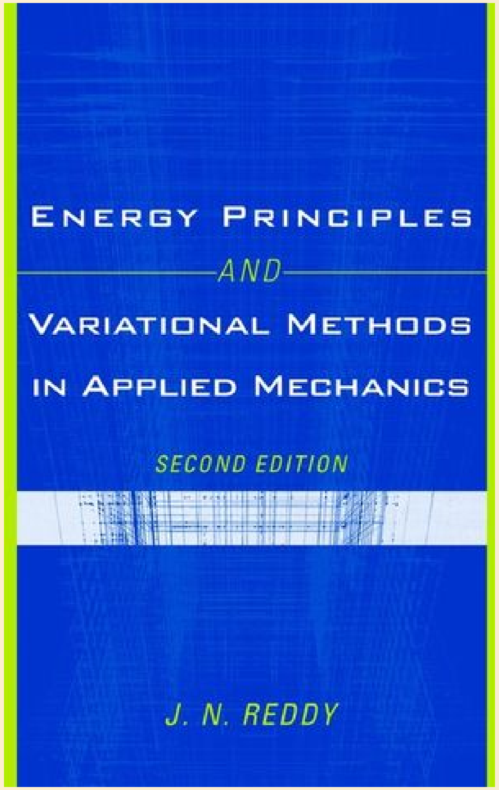
The Finite Element Method

Read: Chapter 9



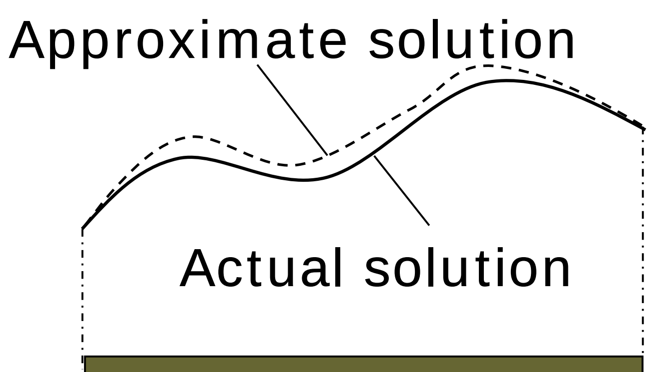
CONTENTS

- Approximate solutions and methods of approximation
- The basic features of the finite element method
- Examples
- Finite element discretization
- Terminology
- Steps involved in the finite element model development

EXACT AND APPROXIMATE SOLUTIONS

An *exact solution* satisfies (a) the differential equation at every point of the domain and (b) boundary conditions on the boundary. An *approximate solution* satisfies the differential equation as well as the specified boundary conditions in some "acceptable sense" (to be made clearer shortly). We seek the approximate solution as a linear combination of unknown parameters c_i and known functions ϕ_i and ϕ_0

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} c_i \phi_i(x) + \phi_0(x)$$



Approximation of the actual solution over the entire domain

DETERMINING APPROXIMATE SOLUTIONS (CONTINUED)

1. Suppose that ϕ_i is selected to satisfy the boundary conditions exactly. Then substitution of $u_N(x)$ into the differential equation

$$-\frac{d}{dx}\left(a(x)\frac{du_N}{dx}\right)+c(x)u_N-f(x)=0$$

will result in a non-zero function on the left side of the equality:

$$-\frac{d}{dx}\left(a(x)\frac{du_N}{dx}\right)+c(x)u_N-f(x)\equiv R(x)\neq 0$$

Then c_i are determined such that the residual, $R(x)$, is zero in some sense.

METHODS OF APPROXIMATION

1. One sense in which the residual, $R(x)$, can be made zero is to require it to be zero at selected number of points. The number of points should be equal to the number of unknowns in the approximate solution

$$u(x) \approx u_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$$

$\phi_j(x)$ and $\phi_0(x)$ are functions to be selected to satisfy the specified boundary conditions and c_j are parameters to be determined such that the residual is *made* zero in some sense.

This way of determining c_i is known as the *Collocation method*. We obtain N algebraic equations in N unknown C's

$$R(x_i) = 0, \quad i = 1, 2, \cdots, N$$

METHODS OF APPROXIMATION (CONTINUED)

2. Another approach in which the residual, $R(x)$, can be made zero is in a least-squares sense; i.e., minimize the integral of the square of the residual with respect to C's.

$$\text{Minimize} \quad J(c_1, c_2, \cdots, c_N) = \int_0^L R^2 \, dx$$

$$\text{or} \quad \frac{\partial J}{\partial c_i} = 2\int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

This method is known as the *least-Squares method*. We obtain N algebraic equations in N unknown C's

$$\int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

Methods of Approximation (Continued)

3. Yet, another approach in which the residual, $R(x)$, can be made zero is in a weighted-residual sense

$$0 = \int_0^L \psi_i R\,dx, \;\; i = 1, 2, \cdots, N$$

where ψ_i are linearly independent set of functions

This method is known as the *Weighted-Residual method*. We obtain N algebraic equations in N unknown C's. In general, weight functions ψ_i are not the same as the approximation functions φ_i. Various special cases are

Petrov-Galerkin Method: $\psi_i \neq \phi_i$

Galerkin Method: $\psi_i = \phi_i$

WEIGHTED-INTEGRAL FORMULATIONS in the Numerical Solution of Differential Eqs.

The approximation methods discussed earlier can be viewed as special cases of the weighted-residual methods of approximation. In particular, we have

- Collocation method $\psi_i(x) = \delta(x - x_i)$
- Least-squares method $\psi_i(x) = \dfrac{\partial R}{\partial c_i}$
- ♦ Petrov-Galerkin method $\psi_i(x) \neq \phi_i(x)$
- ♦ Galerkin Method $\psi_i(x) = \phi_i(x)$

Methods of Approximation
(Continued)

4. Another approach in which the governing equation is cast in a **weak-form** and the weight function is taken the same as the approximation function is known as the *Ritz method*:

$$B(\phi_i, u_N) = \ell(\phi_i), \quad i = 1, 2, \ldots, N$$

The Ritz method is the most commonly used method for all commercial software. In this method, ϕ_0 satisfies only the specified essential (geometric) boundary conditions while ϕ_i satisfies the homogeneous form of the specified essential boundary conditions. The specified natural (force) boundary condition are included in the weak form.

