A REVIEW OF THE EQUATIONS OF MECHANICS

CONTENTS

- Continuum assumption
- Kinematics of deformation
- Kinetics: Stress vector
- Cauchy's formula
- Balance of linear momentum
- Balance of angular momentum
- Conservation of energy
- Work and energy
- Strain energy and complementary strain energy
- Virtual Work

CONTINUUM ASSUMPTION AND CLASSIFICATION OF EQUATIONS OF MECHANICS

Continuum assumption

$$\rho(\mathbf{x}, t) \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$

The study of motion and deformation of a continuum can be broadly classified into four basic categories:

(1) **Kinematics** (strain-displacement equations)
(2) **Kinetics** (balance of linear and angular momentum)
(3) **Thermodynamics** (first and second laws of thermodynamics)
(4) **Constitutive equations** (stress-strain relations)

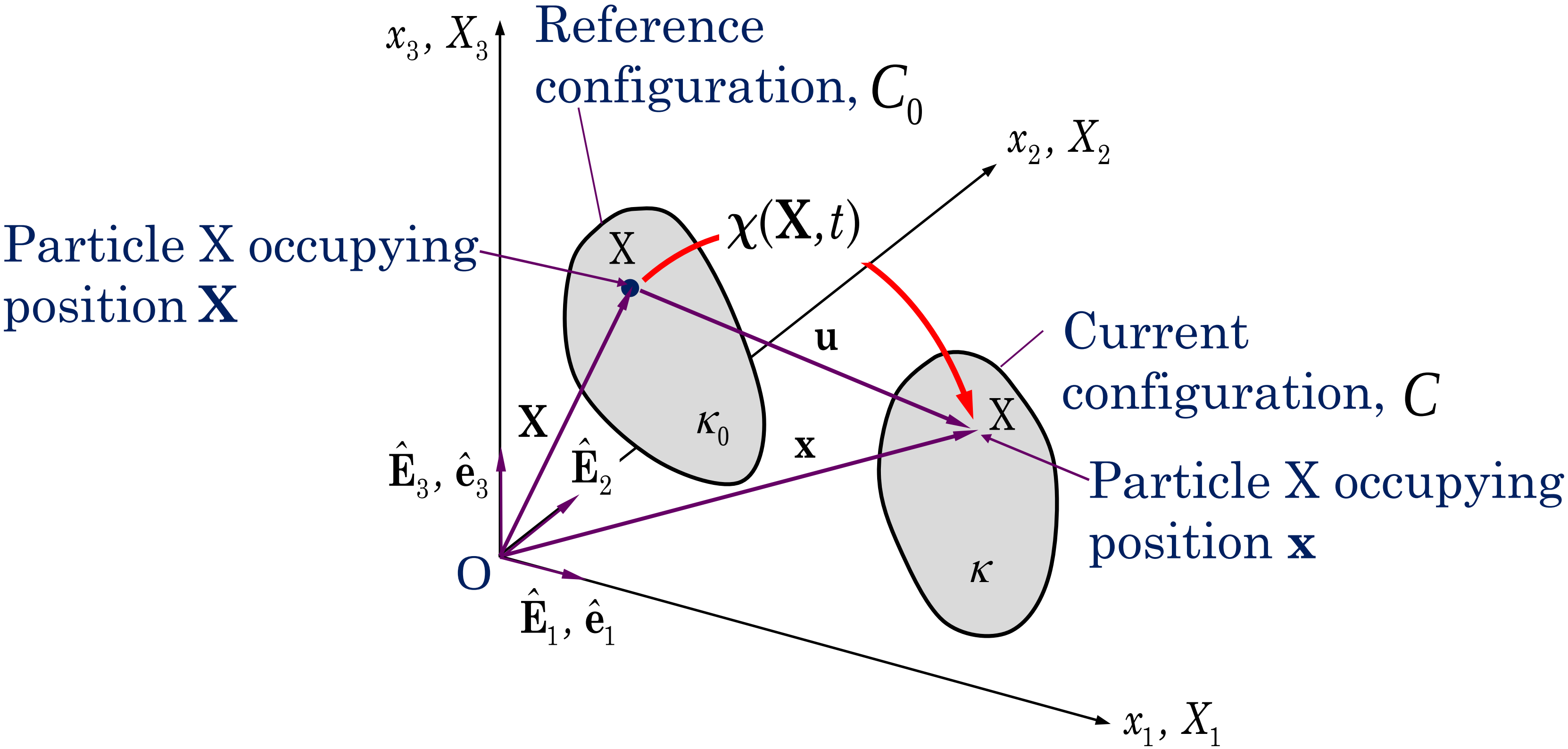
KINEMATICS OF SOLIDS

Material Description: $\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbf{X} = \chi(\mathbf{X}, 0)$

(X_1, X_2, X_3) are the *material coordinates*.

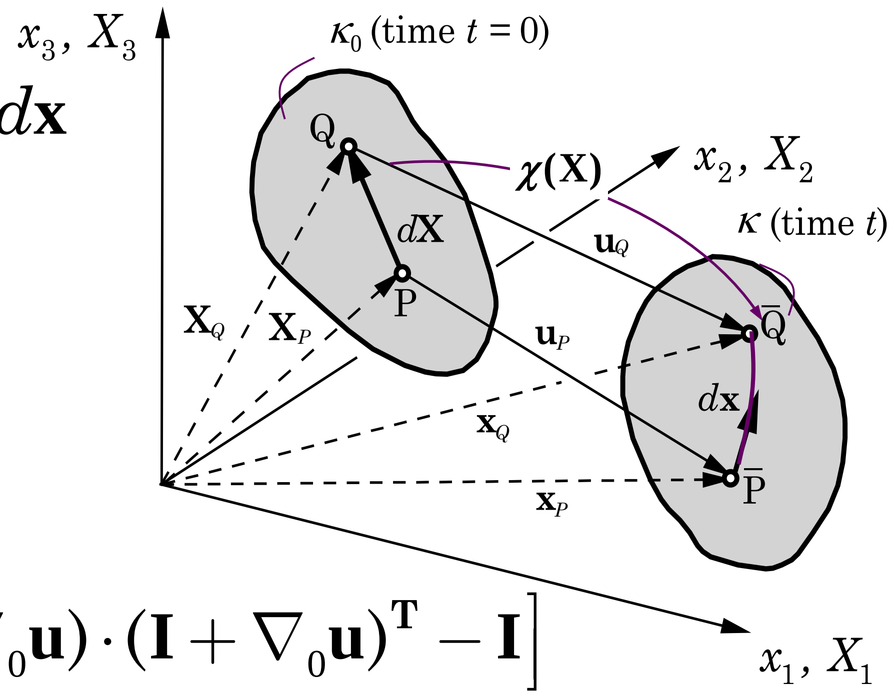
(x_1, x_2, x_3) are the *spatial* (current) *coordinates*.

Displacement vector: $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$



GREEN-LAGRANGE STRAIN TENSOR

$$(dS)^2 = d\mathbf{X}\cdot d\mathbf{X},\ (ds)^2 = d\mathbf{x}\cdot d\mathbf{x}$$



Define the Green-Lagrange strain tensor **E** as

$$(ds)^2-(dS)^2 \equiv 2d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X}$$

where **E** is

$$\mathbf{E} = \tfrac{1}{2}\left(\frac{d\mathbf{x}}{d\mathbf{X}}\cdot\frac{d\mathbf{x}}{d\mathbf{X}} - \mathbf{I}\right) = \tfrac{1}{2}\left[(\mathbf{I}+\nabla_0\mathbf{u})\cdot(\mathbf{I}+\nabla_0\mathbf{u})^{\mathbf{T}} - \mathbf{I}\right]$$

$$= \tfrac{1}{2}\left[\nabla_0\mathbf{u} + (\nabla_0\mathbf{u})^{\mathbf{T}} + (\nabla_0\mathbf{u})\cdot(\nabla_0\mathbf{u})^{\mathbf{T}}\right]$$

