LEAST-SQUARES FINITE ELEMENT MODELS

- General idea of the least-squares formulation applied to an abstract boundary-value problem
- Works of our group
- Application to Poisson’s equation
- Application to flows of viscous incompressible fluids
- Numerical Examples
Abstract nonlinear boundary value problem

\[ \mathcal{L}(u) = f \quad \text{in } \Omega \]
\[ g(u) = h \quad \text{on } \Gamma \]

\( \Omega \): domain of boundary value problem
\( \Gamma \): boundary of \( \Omega \)
\( \mathcal{L} \): first-order nonlinear partial differential operator
\( g \): linear boundary condition operator
\( f, h \): data
Least-Squares Variational Formulation: Abstract Nonlinear Formulation

Abstract least-squares variational principle

\[ J(u; f, h) = \frac{1}{2} \left( \| \mathcal{L}(u) - f \|_{\Omega,0}^2 + \| g(u) - h \|_{\Gamma,0}^2 \right) \]

- Find \( u \in \mathcal{V} \) such that \( J(u; f, h) \leq J(\tilde{u}; f, h) \) for all \( \tilde{u} \in \mathcal{V} \), where \( \mathcal{V} \) is an appropriate vector space, such as \( H^1(\Omega) \)

Necessary condition for minimization:

\[ \mathcal{G}(\tilde{u}, u) = \left( \nabla \mathcal{L}(u) \cdot \tilde{u}, \mathcal{L}(u) - f \right)_{\Omega,0} + \left( g(\tilde{u}), g(u) - h \right)_{\Gamma,0} \]
Two approaches may be adopted when formulating least-squares finite element models of nonlinear problems

- (1) Linearize PDE prior to construction and minimization of least-squares functional $J$
  - Element matrices will always be symmetric
  - Simplest possible form of the element matrices

- (2) Linearize finite element equations following construction and minimization of least-squares functional $J$
  - Approach is consistent with variational setting
  - Finite element matrices are more complicated
  - Resulting coefficient matrix may not be symmetric
Finite Element Implementation

**Spectral/hp Finite Elements**

- One-dimensional high-order Lagrange interpolation functions

\[ \psi_i(\xi) = \frac{(\xi-1)(\xi+1)L_p'(\xi)}{p(p+1)L_p(\xi_i)(\xi-\xi_i)} \]

- Multi-dimensional interpolation functions constructed from tensor products of the one-dimensional functions
- We employ full Gauss Legendre quadrature rules in evaluation of the integrals
APPLICATIONS OF LSFEM TO DATE
by JNReddy and his coauthors

- **Fluid Dynamics (2-D)**
  - Viscous incompressible fluids
  - Viscous compressible fluids (with shocks)
  - Non-Newtonian (polymer and power-law) fluids
  - Coupled fluid flow and heat transfer
  - Fluid-solid interaction

- **Solid Mechanics (static and free vibration analysis)**
  - Beams
  - Plates
  - Shells
  - Fracture mechanics
  - Helmholtz equation
Finite Element Formulations
Of the Poisson Equation

(Primal) Problem:

\[-\nabla^2 u = f \quad \text{in} \quad \Omega\]

\((-\nabla^2 = -\nabla \cdot \nabla)\]

\[u = \hat{u} \quad \text{on} \quad \Gamma_u\]

\[\frac{\partial u}{\partial n} = \hat{g} \quad \text{on} \quad \Gamma_g\]

(Mixed) Problem:

\[v - \nabla u = 0 \quad \text{in} \quad \Omega\]

\[-\nabla \cdot v = f \quad \text{in} \quad \Omega\]

\[u = \hat{u} \quad \text{on} \quad \Gamma_u\]

\[\hat{n} \cdot v = \hat{g} \quad \text{on} \quad \Gamma_g\]
Least-Squares Formulation - Primal

1. \[ I_1(u) = \| -\nabla^2 u - f \|_{0,\Omega}^2 + \| \frac{\partial u}{\partial n} - \hat{g} \|_{0,\Gamma_g}^2 \]

2. **Minimize** \( I_1(u) \)

\[ B_1(u, v) = l_1(v) \]

\[ B_1(u, v) = \left( -\nabla^2 v, -\nabla^2 u \right)_{0,\Omega} + \left( \frac{\partial v}{\partial n}, \frac{\partial u}{\partial n} \right)_{0,\Gamma_g} \]

\[ l_1(v) = \left( \nabla^2 v, f \right)_{0,\Omega} + \left( \frac{\partial v}{\partial n}, g \right)_{0,\Gamma_g} \]
Least Squares Formulation - Mixed

\[ I_m(u) = \| \mathbf{v} - \nabla u \|^2_{0,\Omega} + \| -\nabla \cdot \mathbf{v} - f \|^2_{0,\Omega} + \| \mathbf{n} \cdot \mathbf{v} - \mathbf{g} \|^2_{0,\Gamma_g} \]

Minimize \( I_m : \delta I_m = 0 \) gives

\[ B_m((u, v), (\delta u, \delta v)) = l_m((\delta u, \delta v)) \]
Least-squares Mixed Fe Model

Finite element approximation

\[ u(x) \approx u_h(x) = \sum_{j=1}^{m} u_j \psi_j(x), \quad v(x) \approx v_h(x) = \sum_{j=1}^{n} v_j \varphi_j(x) \]