AN EXAMPLE

For a weighted-residual method:

ϕ_0 satisfy the actual specified BCs

ϕ_i satisfy the homogeneous form of the actual specified BCs

For the Ritz method:

ϕ_0 satisfy the actual specified <u>essential</u> BCs

ϕ_i satisfy the homogeneous form of the actual specified <u>essential</u> BCs

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f_0=0, \quad 0<x<L$$

$$u(0)=u_0, \;\; a\frac{du}{dx}\bigg|_{x=L}=P$$

EXAMPLE (CONTINUED)

For a weighted-residual method:

$$\phi_0(0) = u_0, \quad a\frac{d\phi_0}{dx}\bigg|_{x=L} = P$$

$$\phi_i(0) = 0, \quad a\frac{d\phi_0}{dx}\bigg|_{x=L} = 0$$

For the Ritz (or weak-form Galerkin) method:

$$\phi_0(0) = u_0, \quad \phi_i(0) = 0$$

BASIC DRAWBACKS OF CLASSICAL VARIATIONAL METHODS

The following drawbacks of the classical variational methods make them impractical:

- Difficulty in constructing the approximation functions for arbitrary domains, loads, material properties, and boundary conditions
- Automating the procedure for a class of problems, independent of the boundary conditions
- Not amenable to general purpose computer programs

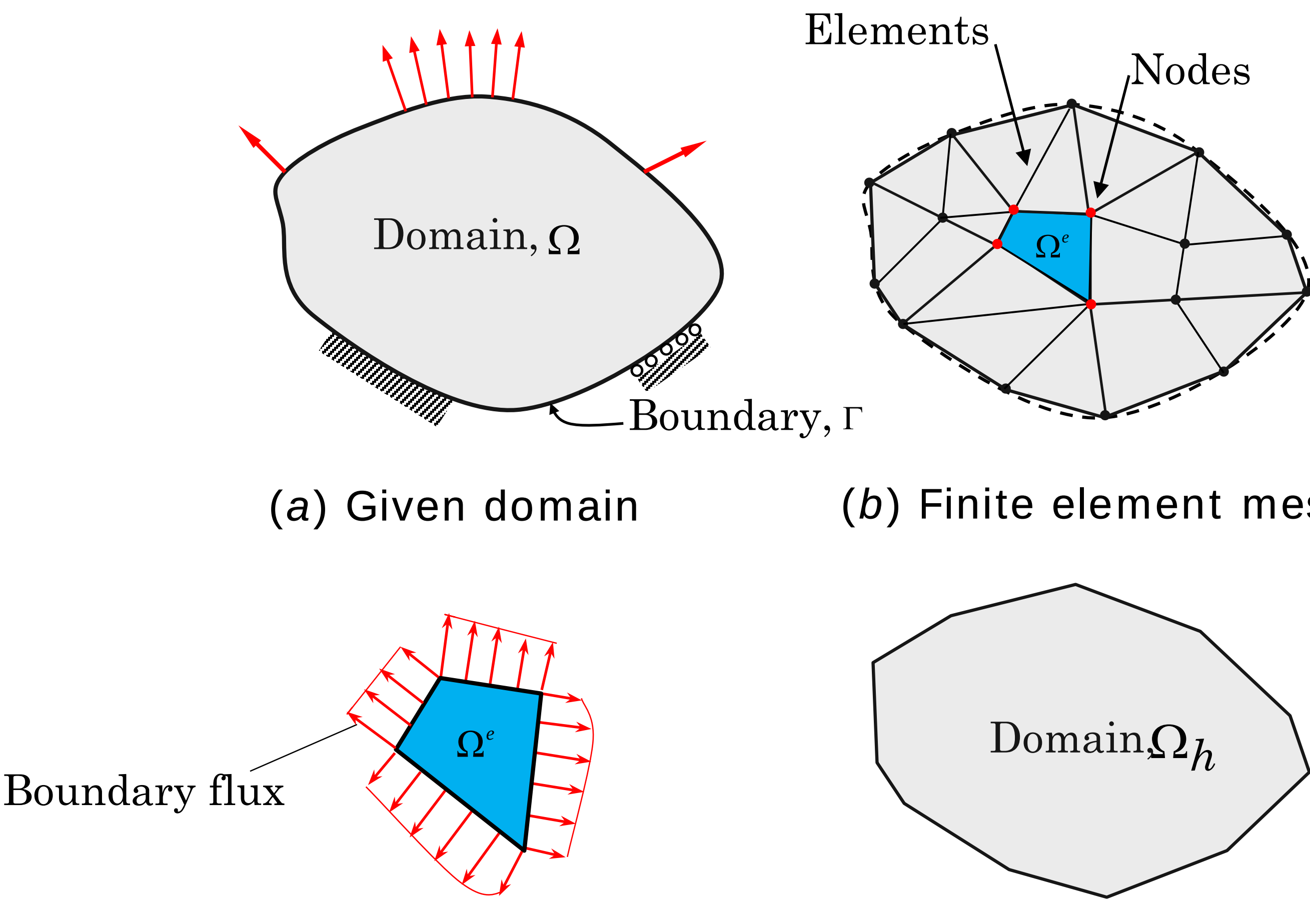
These features led to the development of the finite element method.

BASIC FEATURES OF THE FINITE ELEMENT METHOD (FEM)

- Divide whole into parts (*finite element mesh*)
- Set up the `problem' over a typical part

  (derive a set of relationships between primary and secondary variables)
- Assemble the parts to obtain the solution to the whole