The rectangular Cartesian component form of **E** is

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right)$$

GREEN-LAGRANGE STRAIN TENSOR

The rectangular Cartesian components in explicit form are given by

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right],$$

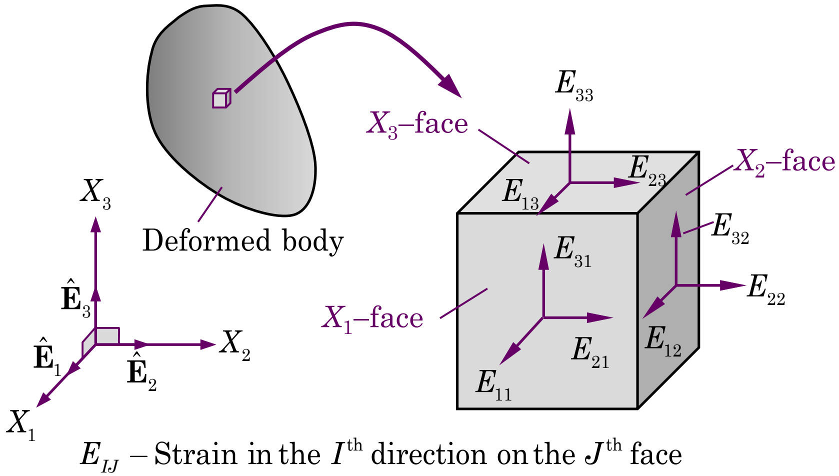
$$E_{22} = \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right]$$

$$E_{33} = \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right]$$

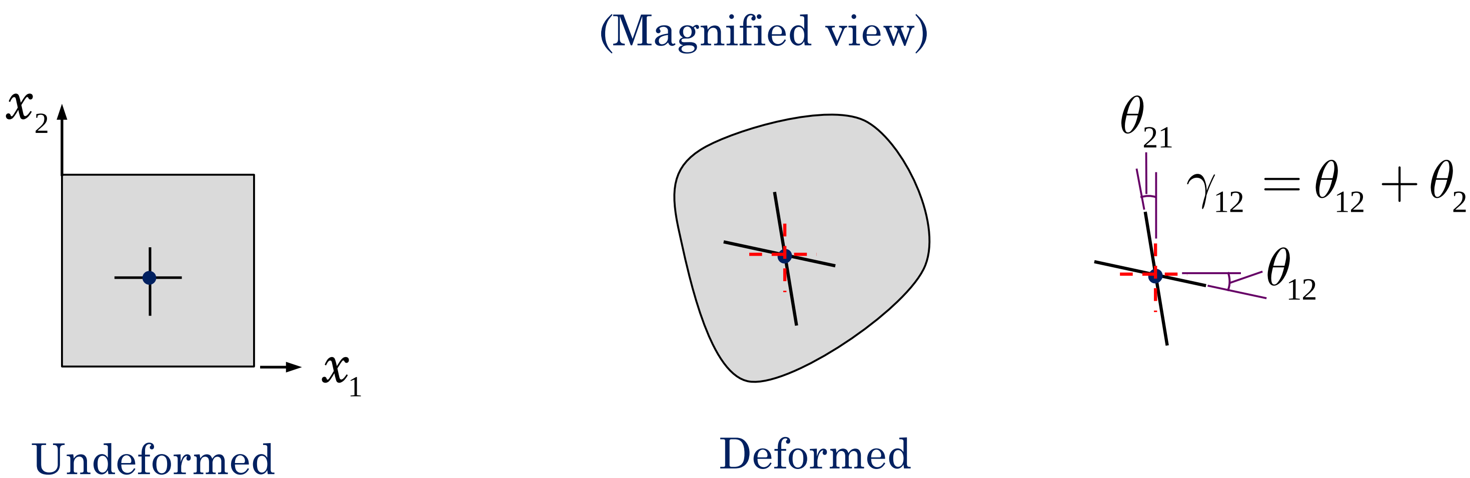
$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right)$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3}\right)$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right)$$



GEOMETRIC INTERPRETATION OF THE STRAIN TENSOR COMPONENTS



$$E_{11} = \text{Change in length of the line element in the } x_1 \text{ ditection divided by the original length}$$

$$2E_{12} = \gamma_{12} = \text{Total change from } \frac{\pi}{2}$$

INFITESIMAL STRAIN TENSOR

If **E** is of the order $O(\epsilon)$ in $\nabla_0 \mathbf{u}$, then we mean

$$\frac{\partial u_I}{\partial X_J} = O(\epsilon) \text{ as } \epsilon \to 0$$

If terms of the order $O(\epsilon^2)$ can be omitted, then

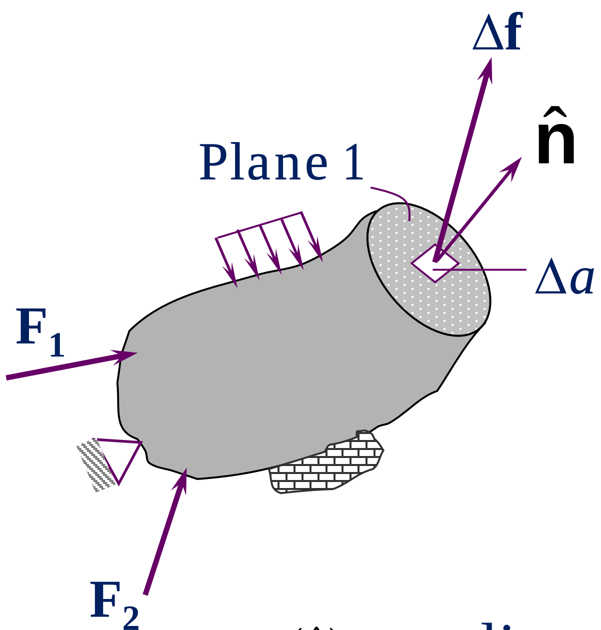
$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right)$$

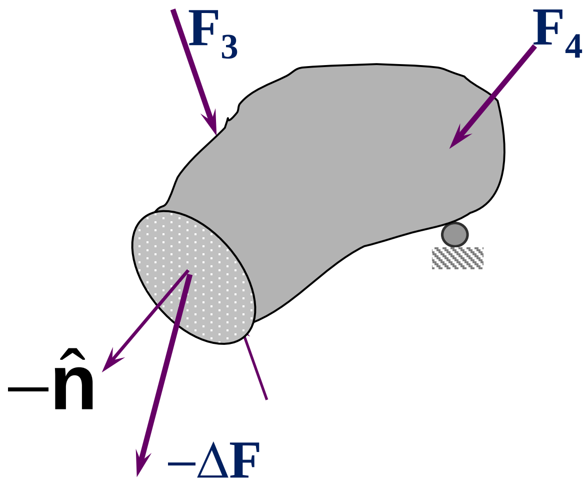
can be approximated as

$$E_{IJ} \approx \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I}\right) = O(\epsilon) \text{ as } \epsilon^2 \to 0$$

$$\mathbf{E} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^{\mathbf{T}}\right],$$ the infinitesimal strain tensor

