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KINEMATICS OF THE LINEARIZED EULER-BERNOULLI BEAM THEORY

Undeformed Beam

Euler-Bernoulli Beam Theory (EBT) is based on the assumptions of:

1. straightness,
2. inextensibility, and
3. normality

Deformed Beam

Strains, displacements, and rotations are small
Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)

Displacement field (constructed using the hypothesis)

\[ u_1(x, z) = u - z \frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x) \]

Linear strains

\[ \varepsilon_{11} = \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} - z \frac{d^2 w}{dx^2}, \]
\[ \gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\frac{dw}{dx} + \frac{dw}{dx} = 0. \]

Constitutive relations

\[ \sigma_{xx} = E \varepsilon_{xx} = E \frac{du}{dx} - Ez \frac{d^2 w}{dx^2}, \]
\[ \sigma_{xz} = G \gamma_{xz} = 0 \]

Notation for stress components
Euler-Bernoulli Beam Theory: Vector Approach

Definition of stress resultants

\[ N = \int \sigma_{xx} \, dA, \quad M = \int \sigma_{xx} \cdot z \, dA, \quad V = \int \sigma_{xz} \, dA. \]
Euler-Bernoulli Beam Theory

Summation of forces in the $x$ and $z$ directions and moments about the $y$-axis.

\[ \sum F_x = 0 : \quad -N + (N + \Delta N) + f \Delta x = 0 \]

\[ \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} + f = 0 \quad \Rightarrow \quad \frac{dN}{dx} + f = 0 \]

\[ \sum F_z = 0 : \quad -V + (V + \Delta V) + q \Delta x - c_f w \Delta x = 0 \]

\[ \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} + q - c_f w = 0 \quad \Rightarrow \quad \frac{dV}{dx} + q - c_f w = 0 \]

\[ \sum M_y = 0 : \quad -V \Delta x - M + (M + \Delta M) + (q \Delta x) \alpha \Delta x - \left( c_f w \Delta x \right) \beta \Delta x = 0 \]

\[ \lim_{\Delta x \to 0} -V + \frac{\Delta M}{\Delta x} + (q \Delta x) \alpha - \left( c_f w \Delta x \right) \beta = 0 \quad \Rightarrow \quad \frac{dM}{dx} - V = 0 \]
Euler-Bernoulli Beam Theory (Continued)

Equilibrium equations

\[
\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + q - c_f w = 0
\]

\[
\int 1 \cdot dA = A, \quad \int z \cdot dA = 0, \quad \int z^2 \cdot dA = I
\]

Stress resultants in terms of deflection

\[
N = \int_A \sigma_{xx} \ dA = \int_A \left( E \frac{du}{dx} - Ez \frac{d^2w}{dx^2} \right) \ dA = EA \frac{du}{dx}
\]

\[
M = \int_A \sigma_{xx} \times z \ dA = \int_A \left( E \frac{du}{dx} - Ez \frac{d^2w}{dx^2} \right) z \ dA = -EI \frac{d^2w}{dx^2}
\]

\[
V = \frac{dM}{dx} = \frac{d}{dx} \left( -EI \frac{d^2w}{dx^2} \right)
\]
Euler-Bernoulli Beam Theory (Continued)

Governing equations in terms of the displacements

For Bars

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2 w}{dx^2} \right) + c_f w - q = 0, \quad 0 < x < L \]

For Beams

\[- \frac{d}{dx} \left( EA \frac{du}{dx} \right) - f = 0, \quad 0 < x < L \]

Axial deformation of a bar

Bending of a beam

Axial displacement is uncoupled from transverse displacement
Kinematics of Timoshenko Beam Theory

Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)  
*Straightness, inextensibility, and normality*

Timoshenko Beam Theory (TBT)  
*Straightness and inextensibility*
Timoshenko Beam Theory