Finite Element Discretization



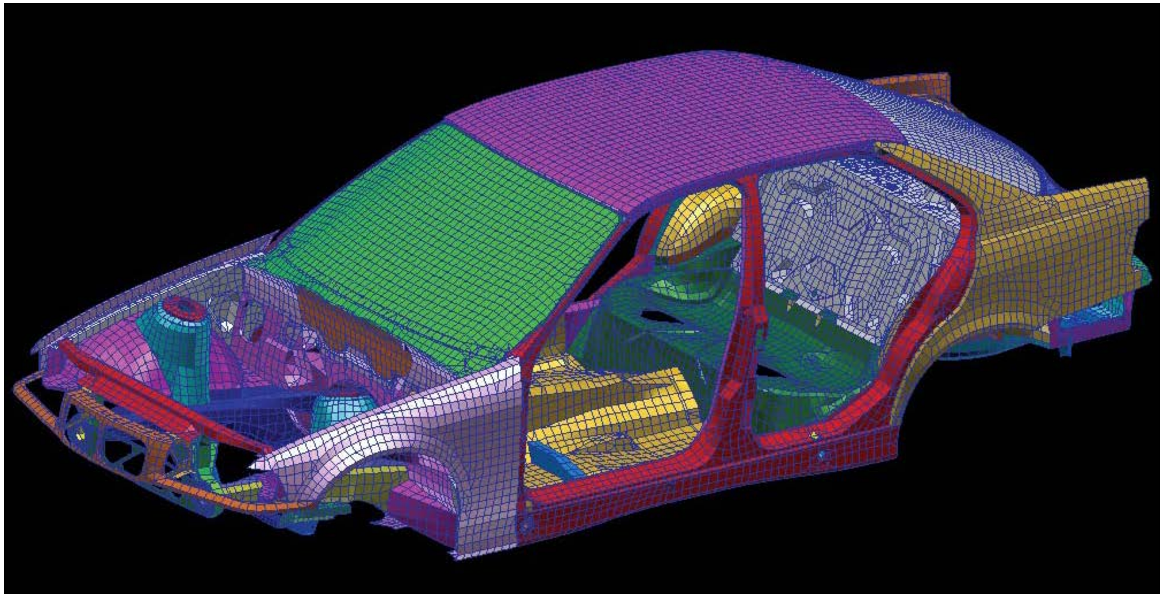
(*a*) Given domain

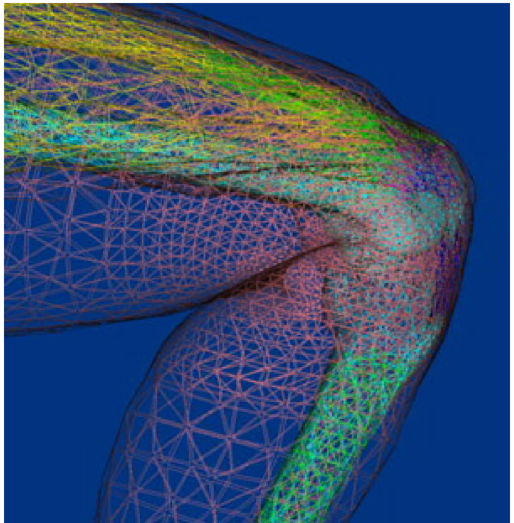
(*b*) Finite element mesh

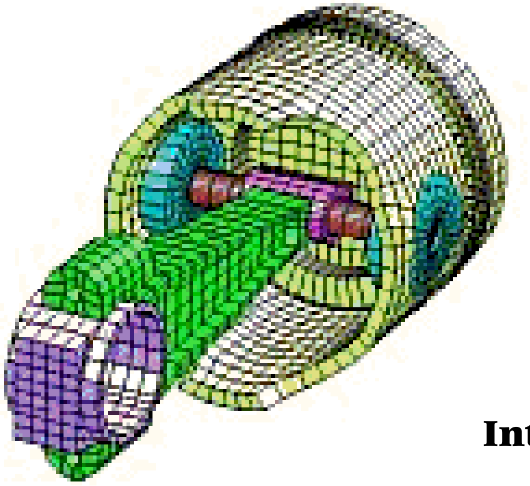
(*c*) Typical element with boundary fluxes

(*d*) Discretized domain

Some Examples of Real-World Finite Element Discretizations







FEM Terminology

- ***Element*** A geometric sub-domain of the region being simulated, with the property that it allows a unique (1) representation of its geometry and (2) derivation of the approximation (interpolation) functions.
- ***Node*** A geometric location in the element which plays a role in the derivation of the interpolation functions and it is the point at which solution is sought.
- ***Mesh*** A collection of elements (or nodes) that replaces the actual domain.
- ***Weak Form*** An integral statement equivalent to the governing equations and *natural* boundary conditions. **More to come.**

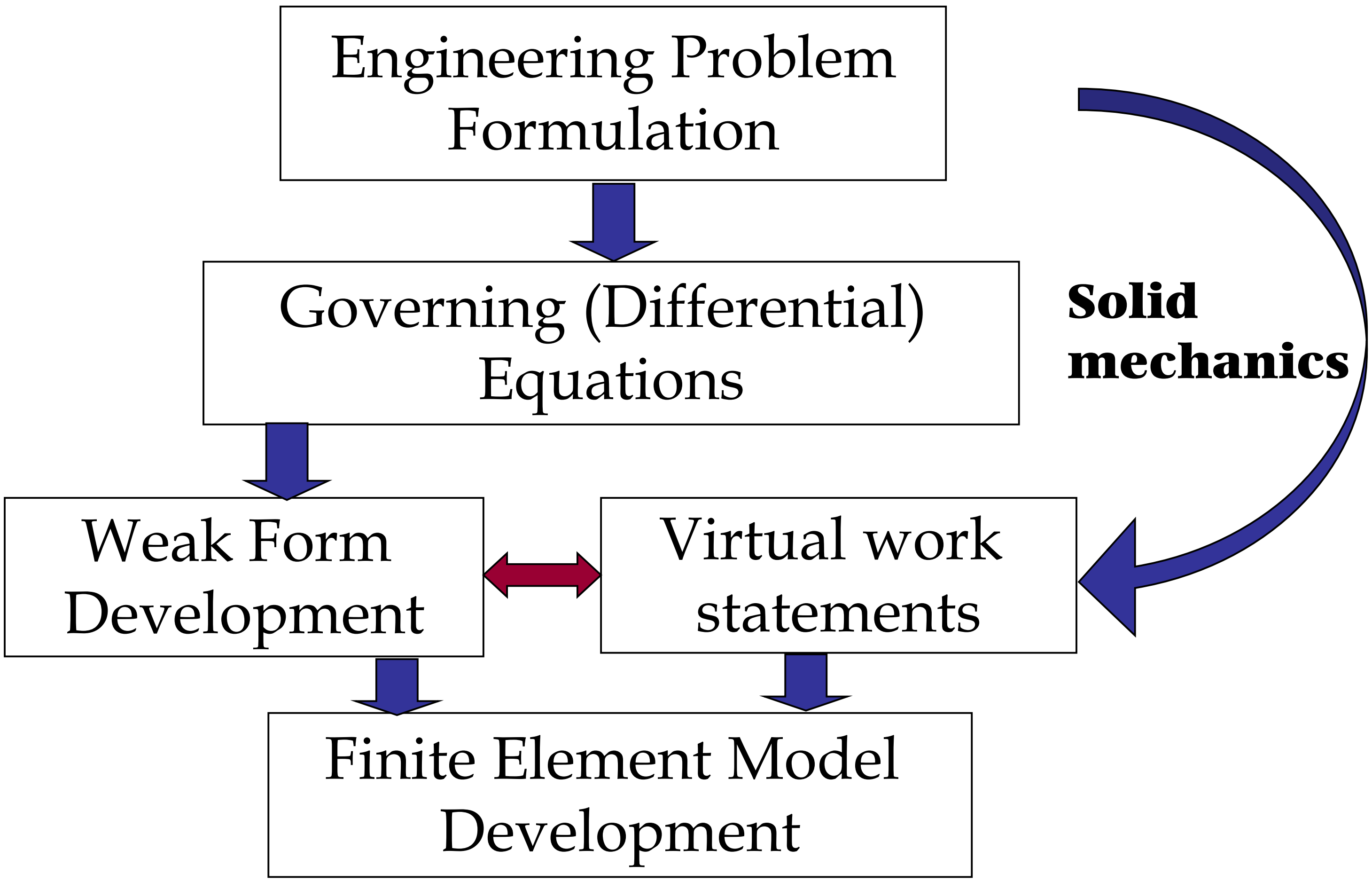
FEM Terminology (continued)