STRESS VECTOR



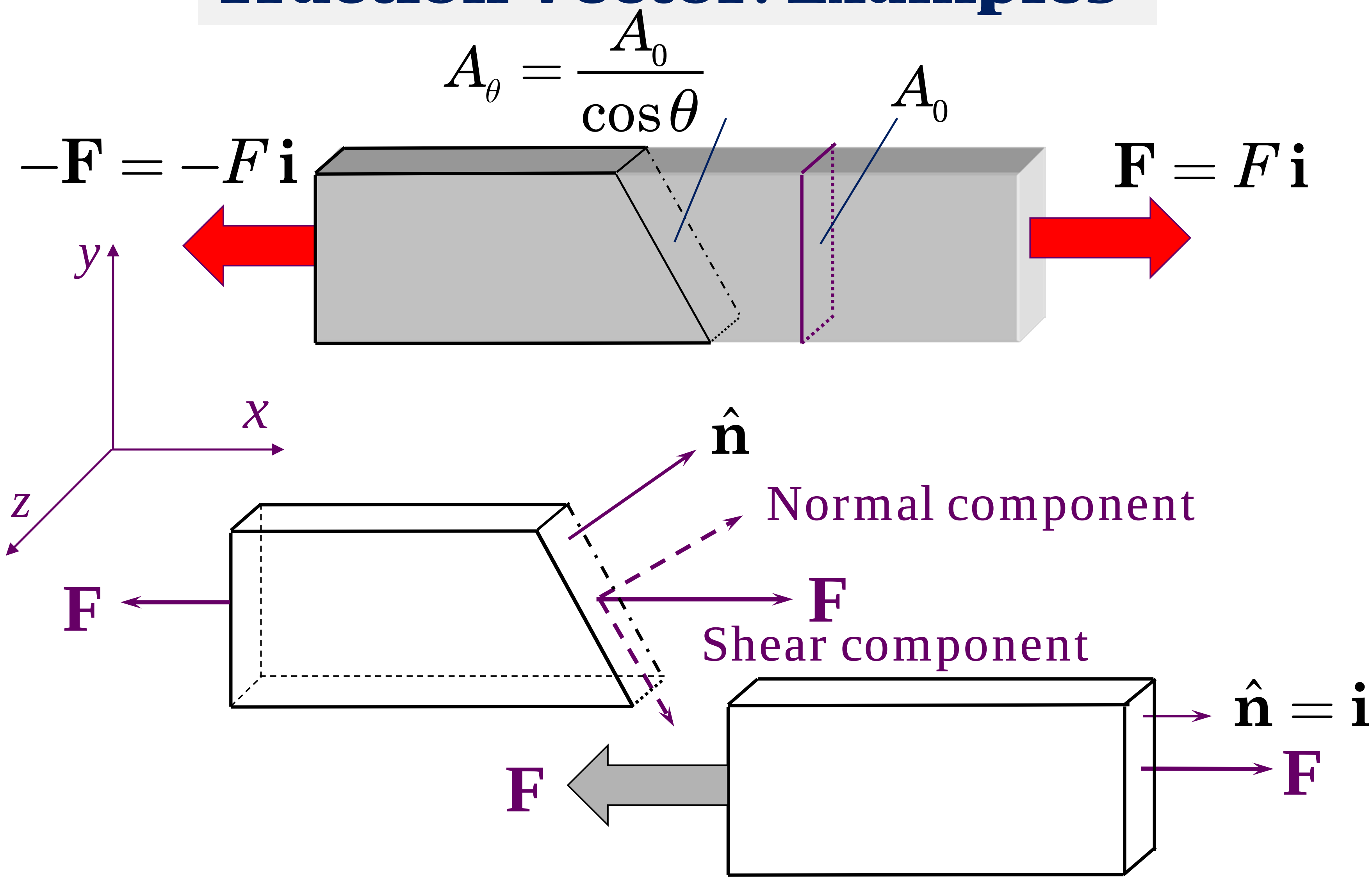


$$\mathbf{t}^{(\hat{\mathbf{n}})} = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}}{\Delta a} \qquad \mathbf{t}^{(-\hat{\mathbf{n}})} = \lim_{\Delta a \to 0} \frac{-\Delta \mathbf{f}}{\Delta a}$$

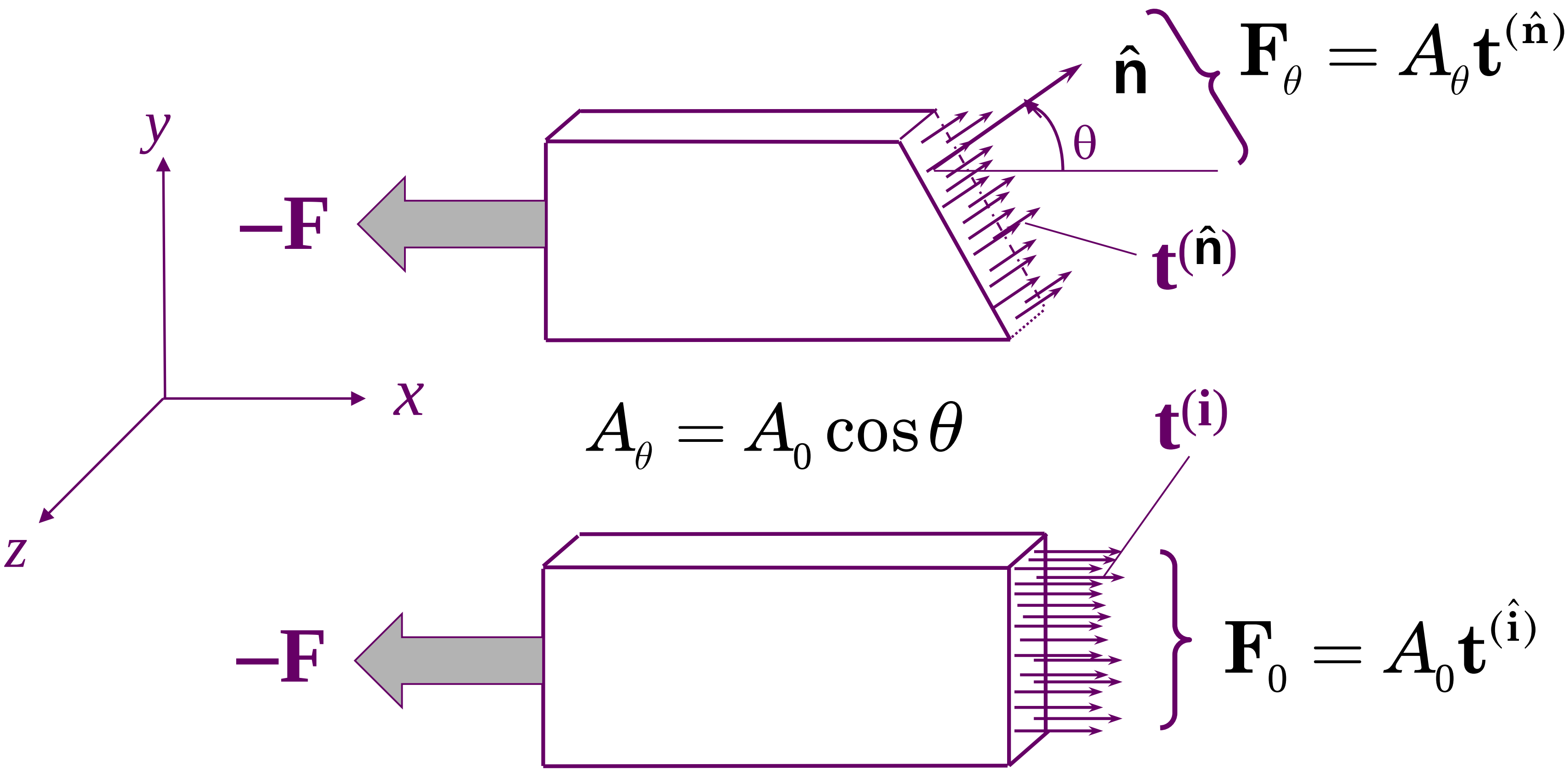
$$\mathbf{t}^{(\hat{\mathbf{n}})} = -\mathbf{t}^{(-\hat{\mathbf{n}})}$$ or $$\mathbf{t}^{(-\hat{\mathbf{n}})} = -\mathbf{t}^{(\hat{\mathbf{n}})}$$

