
u

Beams
w



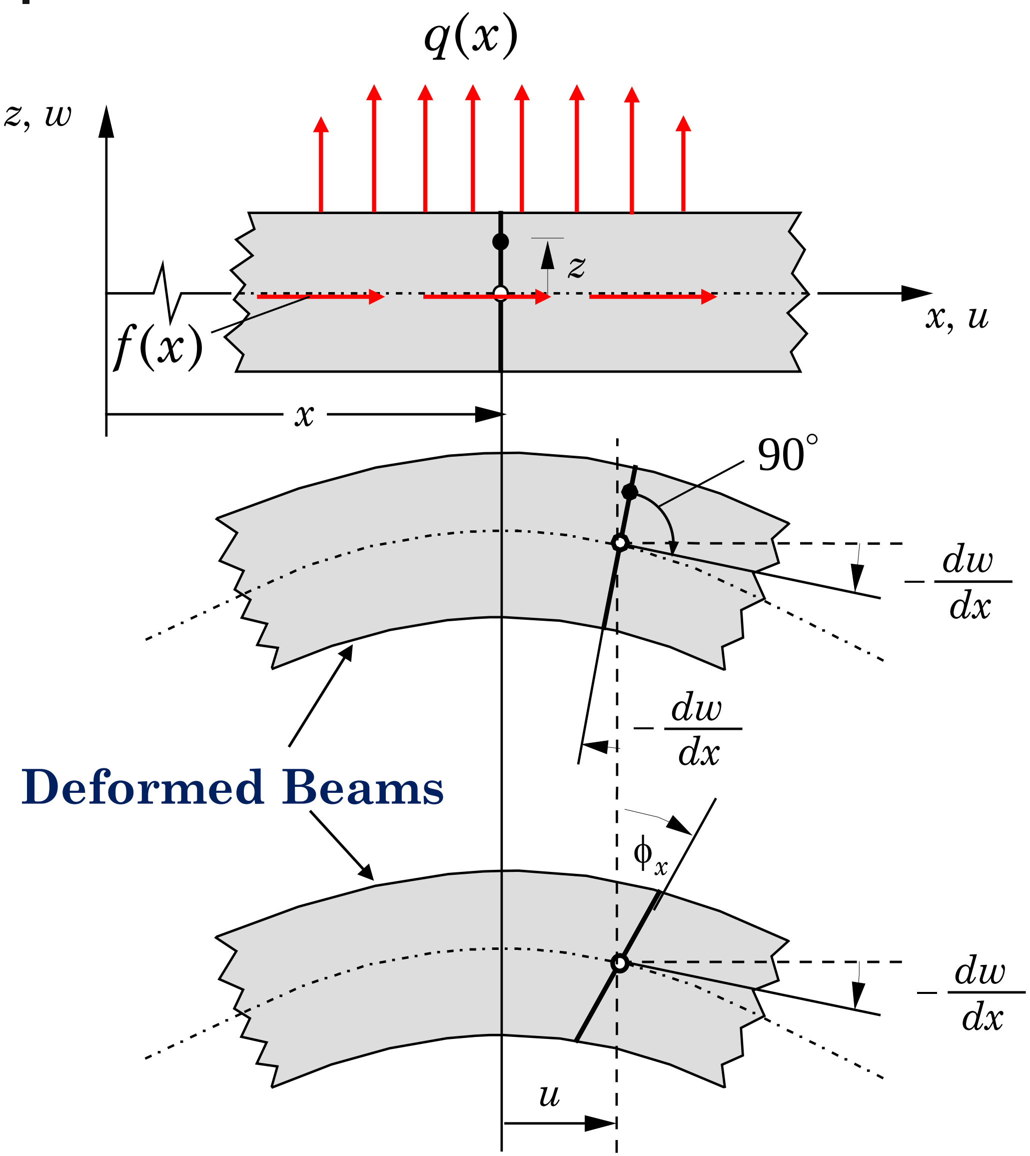
Axial deformation of a bar



Bending of a beam

Axial displacement is uncoupled from transverse displacement

Kinematics of Timoshenko Beam Theory



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)
Straightness and inextensibility