Kinematic Relations

\[ u_1(x,z) = u(x) + z\phi_x(x), \]
\[ u_2 = 0, \quad u_3(x,z) = w(x) \]
\[ \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} + z \frac{d\phi_x}{dx}, \]
\[ \gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \phi_x + \frac{dw}{dx} \]

Constitutive Equations

\[ \sigma_{xx} = E \varepsilon_{xx} = E \left( \frac{du}{dx} + z \frac{d\phi_x}{dx} \right) \]
\[ \sigma_{xz} = G \gamma_{xz} = G \left( \phi_x + \frac{dw}{dx} \right) \]
Timoshenko Beam Theory (Continued)

Equilibrium Equations (same as those from EBT)

\[
\frac{dN}{dx} + f = 0, \quad -\frac{dV}{dx} - q + c_f w = 0,
\]

\[-\frac{dM}{dx} + V = 0\]

Beam Constitutive Equations

\[
N = \int_A \sigma_{xx} \, dA = \int_A E \left( \frac{du}{dx} + z \frac{d\phi_x}{dx} \right) \, dA = EA \frac{du}{dx}
\]

\[
M = \int_A \sigma_{xx}z \, dA = \int_A E \left( \frac{du}{dx} + z \frac{d\phi_x}{dx} \right)z \, dA = EI \frac{d\phi_x}{dx}
\]

\[
V = K_s \int_A \sigma_{xz} \, dA = GK_s \left( \phi_x + \frac{dw}{dx} \right) \int_A \, dA = GAK_s \left( \phi_x + \frac{dw}{dx} \right)
\]
Governing Equations in terms of the displacements

\[
- \frac{d}{dx} \left[ GAK_s \left( \phi_x + \frac{dw}{dx} \right) \right] + c_f w = q \quad (1)
\]

\[
- \frac{d}{dx} \left( EI \frac{d\phi_x}{dx} \right) + GAK_s \left( \phi_x + \frac{dw}{dx} \right) = 0 \quad (2)
\]

We have two second-order equations in two unknowns \((w, \phi_x)\). Next, we develop the weak forms over a typical beam finite element.
Euler-Bernoulli beam theory (pure bending) – $EI$ constant

$$EI \frac{d^4w}{dx^4} + kw - q = 0, \quad 0 < x < a$$

Boundary conditions:

$$(w, V) \text{ and } (\theta_x, M); \quad \theta_x = -\frac{dw}{dx}$$

General solution for uniform load:

$$w(x) = \left( c_1 \sinh \frac{2\beta x}{a} + c_2 \cosh \frac{2\beta x}{a} \right) \sin \frac{2\beta x}{a}$$

$$+ \left( c_3 \sinh \frac{2\beta x}{a} + c_4 \cosh \frac{2\beta x}{a} \right) \cos \frac{2\beta x}{a} + \frac{q_0}{k}$$

$$\beta = \frac{ka^4}{64EI}$$

$c_1$ thru $c_4$ are constants to be determined using boundary conditions.
Simply supported beam: Using symmetry and half beam,

\[ \frac{d^2w}{dx^2} = \frac{d^3w}{dx^3} = 0 \text{ at } x = 0; \quad w = \frac{d^2w}{dx^2} = 0 \text{ at } x = \frac{a}{2} \]

We obtain \( c_2 = c_3 = 0 \), and

\[ c_1 \sin \beta \sinh \beta + c_4 \cos \beta \cosh \beta + \frac{q_0}{k} = 0, \]

\[ c_1 \cos \beta \cosh \beta + c_4 \sin \beta \sinh \beta = 0. \]

Solving these equations, we obtain

\[ c_1 = -\frac{2q_0}{k} \left( \frac{\sin \beta \sinh \beta}{\cos 2\beta + \cosh 2\beta} \right), \quad c_4 = -\frac{2q_0}{k} \left( \frac{\cos \beta \cosh \beta}{\cos 2\beta + \cosh 2\beta} \right) \]