Cauchy's Lemma

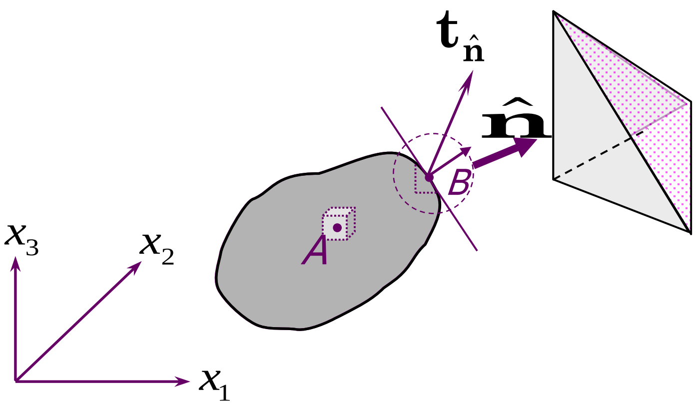
Traction Vector: Examples

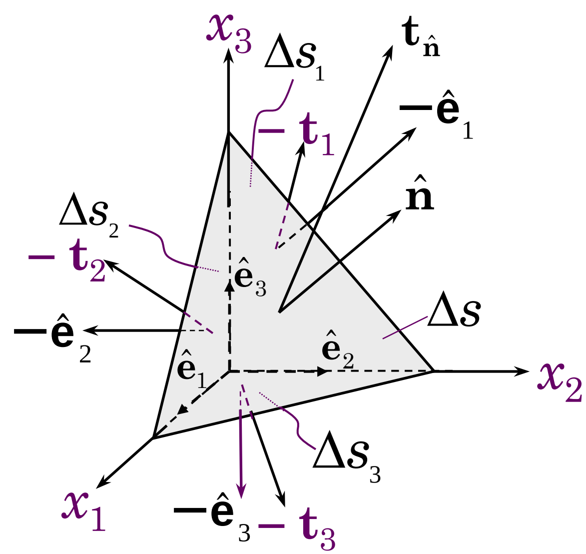


Traction Vector: Examples



CAUCHY'S FORMULA AND STRESS TENSOR





$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a}-\mathbf{f})$$

As $\Delta h \to 0$, we obtain

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i)\mathbf{t}_i \equiv \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$$

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}}\cdot\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\sigma} = \hat{\mathbf{e}}_i\mathbf{t}_i \quad \left[t_i^{(\hat{\mathbf{n}})} = n_j\sigma_{ji},\ \mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_j\right]$$

plane direction

AN EXAMPLE

Problem: Given the following stress tensor components in Cartesian coordinates

$$[\sigma] = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} (\text{MPa})$$

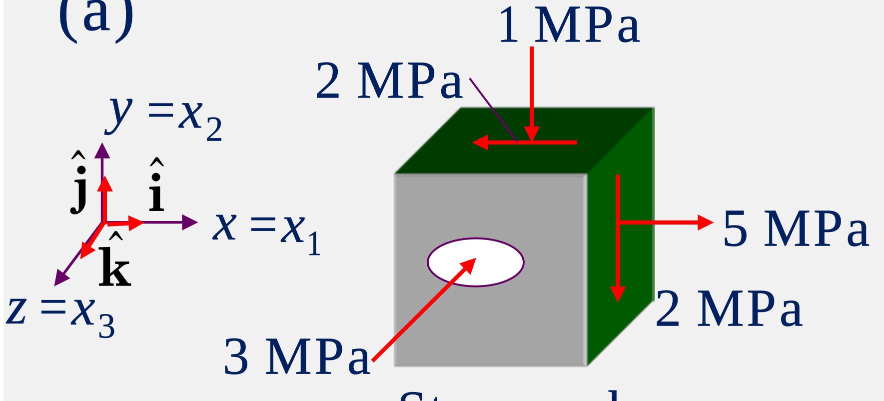
(a) Show the stress components on the stress cube.

(b) Determine the traction vectors $\mathbf{t}^{(\hat{\mathbf{i}})}$, $\mathbf{t}^{(\hat{\mathbf{j}})}$, and $\mathbf{t}^{(\hat{\mathbf{k}})}$

(c) Sketch the traction vectors on the stress cube.

Solution: We have

(a)



Stress cube

(b)

$$\mathbf{t}^{(\hat{\mathbf{i}})} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

$$\mathbf{t}^{(\hat{\mathbf{j}})} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}}$$

$$\mathbf{t}^{(\hat{\mathbf{k}})} = -3\hat{\mathbf{k}}$$

Example (continued)

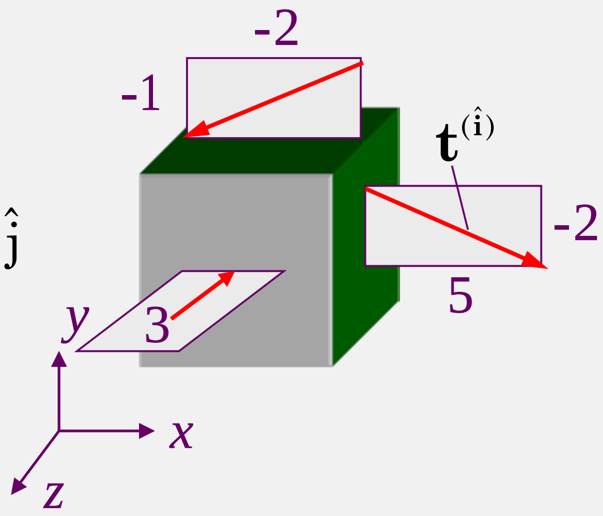
(b) Solution by use of Cauchy's formula

$$[\sigma]=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}(\text{MPa}),\qquad \begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}n_x\\ n_y\\ n_z\end{Bmatrix}$$

(c)

(i) When $\hat{\mathbf{n}}=\hat{\mathbf{i}}\ \ (n_x=1,\ n_y=0,\ n_z=0)$

$$\begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}^{(\hat{\mathrm{i}})}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}1\\ 0\\ 0\end{Bmatrix}=\begin{Bmatrix}5\\ -2\\ 0\end{Bmatrix}\Rightarrow \mathrm{t}^{(\hat{\mathrm{i}})}=5\hat{\mathrm{i}}-2\hat{\mathrm{j}}$$



(ii) When $\hat{\mathbf{n}}=\hat{\mathbf{j}}\ \ (n_x=0,\ n_y=1,\ n_z=0)$

$$\begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}^{(\hat{\mathrm{j}})}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}0\\ 1\\ 0\end{Bmatrix}=\begin{Bmatrix}-2\\ -1\\ 0\end{Bmatrix}\Rightarrow \mathrm{t}^{(\hat{\mathrm{j}})}=-2\hat{\mathrm{i}}-\hat{\mathrm{j}}$$

(iii) When $\hat{\mathbf{n}}=\hat{\mathbf{k}}\ \ (n_x=0,\ n_y=0,\ n_z=1)\Rightarrow \mathbf{t}^{(\hat{\mathbf{k}})}=-3\hat{\mathbf{k}}$

AN EXAMPLE

Problem statement:

With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ psi.}$$

Determine stress vector and its normal and tangential components at the point on the plane

$$\phi(x_1, x_2, x_3) \equiv x_1 + 2x_2 + 2x_3 = \text{constant}$$

which is planning through the point.

EXAMPLE (continued)

Solution:

First, we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal to the plane is

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)$$

The components of the stress vector are

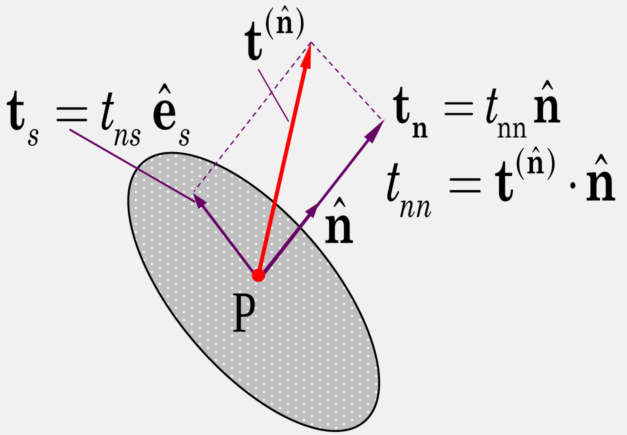
$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi}$$

Solution (continued):

The traction vector normal to the plane is given by

$$\mathbf{t}_{nn} = (\mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \frac{2600}{9}\hat{\mathbf{n}}$$

$$= \frac{2600}{27}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \ \text{psi}$$



and the traction vector projected onto the plane (i.e., shear traction) is given by