Timoshenko Beam Theory

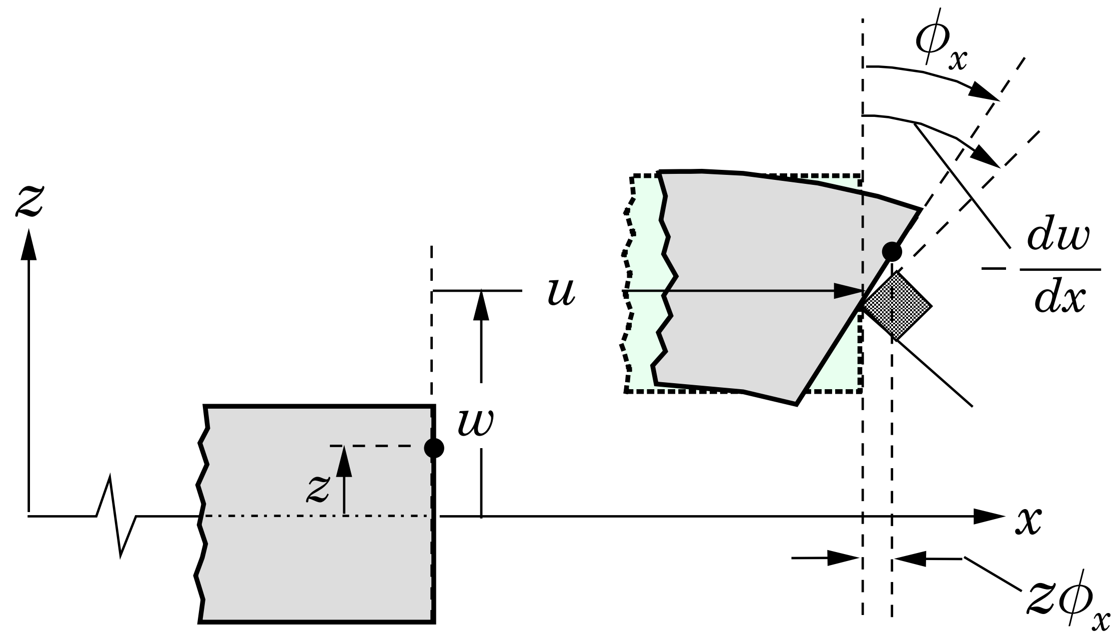
Kinematic Relations

$$u_1(x,z) = u(x) + z\phi_x(x),$$

$$u_2 = 0, \quad u_3(x,z) = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} + z\frac{d\phi_x}{dx},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \phi_x + \frac{dw}{dx}$$



Constitutive Equations

$$\sigma_{xx} = E\,\varepsilon_{xx} = E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right)$$

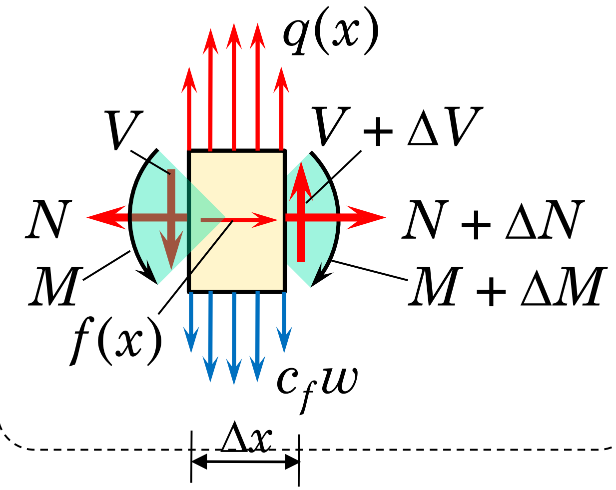
$$\sigma_{xz} = G\,\gamma_{xz} = G\left(\phi_x + \frac{dw}{dx}\right)$$

Timoshenko Beam Theory (Continued)