- ***Finite Element Model*** A set of algebraic equations relating the nodal values of the primary variables (e.g., displacements) to the nodal values of the secondary variables (e.g., forces) in an element.
- *Finite element model* is NOT the same as the *finite element method.* There is only one finite element method but there can be more than one finite element model of a problem (or mathematical model).
- ***Numerical Simulation*** Evaluation of the mathematical model (i.e., solution of the governing equations) using a numerical method and computer.

Major Steps of Finite Element Model Development

- Begin with the *governing equations* of the problem
- Develop its ***weak form*** over a typical finite element
- *Approximate* the solution over each finite element
- Obtain algebraic relations among the *quantities of interest* over each finite element (i.e., finite element model)

Major Steps of Finite Element Model Development



FINIE ELEMENT ANALYSIS OF
1D Problems Governed by Second-Order Equation

CONTENTS

- Model differential equation
- Finite element approximation
- Finite element discretization
- Development of weak form and the definition of primary and secondary variables (duality)
- Finite element model
- Interpolation functions
- Assembly of elements
- Numerical examples

Model Differential Equation Boundary Conditions and data

Governing equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$

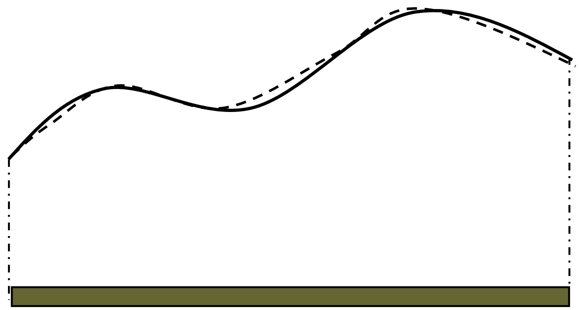
Boundary conditions

$$u=\hat{u} \quad \text{or} \quad a\frac{du}{dx}+b(u-u_0)=P \text{ at a boundary point}$$

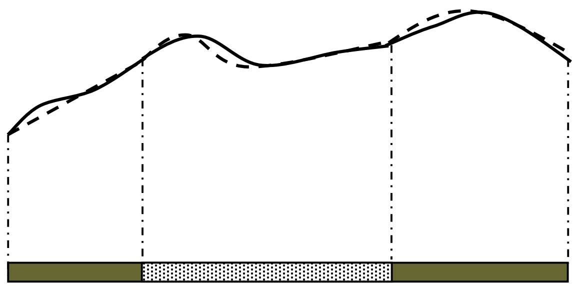
Data (i.e., information you need to solve the problem)

$$L,\ a(x),\ c(x),\ f(x),\ b,\ \hat{u},\ u_0,\ P$$

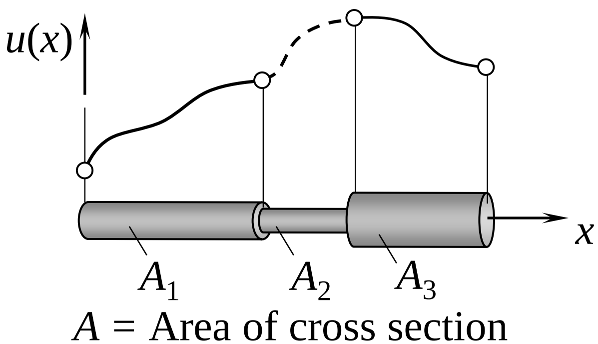
Finite Element Approximation (Need for Seeking Solution on Sub-intervals)

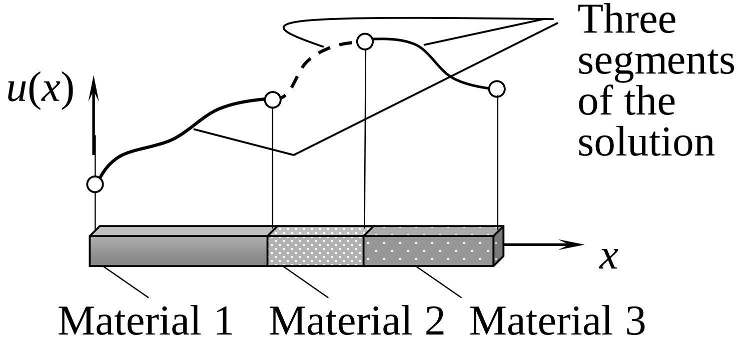
1. 

Approximation of the actual solution over the entire domain requires higher-order (or degree) polynomials.

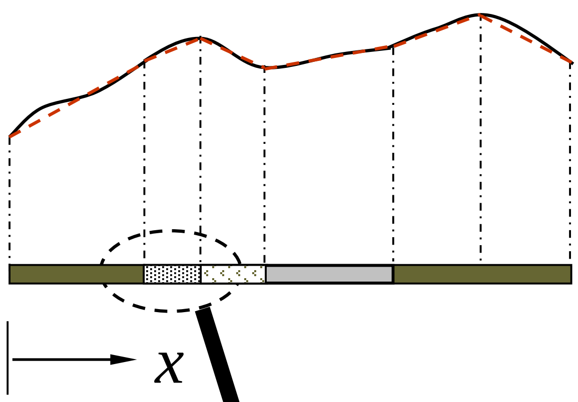
2. 