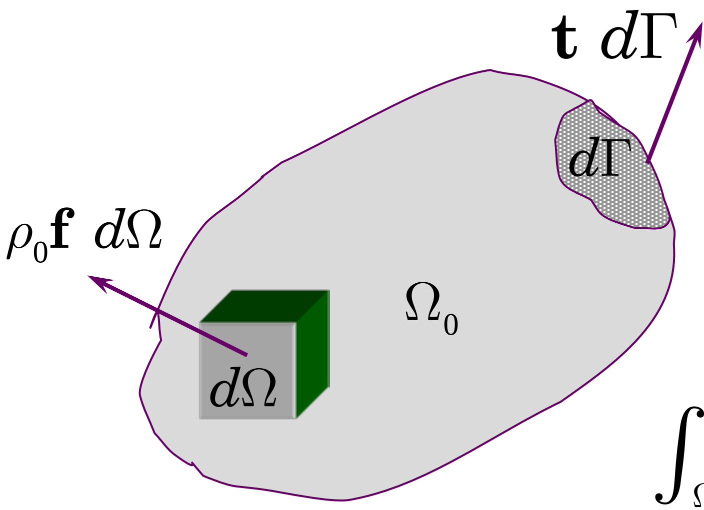
$$\mathbf{t}_{ns} = \mathbf{t}(\hat{\mathbf{n}}) - \mathbf{t}_{nn} = \frac{100}{27}(118\hat{\mathbf{e}}_1 - 16\hat{\mathbf{e}}_2 - 43\hat{\mathbf{e}}_3) \ \text{psi}.$$

The magnitudes are

$$|\mathbf{t}_{nn}| = t_{nn} = \frac{2600}{9} = 288.89 \ \text{psi}, \quad |\mathbf{t}_{ns}| = t_{ns} = 468.91 \ \text{psi}.$$

Balance of Linear Momentum in the Lagrangian Description

The time rate of change of total linear momentum of a given continuum equals the vector sum of all external forces acting on the continuum. This also known as **Newton's Second Law**.



$$\oint_{\Gamma_0} \mathbf{t}\, d\Gamma + \int_{\Omega_0} \rho_0 \mathbf{f}\, d\Omega = \frac{\partial}{\partial t}\int_{\Omega_0} \rho_0 \mathbf{v}\, d\Omega$$

$$\oint_{\Gamma_0} \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\, d\Gamma + \int_{\Omega_0} \rho_0 \mathbf{f}\, d\Omega = \int_{\Omega_0} \rho_0 \frac{\partial \mathbf{v}}{\partial t} d\Omega$$

$$\int_{\Omega_0} \left(\nabla\cdot\boldsymbol{\sigma} + \rho_0 \mathbf{f} - \rho_0 \frac{\partial \mathbf{v}}{\partial t} \right) d\Omega = 0$$

Newton's First Law. Newton's First Law states that an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force.

Balance of Linear Momentum
(continued)

Vector form of the equation of motion

$$\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Cartesian Component Form

$$\left(\hat{\mathbf{e}}_k \frac{\partial}{\partial x_k} \right) \cdot \left(\boldsymbol{\sigma}_{ji} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i \right) + \rho_0 f_i \hat{\mathbf{e}}_i = \rho_0 \frac{\partial^2 (u_i \hat{\mathbf{e}}_i)}{\partial t^2}$$

$$\frac{\partial \sigma_{ji}}{\partial x_j} \hat{\mathbf{e}}_i + \rho_0 f_i \hat{\mathbf{e}}_i = \rho_0 \frac{\partial v_i}{\partial t} \hat{\mathbf{e}}_i \;\Rightarrow\; \frac{\partial \sigma_{ji}}{\partial x_j} + \rho_0 f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Balance of Linear Momentum
(continued)

Cartesian Component Form (expanded form)

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho_0 f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho_0 f_1 = \rho_0 \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho_0 f_2 = \rho_0 \frac{\partial^2 u_2}{\partial t^2}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho_0 f_3 = \rho_0 \frac{\partial^2 u_3}{\partial t^2}$$

Balance of Angular Momentum

The principle of balance of angular momentum can be stated as: *the time rate of change of the total moment of momentum for a continuum is equal to vector sum of the moments of external forces acting on the continuum*. We assume that there are no body (volume dependent) couples **M**:

$$\lim_{\Delta V \to 0} \Delta \mathbf{M} / \Delta V = \mathbf{0}$$

Then the balance of angular momentum requires

$$\oint_{\Gamma} \mathbf{x} \times \mathbf{t}\, d\Gamma + \int_{\Omega} \mathbf{x} \times \mathbf{f}\, d\Omega = \frac{D}{Dt} \int_{\Omega} \mathbf{x} \times \rho \mathbf{v}\, d\Omega$$

which results in the symmetry of the Cauchy stress tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \quad \text{or} \quad \sigma_{ij} = \sigma_{ji}$$

Balance of Energy
(in spatial description)

The first law of thermodynamics can be stated as: *the time rate of the total energy is equal to the sum of the rate of work done by the external forces and the change of heat content per unit mass*. The second law of thermodynamics provides a restriction on the inter-convertibility of energies (e.g., thermal to mechanical). The first law can be expressed as (e_c is the internal energy density, g is the internal heat generation, and **d** is the symmetric part of the velocity gradient)

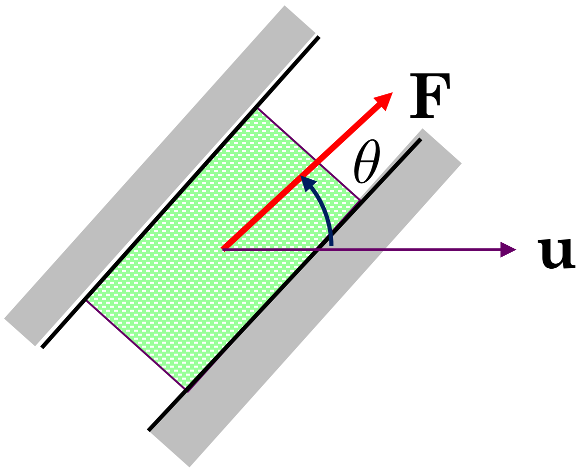
$$\frac{d}{dt}\int_{\Omega}\rho\left(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+e_c\right)d\Omega=\frac{1}{2}\frac{D}{Dt}\int_{\Omega}\rho\mathbf{v}\cdot\mathbf{v}\,d\Omega+\int_{\Omega}\left(\boldsymbol{\sigma}:\mathbf{d}-\nabla\cdot\mathbf{q}+g\right)d\Omega$$

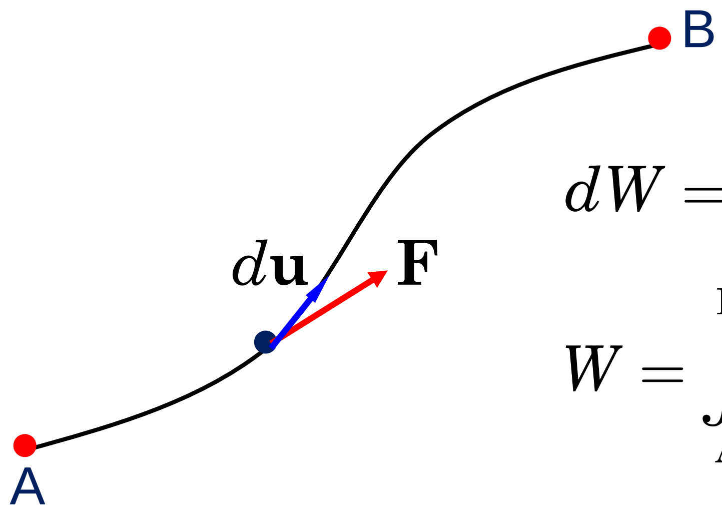
which results in

$$\rho\frac{de_c}{dt}=\boldsymbol{\sigma}:\mathbf{d}-\nabla\cdot\mathbf{q}+g$$

WORK and ENERGY

<u>**Work done**</u> **Magnitude of the force multiplied by the magnitude of the displacement in the direction of the force:** $W = \mathbf{F} \cdot \mathbf{u}$