Equilibrium Equations (same as those from EBT)

$$\frac{dN}{dx} + f = 0, \quad -\frac{dV}{dx} - q + c_f w = 0,$$

$$-\frac{dM}{dx} + V = 0$$



Beam Constitutive Equations

$$N = \int_A \sigma_{xx}\, dA = \int_A E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right) dA = EA\frac{du}{dx}$$

$$M = \int_A \sigma_{xx} z\, dA = \int_A E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right) z\, dA = EI\frac{d\phi_x}{dx}$$

$$V = K_s \int_A \sigma_{xz}\, dA = GK_s\left(\phi_x + \frac{dw}{dx}\right)\int_A dA = GAK_s\left(\phi_x + \frac{dw}{dx}\right)$$

Timoshenko Beam Theory (Continued)

Governing Equations in terms of the displacements

$$-\frac{d}{dx}\left[GAK_s\left(\phi_x+\frac{dw}{dx}\right)\right]+c_f w=q \qquad (1)$$

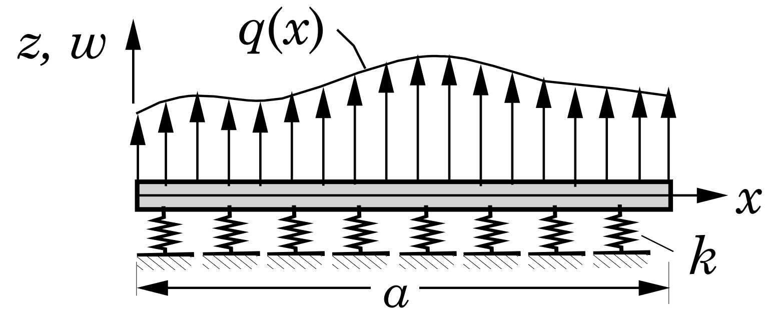
$$-\frac{d}{dx}\left(EI\frac{d\phi_x}{dx}\right)+GAK_s\left(\phi_x+\frac{dw}{dx}\right)=0 \qquad (2)$$

We have two second-order equations in two unknowns (w, ϕ_x). Next, we develop the weak forms over a typical beam finite element.

ANALYTICAL SOLUTIONS

Euler-Bernoulli beam theory (pure bending) – *EI* constant

$$EI\frac{d^4w}{dx^4} + kw - q = 0, \quad 0 < x < a$$



Boundary conditions:

$$(w, V) \text{ and } (\theta_x, M); \quad \theta_x = -\frac{dw}{dx}$$

General solution for uniform load:

$$w(x) = \left(c_1 \sinh\frac{2\beta x}{a} + c_2 \cosh\frac{2\beta x}{a}\right)\sin\frac{2\beta x}{a} + \left(c_3 \sinh\frac{2\beta x}{a} + c_4 \cosh\frac{2\beta x}{a}\right)\cos\frac{2\beta x}{a} + \frac{q_0}{k}$$

$$\beta = \frac{ka^4}{64EI}$$

c_1 thru c_4 are constants to be determined using boundary conditions.

ANALYTICAL SOLUTIONS (continued)

Simply supported beam: Using symmetry and half beam,

$$\frac{dw}{dx} = \frac{d^3w}{dx^3} = 0 \ \text{ at } x = 0; \quad w = \frac{d^2w}{dx^2} = 0 \text{ at } x = \frac{a}{2}$$

We obtain $c_2 = c_3 = 0$, and

$$c_1 \sin\beta \sinh\beta + c_4 \cos\beta \cosh\beta + \frac{q_0}{k} = 0,$$
$$c_1 \cos\beta \cosh\beta + c_4 \sin\beta \sinh\beta = 0.$$

Solving these equations, we obtain

$$c_1 = -\frac{2q_0}{k}\left(\frac{\sin\beta \sinh\beta}{\cos 2\beta + \cosh 2\beta}\right), \quad c_4 = -\frac{2q_0}{k}\left(\frac{\cos\beta \cosh\beta}{\cos 2\beta + \cosh 2\beta}\right)$$

Similarly, one can obtain solutions for other boundary conditions.