Finite element model

\[
\begin{bmatrix}
K_{11} & K_{12} \\
(K_{12})^T & K_{22}
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

\[ K_{ij}^{11} = \int_{\Omega} \nabla\psi_i \cdot \nabla\psi_j \, dx, \quad K_{ij}^{12} = -\int_{\Omega} \nabla\psi_i \cdot \varphi_j \, dx = K_{ji}^{21} \]

\[ K_{ij}^{22} = \int_{\Omega} (\varphi_i \varphi_j + \nabla \varphi_i \cdot \nabla \varphi_j) \, dx + \int_{\Gamma} (\hat{n} \cdot \varphi_i)(\hat{n} \cdot \varphi_j) \, ds \]

\[ F_i^1 = 0, \quad F_i^2 = -\int_{\Omega} f \nabla \cdot \varphi_i \, dx + \int_{\Gamma} \hat{n} \cdot \varphi_i \hat{g} \, ds \]
Example (Using LSFEM Mixed Model):

Differential Equation

$$-\nabla^2 u = f \text{ in } -1 \leq x, y \leq 1$$

Boundary Conditions

$$\frac{\partial u}{\partial y} \equiv u = 0 \text{ on } y = \pm 1$$

$$\frac{\partial u}{\partial x} \equiv w = q * (y) = 0 \text{ on } x = -1$$

$$u = u * (y) = 8 \cos \pi y \text{ on } x = 1$$

Analytical solution:

$$u(x, y) = (7x + x^7) \cos \pi y$$
Plots of the $L_2$-Error norms as a function of $p$
LEAST-SQUARES FORMULATION
OF VISCOSOUS INCOMPRESSIBLE FLUIDS

Governing equations (Navier-Stokes equations)

\[
(u \cdot \nabla)u + \nabla p - \frac{1}{Re} \nabla \cdot [(\nabla u) + (\nabla u)^T] = f \quad \text{in} \quad \Omega
\]

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega
\]

\[
u = \hat{u} \quad \text{on} \quad \Gamma_u
\]

\[
\hat{n} \cdot \sigma = \hat{t} \quad \text{on} \quad \Gamma_\sigma
\]
VELOCITY-PRESSURE-VORTICITY FORMULATION OF N-S EQUATIONS FOR VISCOUS INCOMPRESSIBLE FLUIDS

\[(u \cdot \nabla)u + \nabla p - \frac{1}{Re} \nabla \times \omega = f\]

\[\omega - \nabla \times u = 0\]

\[\nabla \cdot u = 0\]

\[\nabla \cdot \omega = 0 \quad \text{in} \quad \Omega\]

\[u = \hat{u} \quad \text{on} \quad \Gamma_u\]

\[\omega = \hat{\omega} \quad \text{on} \quad \Gamma_\omega\]

\[J(u, p, \omega; f) = \frac{1}{2} \left( \left\| (u \cdot \nabla) u + \nabla p + \frac{1}{Re} \nabla \times \omega - f \right\|^2 + \left\| \omega - \nabla \times u \right\|^2 \right.\]

\[\left. + \left\| \nabla \cdot u \right\|^2 + \left\| \nabla \cdot \omega \right\|^2 \right)\]
NUMERICAL EXAMPLES
Lid-driven Cavity-1
Lid-driven Cavity-3

Finite element mesh (20x20)

Stream function (Re=5,000, ~)

Fluid Flow (LSFEM) 18
Re = 10^4
Oscillatory flow of a viscous incompressible fluid in a lid-driven cavity

Oscillatory Lid-Driven Cavity Flow
Velocity Vector Field and Vorticity Contours

Non-stationary incompressible N-S equations, Re = 400
Least-Squares space / time decoupled formulation
6 x 6 mesh with p = 5

J. P. Fox
TAMU
Flow of a viscous fluid in a narrow channel (backward facing step)
Flow of a Viscous Incompressible Fluid around a Cylinder-1

Mesh (501 elements; \( p=4 \))

Close-up of mesh around the cylinder
Circular Cylinder in Crossflow

Vorticity Contours

Non-stationary incompressible N-S equations, Re = 100
Least-Squares time / space decoupled formulation
1200 elements with p = 2
Robust at moderately high Reynolds numbers: Re = 100 – 10^4

High p-level solution: p = 4, 6, 8, 10

No filters or stabilization are needed
Flow of a viscous fluid past a circular cylinder

Incompressible flow past two circular cylinders in a side-by-side arrangement surface-to-surface gap, S/D = 0.85, Re = 100

Vorticity contours

Least-squares finite element formulation
p-levels of 4/4/2 in space-time

J.P. Pontaza, 20
Steady Flow Past a Circular Cylinder-4
Flow Past Two Circular Cylinders

Incompressible flow past two circular cylinders in a side-by-side arrangement with a surface-to-surface gap, S/D=0.85, Re=100. Velocity magnitude contours showing the "bistable gap jet".

Least-squares finite element formulation with p-levels of 4/4/2 in space-time.
Motion of a Cylinder in a Square Cavity

Initial boundary value problem

**Problem parameters**
- $\rho = 1$, $\text{Re} = 100$
- $v_{\text{walls}} = 0$, $v_{\text{cyl}} = 1.0$
- $t \in (0, 0.70]$  

**Finite element discretization**
- $\text{NE} = 400$, $p$-level = 4
- 31,360 degrees of freedom
- Time step: $\Delta t = 0.005$
- $\alpha = 0.5$ ($\alpha$-family)
- $\varepsilon = 10^{-6}$ (nonlinear iteration)
Motion of a Cylinder in a Square Cavity
Motion of a Cylinder in a Square Cavity

- Pseudo-elasticity technique used to update mesh at each time step
- Non-uniform Young's modulus specified for each element
Fluid-Solid Interaction

(movement of a rigid solid circular cylinder in a viscous fluid)