Similarly, one can obtain solutions for other boundary conditions.
Solution with $k = 0$: Successive integrations yield

\[
\frac{d}{dx} \left( EI \frac{d^2 w}{dx^2} \right) = \int q(x) \, dx + C_1,
\]

\[
EI \frac{d^2 w}{dx^2} = \int \left( \int q(x) \, dx \right) \, dx + C_1 x + C_2,
\]

\[
\frac{dw}{dx} = \int \frac{1}{EI} \left[ \int \left( \int q(x) \, dx \right) \, dx \right] \, dx + C_1 \int \frac{x}{EI} \, dx + C_2 \int \frac{1}{EI} \, dx + C_3,
\]

\[
w = \int \left\{ \int \frac{1}{EI} \left[ \int \left( \int q(x) \, dx \right) \, dx \right] \, dx \right\} \, dx + C_1 \int \left( \int \frac{x}{EI} \, dx \right) \, dx
\]

\[
+ C_2 \int \left( \int \frac{1}{EI} \, dx \right) \, dx + C_3 x + C_4.
\]

The constants of integration can be determined using the boundary conditions.
The boundary conditions on the moment give

\[ M(0) = M(\alpha) = 0 \Rightarrow C_2 = 0, C_1 = -\frac{q_0\alpha}{2}, \]

\[ w(x) = \frac{q_0x^4}{24EI} - \frac{q_0\alpha x^3}{12EI} + C_3x + C_4. \]

\[ w(0) = 0 \quad w(\alpha) = 0 \Rightarrow C_4 = 0, C_3 = (q_0\alpha^3 / 24EI). \]

The solutions for the bending moment and deflection become

\[ M(x) = -EI \frac{d^2w}{dx^2} = \frac{q_0}{2} x(\alpha - x), \quad 0 < x < \alpha. \]

\[ w(x) = \frac{q_0x^4}{24EI} - \frac{q_0\alpha x^3}{12EI} + \frac{q_0\alpha^3}{24EI} = \frac{q_0}{24EI} (x^4 - 2x^3\alpha + \alpha^3x). \]

The maximum values are

\[ w_{\text{max}} = w(0.5\alpha) = \frac{5q_0\alpha^4}{384EI}; \quad M(0.5\alpha) = \frac{q_0\alpha^2}{8}. \]
NAVIER SOLUTIONS
for simply supported beams

Euler-Bernoulli beam theory (pure bending) – $EI$ constant

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) + kw - q = 0, \quad 0 < x < \alpha \]

Simply supported: \( w = 0, \quad M = -EI \frac{d^2w}{dx^2} = 0 \) at \( x = 0, \alpha \)

Solution form: \( w = \sum_{n=1}^{\infty} W_n \sin \alpha_n x, \quad \alpha_n = \frac{n\pi}{\alpha} \)

\[ \frac{d^2w}{dx^2} = -\sum_{n=1}^{\infty} \alpha_n^2 W_n \sin \alpha_n x \]
Navier’s Solution of Simply supported Beams

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 w}{dx^2} \right) + kw - q = 0
\]

\[
\Rightarrow \sum_{n=1}^{\infty} W_n \left[ (-1)^2 EI \alpha_n^4 + k \right] \sin \alpha_n x = q(x)
\]

\[
q(x) = \sum_{n=1}^{\infty} Q_n \sin \alpha_n x, \quad Q_n = \frac{2}{a} \int_0^a q(x) \sin \alpha_n x \, dx
\]

\[
\sum_{n=1}^{\infty} \left\{ W_n \left[ (-1)^2 EI \alpha_n^4 + k \right] - Q_n \right\} \sin \alpha_n x = 0
\]

\[
W_n = \frac{Q_n}{\left[ (-1)^2 EI \alpha_n^4 + k \right]}
\]

\[
\omega = \sum_{n=1}^{\infty} W_n \sin \alpha_n x, \quad \alpha_n = \frac{n\pi}{a}
\]