ANALYTICAL SOLUTIONS (by integration)

Solution with $k = 0$: Successive integrations yield

$$\frac{d}{dx}\Big(EI\frac{d^2w}{dx^2}\Big) = \int q(x)\,dx + C_1,$$

$$EI\frac{d^2w}{dx^2} = \int\Big(\int q(x)\,dx\Big)dx + C_1x + C_2,$$

$$\frac{dw}{dx} = \int\frac{1}{EI}\Big[\int\Big(\int q(x)\,dx\Big)dx\Big]dx + C_1\int\frac{x}{EI}dx + C_2\int\frac{1}{EI}dx + C_3,$$

$$w = \int\Big\{\int\frac{1}{EI}\Big[\int\Big(\int q(x)dx\Big)dx\Big]dx\Big\}dx + C_1\int\Big(\int\frac{x}{EI}dx\Big)dx$$

$$+ C_2\int\Big(\int\frac{1}{EI}dx\Big)dx + C_3x + C_4.$$

The constants of integration can be determined using the boundary conditions

ANALYTICAL SOLUTIONS (by integration)

Simply supported beam under uniform load:

The boundary conditions on the moment give

$$M(0) = M(a) = 0 \Rightarrow C_2 = 0, C_1 = -\frac{q_0 a}{2},$$

$$w(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 a x^3}{12EI} + C_3 x + C_4.$$

$$w(0) = 0 \quad w(a) = 0 \Rightarrow C_4 = 0, C_3 = (q_0 a^3 / 24EI).$$

The solutions for the bending moment and deflection become

$$M(x) = -EI\frac{d^2 w}{dx^2} = \frac{q_0}{2}x(a - x), \;\; 0 < x < a.$$

$$w(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 a x^3}{12EI} + \frac{q_0 x a^3}{24EI} = \frac{q_0}{24EI}(x^4 - 2x^3 a + a^3 x).$$

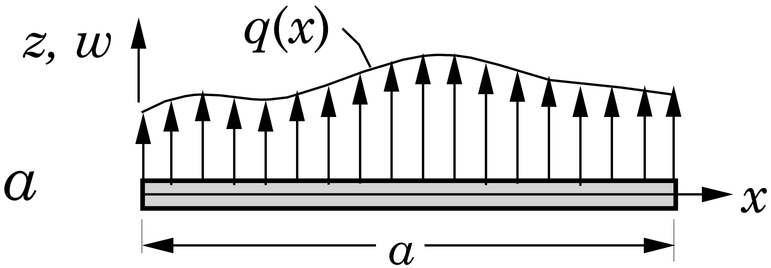
The maximum values are

$$w_{\max} = w(0.5a) = \frac{5 q_0 a^4}{384EI}; \quad M(0.5a) = \frac{q_0 a^2}{8}$$

NAVIER SOLUTIONS
for simply supported beams

Euler-Bernoulli beam theory (pure bending) – *EI* constant

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+kw-q=0,\quad 0<x<a$$



Simply supported: $w=0,\ M\equiv -EI\dfrac{d^2w}{dx^2}=0\ \text{ at }\ x=0,a$

Solution form: $w=\displaystyle\sum_{n=1}^{\infty}W_n\sin\alpha_n x,\ \ \alpha_n=\frac{n\pi}{a}$

$$\frac{d^2w}{dx^2}=-\sum_{n=1}^{\infty}\alpha_n^2W_n\sin\alpha_n x$$

Navier's Solution of Simply supported Beams

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+kw-q=0$$

$$\Rightarrow\sum_{n=1}^{\infty}W_n\left[(-1)^2 EI\alpha_n^4+k\right]\sin\alpha_n x=q(x)$$

$$q(x)=\sum_{n=1}^{\infty}Q_n\sin\alpha_n x,\ Q_n=\frac{2}{a}\int_0^a q(x)\sin\alpha_n x\ dx$$

$$\sum_{n=1}^{\infty}\left\{W_n\left[(-1)^2 EI\alpha_n^4+k\right]-Q_n\right\}\sin\alpha_n x=0$$

$$W_n=\frac{Q_n}{\left[(-1)^2 EI\alpha_n^4+k\right]}$$

$$w=\sum_{n=1}^{\infty}W_n\sin\alpha_n x,\quad \alpha_n=\frac{n\pi}{a}$$