Actual solution may be defined by piecewise continuous functions because of discontinuity of the data.



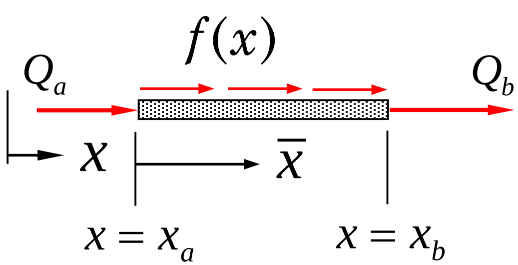


FINITE ELEMENT DISCRETIZATION



Approximation over subintervals (or elements) allows lower-order approximation of the actual solution

A "free-body-diagram" of typical element (geometry and 'forces')



Q_a, Q_b end forces or heats (scalar)

$h = x_b - x_a$ = element length

Approximate Solution and Residual of the Approximation

Approximate solution: $u(x) \approx u_h(x)$

$$\left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)+c(x)u_h - f(x)\right] = R(x) \neq 0$$

We seek to make $R(x)$ zero in a weighted-residual sense:

$$0 = \int_{x_a}^{x_b} w_i \; R(x)\, dx,$$

w_i is the weight function from a set of weight functions $\{w_i\}$

$$= \int_{x_a}^{x_b} w_i \left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)+c(x)u_h - f(x)\right] dx$$

Trading of Differentiation between the weight function and the variable

Product rule of differentiation (an identity) and integration-by-parts

$$\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right) = \frac{dw_i}{dx}a(x)\frac{du_h}{dx} + w_i\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)$$

$$-w_i\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right) = -\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right) + \frac{dw_i}{dx}a(x)\frac{du_h}{dx}$$

$$\int_{x_a}^{x_b} -w_i\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)dx = \int_{x_a}^{x_b}\left[-\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right) + \frac{dw_i}{dx}a(x)\frac{du_h}{dx}\right]dx$$

$$= \int_{x_a}^{x_b} a(x)\frac{dw_i}{dx}\frac{du_h}{dx}\,dx - \left(w_i\, a(x)\frac{du_h}{dx}\right)_{x_a}^{x_b}$$

Identification of the Primary and Secondary Variables

Examine the boundary term(s) obtained during integration-by-parts:

$$\left[w_i \cdot a \frac{du_h}{dx} \right]_{x_a}^{x_b}$$

Secondary variable

The expression always contains the weight function w and a coefficient that depends on the dependent unknown. In this case the coefficient is $a(du_h/dx)$. We will term the coefficient a secondary variable (a name we choose to give).

The weight function w_i in the boundary term when replaced with the dependent variable u_h of the problem is termed the primary variable.

$u_h \longrightarrow$ Primary variable

$$\left[w_i \cdot a \frac{du_h}{dx} \right]_{x_a}^{x_b}$$

WEAK FORM DEVELOPMENT OVER AN ELEMENT

$$0 = \int_{x_a}^{x_b} w_i \left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx} \right) + c(x)u_h - f(x) \right] dx \qquad w - \text{weight function}$$

$$= \int_{x_a}^{x_b} \left[a\frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h - w_i f \right] dx - \left[w_i \cdot a\frac{du_h}{dx} \right]_{x_a}^{x_b} \quad \text{secondary variable}$$

$$= \int_{x_a}^{x_b} \left[a\frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h - w_i f \right] dx - w_i(x_a) \cdot \left(-a\frac{du_h}{dx} \right)_{x_a} - w_i(x_b) \cdot \left(a\frac{du_h}{dx} \right)_{x_b}$$

$$= \int_{x_a}^{x_b} \left[a\frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h - w_i f \right] dx - w_i(x_a) Q_{\mathbf{a}} - w_i(x_b) \cdot Q_b$$

Final Weak Form

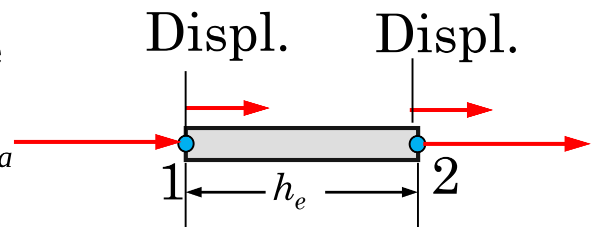
$$\boxed{0 = \int_{x_a}^{x_b} \left[a\frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h - w_i f \right] dx - w_i(x_a) Q_{\mathbf{a}} - w_i(x_b) \cdot Q_b}$$

Interpretation of the secondary variables

Axial deformation of a bar

Force

$$\left(-a\frac{du_h}{dx}\right)_{x_a} \equiv Q_a$$



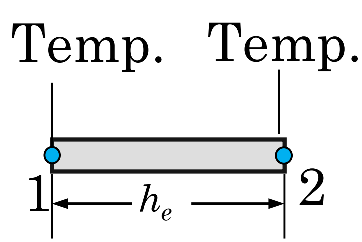
Force

$$\left(a\frac{du_h}{dx}\right)_{x_b} \equiv Q_b$$

Axial heat flow

Heat input

$$\left(-a\frac{du_h}{dx}\right)_{x_a} \equiv Q_a$$



Heat input

$$\left(a\frac{du_h}{dx}\right)_{x_b} \equiv Q_b$$

Primary and Secondary Variables
(Some Remarks)

Primary variables and secondary variables always appear in pairs. They are like `cause' and `effect' (i.e., one is the result of the other). For example, when u_h is the temperature, $a(du_h/dx)$ is heat (and heat causes temperature). When u_h is the displacement, $a(du_h/dx)$ is the force. This **duality** exists in every engineering problem.

Essential and Natural Boundary Conditions

Essential Boundary Conditions: Specifying a primary variable at a boundary point of the domain is called an essential (or Dirichlet) boundary condition.

Natural Boundary Conditions: Specifying a secondary variable at a boundary point of the domain is called a natural (or Neumann) boundary condition.