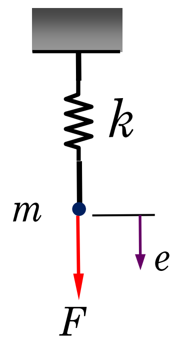


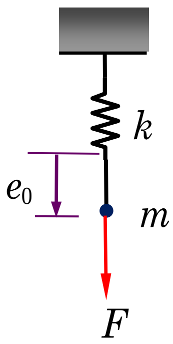
$$dW = \mathbf{F} \cdot d\mathbf{u}$$

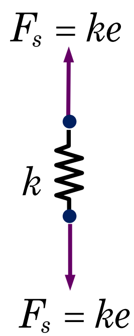
$$W = \int_{A}^{B} \mathbf{F} \cdot d\mathbf{u}$$

WORK and ENERGY

Energy is the capacity to do work







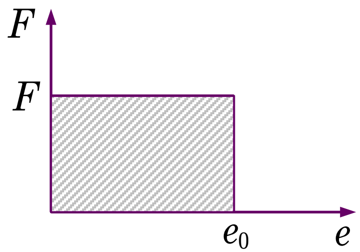
Work done

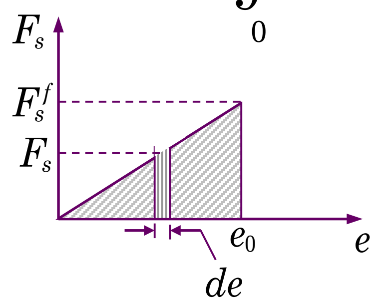
$$W = Fe_0$$

$$dE = F_s \cdot de, \quad F_s = ke$$

$$E = \int_0^{e_0} F_s \cdot de = \left[\tfrac{1}{2} k e^2\right]_0^{e_0} = \tfrac{1}{2} k e_0^2$$

(strain energy)



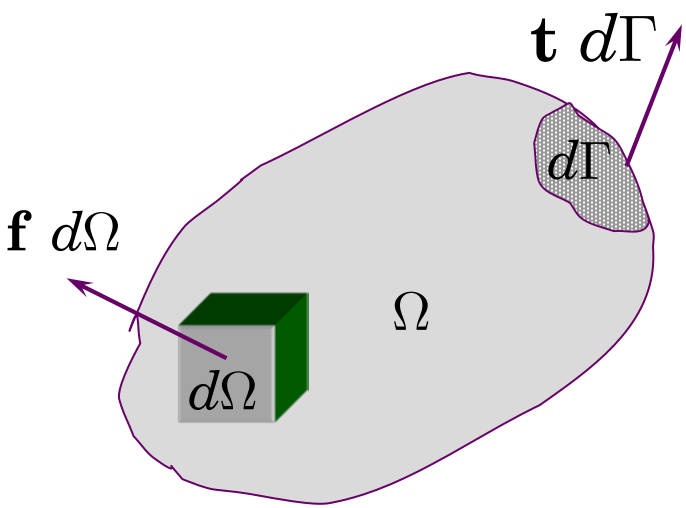


Note that

$$\left.\frac{dE}{de}\right|_{e=e_0} = W$$

EXTERNAL AND INTERNAL WORK IN A DEFORMABLE BODY

Work done by external forces

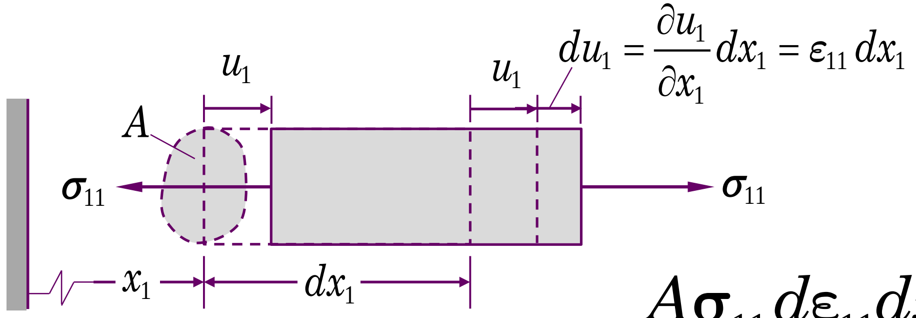


$$W_E = -\left[\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,d\Omega + \oint_\Gamma \mathbf{t}(s)\cdot\mathbf{u}(s)\,d\Gamma\right]$$

In calculating the external work done, the applied (external) forces (or moments) are assumed to be independent of the displacements (or rotations) they cause in a body.

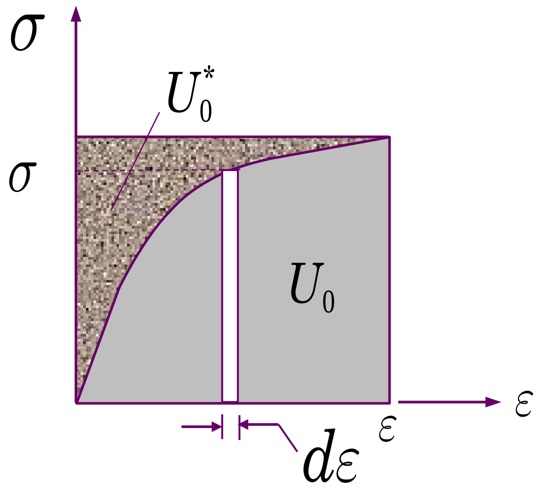
STRAIN ENERGY DENSITY AND STRAIN ENERGY

Work done by internal forces (1D)



$$A\boldsymbol{\sigma}_{11} d\boldsymbol{\varepsilon}_{11} dx_1 = \boldsymbol{\sigma}_{11} d\boldsymbol{\varepsilon}_{11}(A dx_1)$$

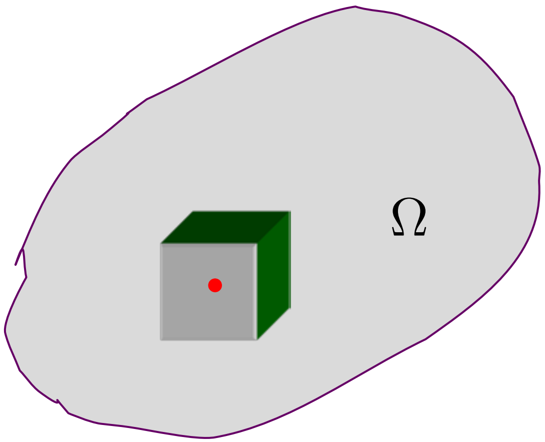
$$\equiv dU_0(A dx_1)$$



$$U_0(\boldsymbol{\varepsilon}_{11}) = \int_0^{\varepsilon_{11}} dU_0,\; U = \int_\Omega U_0 \; d\Omega$$

STRAIN ENERGY DENSITY AND STRAIN ENERGY

Strain energy of a 3D solid



$$U_0(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij},\quad U = \int_\Omega U_0\, d\Omega$$

$$U = \int_\Omega \left(\int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij} \right) d\Omega$$

For a linear elastic body, we have

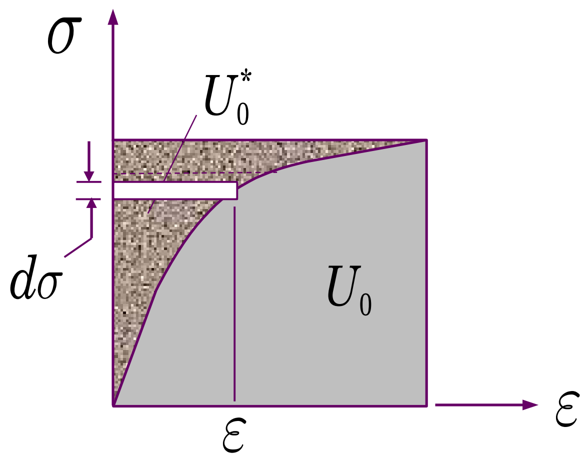
$$U = \frac{1}{2} \int_\Omega \boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x})\, d\Omega$$

If the only energy stored in the body is the strain energy, we write

$$U = W_I$$

COMPLEMENTARY STRAIN ENERGY DENSITY AND COMPLEMENTRY STRAIN ENERGY

Complementary strain energy for 1D



$$A\varepsilon_{11} d\sigma_{11} dx_1 = \varepsilon_{11} d\sigma_{11}(A dx_1) \equiv dU_0^*(A dx_1)$$

$$U_0^*(\sigma_{11}) = \int_0^{\sigma_{11}} dU_0^*, \; U^* = \int_\Omega U_0^* \, d\Omega$$