Linear, Bilinear Forms, and the Variational Problem

Weak Form

$$0 = \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + c w_i u_h - w_i f \right] dx - w_i(x_a) Q_a - w_i(x_b)\cdot Q_b$$

$$= \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + c w_i u_h \right] dx - \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b)\cdot Q_b \right]$$

$$= B(w_i, u_h) - l(w_i)$$

Bilinear Form and Linear Form

$$B(w_i, u_h) = \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + c w_i u_h \right] dx, \quad l(w_i) = \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b)\cdot Q_b \right]$$

Variational Problem: Find u such that

$$B(w_i, u_h) = l(w_i) \quad \text{holds for all} \quad w_i$$

Equivalence Between Minimum of a the Total Potential Energy and Weak Form

Total potential energy (of uniaxial members):

$$\Pi = U + W_E$$

$$= \frac{1}{2}\int_{x_a}^{x_b}\left[EA\left(\frac{du}{dx}\right)^2 + c_f u^2\right]dx - \left(\int_{x_a}^{x_b} uf\ dx + \sum_{i=1}^{n} u_i^e Q_i^e\right)$$

$$\delta\Pi = \int_{x_a}^{x_b}\left[EA\left(\frac{d\delta u}{dx}\right)\left(\frac{du}{dx}\right) + c_f \delta u\, u\right]dx - \left(\int_{x_a}^{x_b} \delta u\, f\ dx + \sum_{i=1}^{n} \delta u_i^e\, Q_i^e\right)$$

Now let $\delta u = w_i$. Then $\delta\Pi = 0$ gives the weak form:

$$0 = \int_{x_a}^{x_b}\left[EA\left(\frac{dw_i}{dx}\right)\left(\frac{du}{dx}\right) + c_f w_i\, u\right]dx - \left(\int_{x_a}^{x_b} w_i\, f\ dx + \sum_{j=1}^{n} w_i(x_j) Q_j^e\right)$$

Equivalence Between Minimum of a Quadratic Functional and Weak Form

Replace w with δu

$$0 = \int_{x_a}^{x_b} \left[a \frac{d\delta u}{dx}\frac{du}{dx} + c\delta u\, u - \delta u\, f \right] dx - \delta u(x_a) Q_a - \delta u(x_b) Q_b$$

$$= \int_{x_a}^{x_b} \left[a \frac{d\delta u}{dx}\frac{du}{dx} + c\delta u\, u \right] dx - \left[\int_{x_b}^{x_a} \delta u\, f\, dx + \delta u(x_a) Q_a + \delta u(x_b) Q_b \right]$$

$$= \frac{1}{2}\delta \int_{x_a}^{x_b} \left[a \left(\frac{du}{dx}\right)^2 + c\, u^2 \right] dx - \delta \left[\int_{x_b}^{x_a} u\, f\, dx + u(x_a) Q_a + u(x_b) Q_b \right]$$

$$= \delta\left[\tfrac{1}{2} B(u,u) - l(u) \right] = \delta I(u)$$

or [when $B(w,u)$ is bilinear and symmetric]

$$\boxed{I(u) = \tfrac{1}{2} B(u,u) - l(u)}$$

FINITE ELEMENT MODEL

(is a set of algebraic relations between the primary and the secondary variables at the nodes)

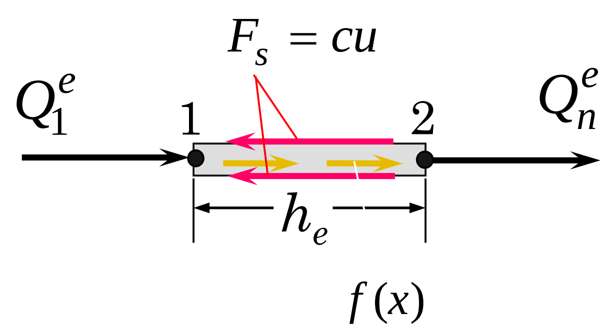
Finite element approximation $u(x) \approx u_h^e(x) = \sum_{j=1} u_j^e \psi_j^e(x)$

$w_i(x) = \psi_i^e(x)$

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw_i}{dx} \frac{du_h}{dx} + c w_i u_h \right) dx - \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b) \cdot Q_b \right]$$

$$= \sum_{j=1}^{n} u_j \int_{x_a}^{x_b} \left[a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c \psi_i \psi_j \right] dx - \left[\int_{x_b}^{x_a} \psi_i f \, dx + \psi_i(x_a) Q_1^e + \psi_i(x_b) Q_n^e \right]$$

$$\sum_{j=1}^{n} K_{ij}^e u_j^e = F_i^e \quad \Rightarrow \quad [K^e]\{u^e\} = \{F^e\}$$

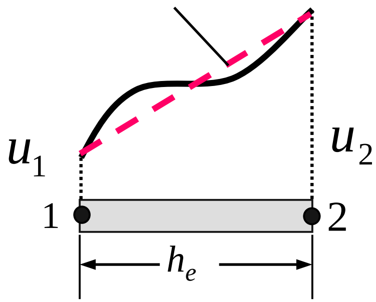


$$K_{ij}^e = \int_{x_a}^{x_b} \left(a_e \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx,$$

$$F_i^e = \int_{x_a}^{x_b} f_e \psi_i \, dx + \psi_i(x_1^e) Q_1^e + \psi_i(x_2^e) Q_2^e + \cdots + \psi_i(x_n^e) Q_n^e$$

Approximation (interpolation or shape) Functions for *Linear* Element

$$u(x) \approx u_h(x) = c_1 + c_2 x$$



$$u_h(x) = c_1 + c_2 x$$

Rewrite c_1 and c_2 in terms of u_1 and u_2

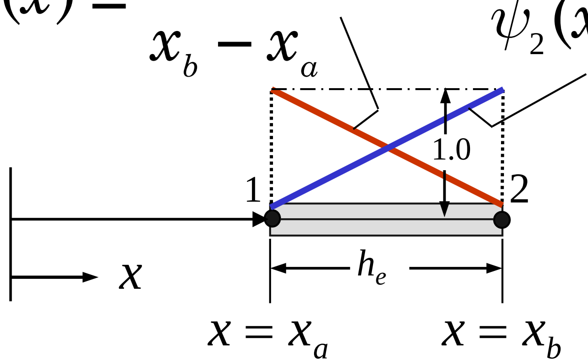
$$u_h(x_b) \equiv u_b = u_2 \quad u_h(x_a) \equiv u_a = u_1$$

$$u_h(x_a) = u_1 = c_1 + c_2 x_a; \quad u_h(x_b) = u_2 = c_1 + c_2 x_b$$

$$\rightarrow c_1 = \frac{u_1 x_b - u_2 x_a}{x_b - x_a}, \quad c_2 = \frac{u_2 - u_1}{x_b - x_a}$$

$$\psi_1(x) \equiv \frac{x_b - x}{x_b - x_a}$$

$$\psi_2(x) \equiv \frac{x - x_a}{x_b - x_a}$$



$$u(x) \approx u_h(x) = c_1 + c_2 x$$

$$= \frac{u_1 x_b - u_2 x_a}{x_b - x_a} + \frac{u_2 - u_1}{x_b - x_a} x$$

$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

Interpolation Properties of the Approximation Functions

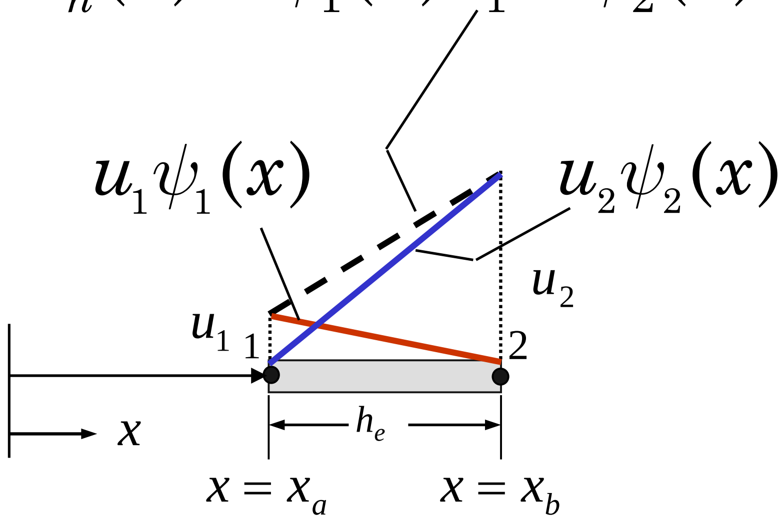
$$\psi_i(x_j) \equiv \delta_{ij} = \begin{cases} 0, & if \quad i \neq j \\ 1, & if \quad i = j \end{cases} \qquad x_1 \equiv x_a\,, \;\; x_2 \equiv x_b$$

Interpolation property

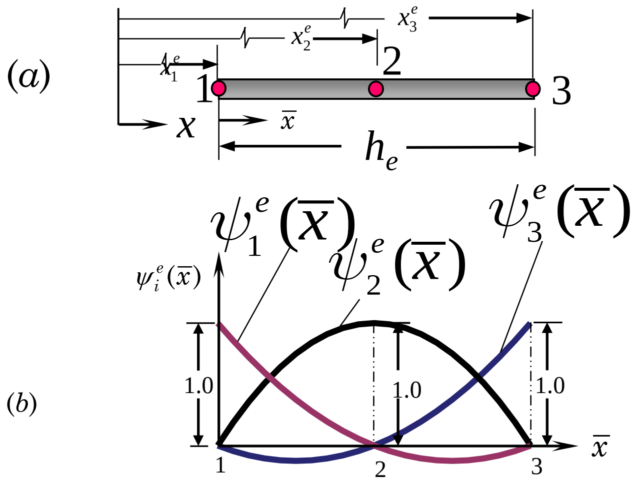
$$u_h(x) = \psi_1(x)u_1 + \psi_2(x)u_2$$

Partition of unity

$$\sum_{j=1}^{n} \psi_j(x) = 1$$



Derivation of Approximation Functions (*Quadratic* Element)



$$u(x) \approx u_h(x) = c_1 + c_2 x + c_3 x^2$$

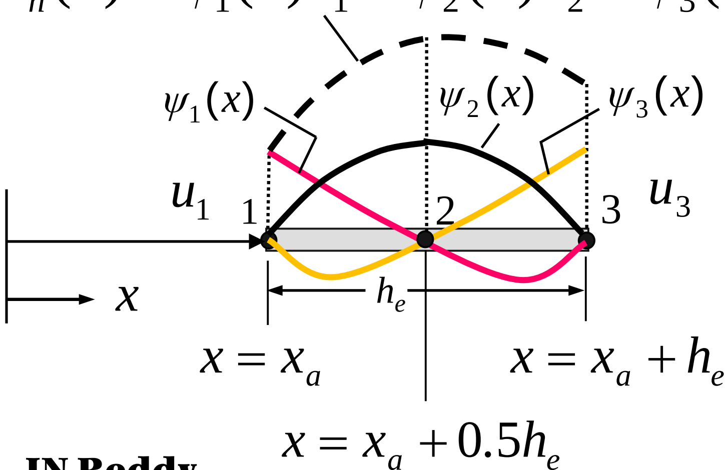
Derivation using the interpolation property

$$\psi_1(\bar{x}) = \alpha_1 (h - \bar{x})(0.5h - \bar{x}),\ \psi_1(0) = 1 \to \alpha_1 = \frac{2}{h^2}$$

$$\psi_2(\bar{x}) = \alpha_2 (h - \bar{x})(\bar{x} - 0),\ \psi_2(0.5h) = 1 \to \alpha_2 = \frac{4}{h^2}$$

$$\psi_3(\bar{x}) = \alpha_3 (\bar{x} - 0)(0.5h - \bar{x}),\ \psi_3(h) = 1 \to \alpha_3 = -\frac{2}{h^2}$$

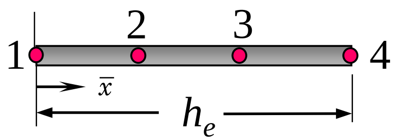
$$u_h(x) = \psi_1(x) u_1 + \psi_2(x) u_2 + \psi_3(x) u_3$$



$$\psi_1(\bar{x}) = \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{2\bar{x}}{h}\right),\ \psi_2(\bar{x}) = 4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right)$$

$$\psi_3(\bar{x}) = -\frac{\bar{x}}{h}\left(1 - \frac{2\bar{x}}{h}\right)$$

Derivation of Approximation Functions
(*Cubic* Element)



$$u(x) \approx u_h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

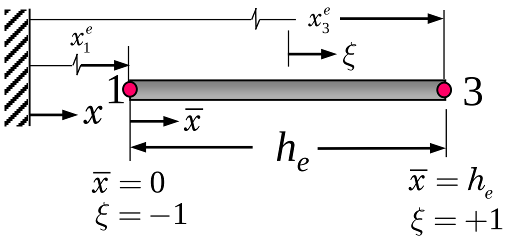
Derivation using the interpolation property

$$\psi_1(\bar{x}) = \alpha_1 \left(\bar{x} - \frac{h}{3}\right)\left(\bar{x} - \frac{2h}{3}\right)(\bar{x} - h)$$

$$\psi_1(0) = 1 \rightarrow \alpha_1 = -\frac{9}{2h^3}$$

$$\psi_1(\bar{x}) = \left(1 - \frac{3\bar{x}}{h}\right)\left(1 - \frac{3\bar{x}}{2h}\right)\left(1 - \frac{\bar{x}}{h}\right)$$