Complementary strain energy for 3D

$$U_0^*(\varepsilon_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{ij} \, d\sigma_{ij}, \; U^* = \int_\Omega U_0^* \, d\Omega$$

STRAIN ENERGY AND COMPLEMENTRY STRAIN ENERGY OF STRAIGHT E-B BEAMS

The strain energy density and strain energy of the Euler-Bernoulli (E-B) beam are (linear elastic material)

$$U_0 = \int_0^{\varepsilon_{xx}} \sigma_{xx} d\varepsilon_{xx} = \int_0^{\varepsilon_{xx}} E(x)\varepsilon_{xx}\, d\varepsilon_{xx} = \frac{E}{2}\varepsilon_{xx}^2 = \frac{E}{2}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right)^2$$

$$U = \int_v U_0 d\Omega = \int_0^L \int_A \frac{E}{2}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right)^2 dAdx = \frac{1}{2}\int_0^L \left[A_{xx}\left(\frac{du}{dx}\right)^2 + D_{xx}\left(\frac{d^2w}{dx^2}\right)^2\right]dx$$

$$A_{xx} = \int_A E(x)dA = EA, \quad D_{xx} = \int_A z^2 E(x)dA = EI$$

The complementary strain energy density and complementary strain energy of the Euler-Bernoulli beam are

$$U_0^* = \int_0^{\sigma_{xx}} \varepsilon_{xx}\, d\sigma_{xx} + 2\int_0^{\sigma_{xz}} \varepsilon_{xz}\, d\sigma_{xz} = \frac{\sigma_{xx}^2}{2E} + \frac{\sigma_{xz}^2}{2G} = \frac{1}{2E}\left(\frac{N}{A}\right)^2 + \frac{1}{2G}\left(\frac{VQ}{Ib}\right)^2$$

$$U^* = \int_0^L \int_A U_0^* dAdx = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{f_s V^2}{GA}\right)dx, \quad f_s = \frac{A}{I^2 b^2}\int_A Q^2(z)dA$$

TOTAL POTENTIAL ENERGY AND COMPLEMENTRY ENERGY

The potential energy of a 3D solid $(W_I = U,\ W_E = V_E)$

$$\Pi(\mathbf{u}) = U + V_E = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} d\Omega - \left[\int_\Omega \mathbf{f}\cdot\mathbf{u} d\Omega + \oint_\Gamma \mathbf{t}\cdot\mathbf{u} d\Gamma\right]$$

The complementary energy of a 3D solid

$$\Pi^*(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) d\Omega - \left[\int_\Omega \mathbf{f}\cdot\mathbf{u} d\Omega + \oint_\Gamma \mathbf{t}\cdot\mathbf{u} d\Gamma\right]$$

For an E-B beam, they are

$$\Pi(u,w) = \frac{1}{2}\int_0^L \left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2w}{dx^2}\right)^2\right] dx - \int_0^L (fu + qw) dx + \ldots$$

$$\Pi^*(\sigma_{xx}, \sigma_{xz}) = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{f_s V^2}{GA}\right) dx - \int_0^L (fu + qw) dx + \ldots$$

THE PRINCIPLE OF MINIMUM TOTAL POTENTIAL ENERGY

Minimum nature of PE:

$$\Pi(\bar{u},\bar{w}) = \frac{1}{2}\int_0^L \left[EA\left(\frac{d\bar{u}}{dx}\right)^2 + EI\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right]dx - \int_0^L (f\bar{u} + q\bar{w})\,dx$$

$$= \frac{1}{2}\int_0^L \left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2 w}{dx^2}\right)^2\right]dx - \int_0^L (fu + qw)\,dx$$

$$+ \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2 w_0}{dx^2}\right)^2\right]dx - \int_0^L (\alpha f u_0 + \beta q w_0)\,dx$$

$$+ \int_0^L \left[\alpha EA\left(\frac{du}{dx}\frac{du_0}{dx}\right) + \beta EI\left(\frac{d^2 w}{dx^2}\frac{d^2 w_0}{dx^2}\right)\right]dx$$

$$= \Pi(u,w) + \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2 w_0}{dx^2}\right)^2\right]dx + \alpha(0) + \beta(0)$$

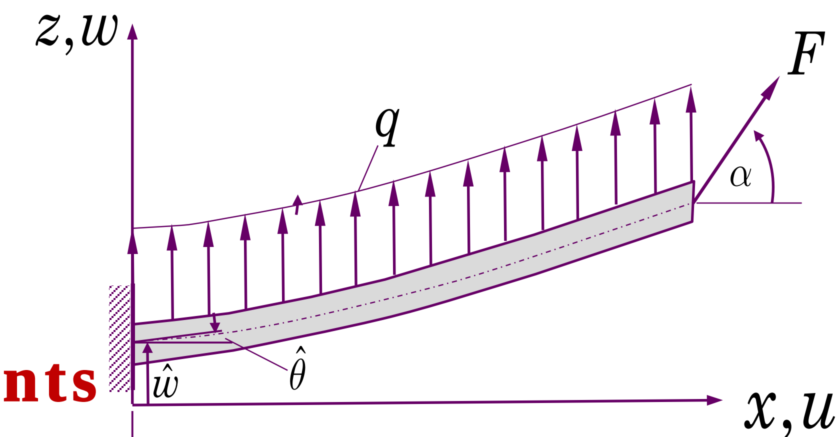
$$\Pi(\bar{u},\bar{w}) \geq \Pi(u,w)$$

because u and w satisfy the equilibrium equations

VIRTUAL WORK

Virtual displacements are those which satisfy the homogeneous form of the specified kinematic boundary conditions, but otherwise arbitrary.

$$u(0) = \hat{u} = 0,\; w(0) = \hat{w},\; -\left.\frac{dw}{dx}\right|_{x=0} = \hat{\theta}$$



Set of admissible displacements

$$u_1(x) = \hat{u} + a_1 x, \quad w_1(x) = \hat{w} + \hat{\theta} x + b_1 x^2$$

Set of admissible virtual displacements

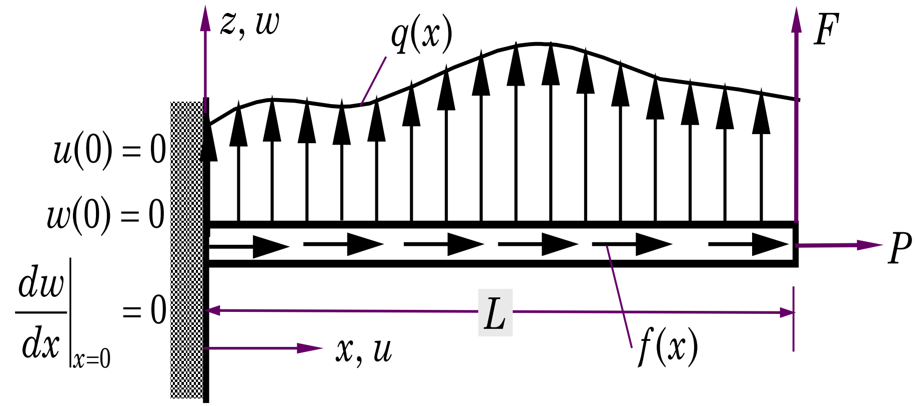
$$\delta u_1 = a_1 x,\; \delta w_1 = b_1 x^2;\quad \delta u_2 = a_1 x + a_2 x^2,\; \delta w_2 = b_1 x^2 + b_2 x^3$$

Virtual work done by actual forces (q, F) in moving through their respective displacements is

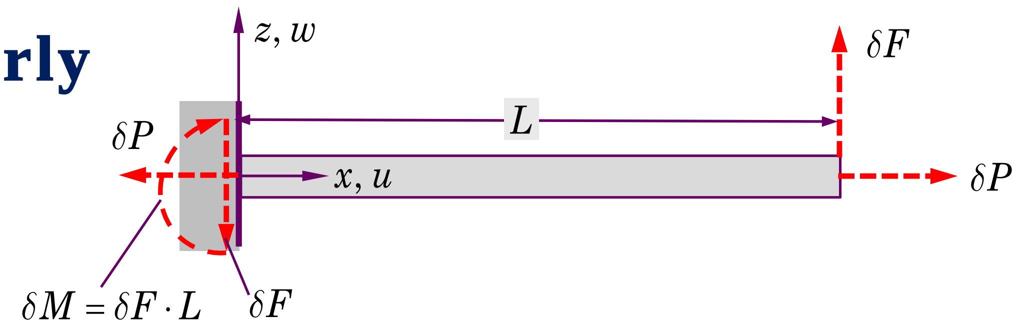
$$\delta W = \int_0^L q(s)\,\delta w(s)\,ds + F\cos\alpha\cdot\delta u(L) + F\sin\alpha\cdot\delta w(L)$$

COMPLEMENTARY VIRTUAL WORK

Virtual forces are those which satisfy the self-equilibrium conditions, but otherwise arbitrary.



The set $(\delta P, \delta F)$ is clearly is in self-equilibrium.



The virtual work done by virtual forces in moving through actual Displacements is

$$\delta W^* = -\left[\delta P \cdot u(L) + \delta F \cdot w(L) - \delta P \cdot u(0) - \delta F \cdot w(0) + \delta M \cdot \theta(0)\right]$$

$$= -\left[\delta P \cdot u(L) + \delta F \cdot w(L)\right]$$

VIRTUAL WORK DONE

Internal virtual work done in a 3D body

$$\delta W_I = \int_\Omega \delta U_0 \, d\Omega = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) : \delta \boldsymbol{\varepsilon} \, d\Omega = \int_\Omega \sigma_{ij}(\varepsilon_{kl}) \delta \varepsilon_{ij} \, d\Omega$$

Internal complementary virtual work done

$$\delta W_I^* = \int_\Omega \delta U_0^* \, d\Omega = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \delta \boldsymbol{\sigma} \, d\Omega = \int_\Omega \varepsilon_{ij}(\sigma_{kl}) \delta \sigma_{ij} \, d\Omega$$

External virtual work done in a 3D body

$$\delta W_E = \int_\Omega \mathbf{f} \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_\sigma} \mathbf{t} \cdot \delta \mathbf{u} \, d\Gamma$$

External complementary virtual work done

$$\delta W_E^* = \int_\Omega \delta \mathbf{f} \cdot \mathbf{u} \, d\Omega + \int_{\Gamma_u} \delta \mathbf{t} \cdot \mathbf{u} \, d\Gamma$$

Total virtual virtual work done

$$\delta W = \delta W_I + \delta W_E; \quad \delta W^* = \delta W_I^* + \delta W_E^*$$

VIRTUAL WORK DONE FOR E-B BEAMS

Internal virtual work done

$$\delta W_I = \int_0^L \left(N \frac{d\delta u}{dx} - M \frac{d^2 \delta w}{dx^2} \right) dx$$

Internal complementary virtual work done

$$\delta W_I^* = \int_0^L \left(\varepsilon_{xx}^{(0)} \delta N + \varepsilon_{xx}^{(1)} \delta M + 2\varepsilon_{xz}^{(1)} \delta V \right) dx$$

External virtual work done in a 3D body

$$\delta W_E = -\left[\int_a^b f(x) \delta u(x)\, dx + \int_c^d \mathrm{q}(x) \delta w(x)\, dx + \text{VW point forces} \right]$$

External complementary virtual work done

$$\delta W_E^* = -\left[\int_a^b \delta f(x) u(x)\, dx + \int_c^d \delta q(x) w(x)\, dx + \text{VW} \right]$$

ANALOGY BETWEEN TOTAL DIFFERENTIAL AND VARIATIONAL OPERTOR

Analogy

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

$$\delta F = \frac{\partial F}{\partial x}\cancelto{0}{\delta x} + \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$

Properties

(1) $\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2,$ (2) $\delta(F_1 F_2) = \delta F_1 F_2 + F_1 \delta F_2$

(3) $\delta\left(\frac{F_1}{F_2}\right) = \frac{\delta F_1 F_2 - F_1 \delta F_2}{F_2^2},$ (4) $\delta(F_1)^n = n(F_1)^{n-1}\delta F_1$

(1) $\delta\left(\frac{du}{dx}\right) = \alpha\frac{dv}{dx} = \frac{d}{dx}(\alpha v) = \frac{d}{dx}(\delta u)$

(2) $\delta\left(\int_0^a u\,dx\right) = \alpha\int_0^a v\,dx = \int_0^a \alpha v\,dx = \int_0^a \delta u\,dx$

FIRST VARIATION OF A FUNCTIONAL

A functional

A *functional* F is a mapping (or operator) from a vector space U into the real number field R. Thus, if $u \in U$ (i.e. u is an element of U), then $F(u)$ is a real number.

The First Variation of a Functional

$$I(u) = \int_a^b F(x,u,u')\,dx, \quad u' = \frac{du}{dx}$$

$$\delta I(u;\delta u) = \delta \int_a^b F(x,u,u')\,dx = \int_a^b \delta F\,dx = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \delta u\,dx + \left[\delta u \cdot \frac{\partial F}{\partial u'} \right]_{x=a}^{x=b}$$

FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS AND EULER EQUATIONS

Lemma: If G is an integrable function and $\eta(x)$ is arbitrary in $a < x < b$ and $\eta(a)$ is arbitrary, then the statement

$$\int_a^b G(x)\eta(x)dx + B(a)\eta(a) = 0$$

implies that $G(x) = 0 \;\; a < x < b$ and $B(a) = 0$

which are called the **Euler equations.** If $\delta I = 0$ and δu is arbitrary in (a,b) and at $x = a$ and $x = b$, then

$$\delta I(u;\delta u) = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u\, dx + \left[\delta u \cdot \frac{\partial F}{\partial u'}\right]_{x=a}^{x=b} = 0$$

$$\Rightarrow \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] = 0 \;\; a < x < b; \quad \left[\frac{\partial F}{\partial u'}\right]_{x=a}^{x=b} = 0$$

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

The Principle

$$\delta W = \delta W_I + \delta W_E = 0$$

$$0 = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, d\Omega - \left[\int_\Omega \mathbf{f} \cdot \boldsymbol{\delta}\mathbf{u}\, d\Omega + \int_{\Gamma_\sigma} \hat{\mathbf{t}} \cdot \boldsymbol{\delta}\mathbf{u}\, d\Gamma \right]$$

Application to 3D Linear Elasticity

$$0 = \frac{1}{2}\int_\Omega \left[\sigma_{ij} \left(\delta u_{i,j} + \delta u_{j,i} \right) \right] d\Omega - \left[\int_\Omega f_i \delta u_i \, d\Omega + \int_{\Gamma_\sigma} \hat{t}_i \delta u_i \, d\Gamma \right]$$

$$= -\int_\Omega \left(\sigma_{ji,j} + f_i \right) \delta u_i \, d\Omega - \int_{\Gamma_\sigma} \hat{t}_i u_i \, ds + \int_\Gamma \sigma_{ji} n_j \delta u_i \, d\Gamma$$

$$= -\int_\Omega \left(\sigma_{ji,j} + f_i \right) \delta u_i \, d\Omega + \int_{\Gamma_\sigma} \left(\sigma_{ij} n_j - \hat{t}_i \right) \delta u_i \, d\Gamma + \int_{\Gamma_u} \sigma_{ij} n_j \, \underbrace{\delta u_i}_{0} \, d\Gamma$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \ \text{ in } \Omega; \ \ \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} - \hat{\mathbf{t}} = \mathbf{0} \ \text{ on } \Gamma_{\boldsymbol{\sigma}}$$

SUMMARY

In these lectures we have covered the following topics with some examples:

- Continuum assumption
- Kinematics of deformation: Introduced the Green-Lagrange strain tensor
- Kinetics: Defined Cauchy stress vector
- Cauchy's formula is derived and Cauchy stress tensor is introduced
- Balance of linear and angular momenta
- Conservation of energy (the first law)
- Concepts of Work and Energy
- Principles of the minimum total potential energy and virtual work