Approximation Functions in terms of the local coordinates



Relations between various coordinates

$$x = \bar{x} + x_1^e$$

$$x = x_1^e \tfrac{1}{2}(1-\xi) + x_2^e \tfrac{1}{2}(1+\xi)$$

$$= \frac{x_1^e + x_2^e}{2} + \frac{h_e}{2}\xi$$

$$\left.\begin{aligned} \psi_1(x) &= \frac{1}{h_e}(x_2^e - x), \\ \psi_2(\bar{x}) &= \frac{1}{h_e}(x - x_1^e) \end{aligned}\right\} \quad x_1^e < x < x_2^e$$

$$\psi_1(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \;\; \psi_2(\bar{x}) = \frac{\bar{x}}{h_e}, \qquad 0 < \bar{x} < h_e$$

$$\psi_1(\xi) = \tfrac{1}{2}(1-\xi), \;\; \psi_2(\bar{x}) = \tfrac{1}{2}(1+\xi), \;\; -1 < \xi < 1$$

NUMERICAL EVALUATION OF COEFFICIENTS in various coordinates

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left(a(x)\frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c(x)\psi_i(x)\psi_j(x) \right) dx$$

$$= \int_0^{h_e} \left(a(\bar{x})\frac{d\psi_i}{d\bar{x}}\frac{d\psi_j}{d\bar{x}} + c(\bar{x})\psi_i(\bar{x})\psi_j(\bar{x}) \right) d\bar{x}$$

$$F_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx + \ Q_i = \int_0^{h_e} f(\bar{x})\psi_i(\bar{x})\, d\bar{x} + \ Q_i$$

$$F_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx + \ Q_i = \int_{-1}^{+1} f(\xi)\psi_i(\xi)\, Jd\xi + \ Q_i$$

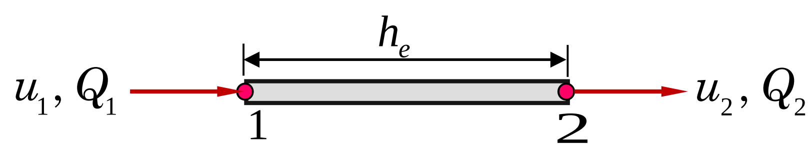
$$J = \frac{dx}{d\xi} = \frac{h_e}{2}$$

NUMERICAL EVALUATION OF COEFFICIENTS
for element-wise constant data

$$\text{For constant}\,data: \quad a = a_e,\, c = c_e,\, f = f_e$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c\psi_i\psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f\psi_i \, dx + Q_i$$

Linear element:



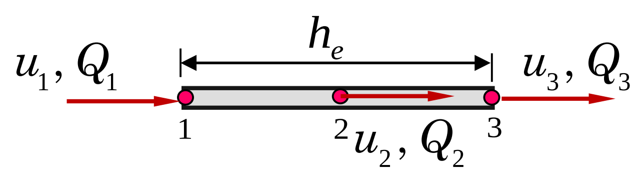
$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

NUMERICAL EVALUATION OF COEFFICIENTS
for element-wise constant data

$$\text{For constant } data: \quad a = a_e,\ c = c_e,\ f = f_e$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c\psi_i\psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f\psi_i \, dx + Q_i$$

Quadratic element:

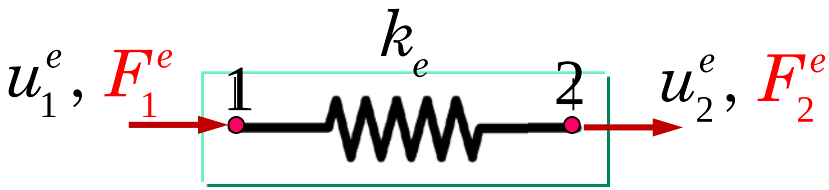


known

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

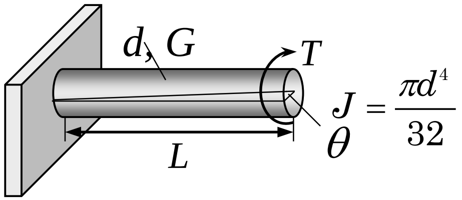
Other (Discrete) Elements

Spring element:

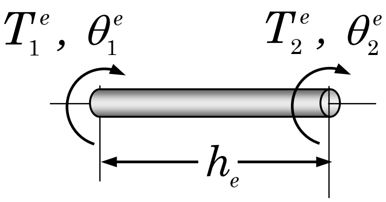


$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix}$$

Torsion element:



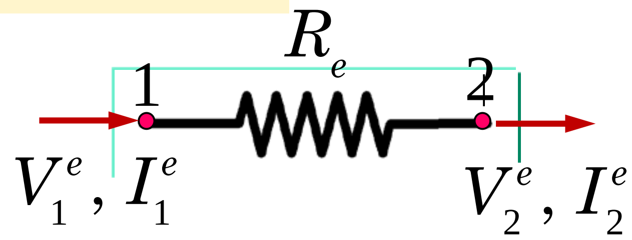
$$[K^e] = \frac{G_e J_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix}$$



Other (Discrete) Elements

(continued)

Electrical element: $k_e = \dfrac{1}{R_e}$

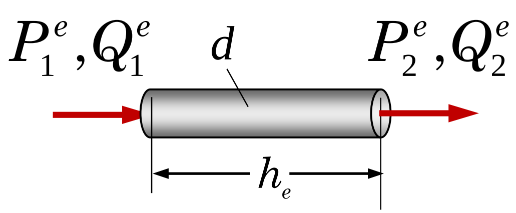


$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} I_1^e \\ I_2^e \end{Bmatrix}$$

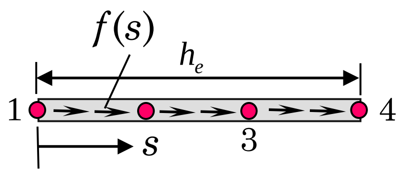
R_e = Electrical resistance

Pipe flow element: $k_e = \dfrac{\pi d^4}{128 \mu h_e}$

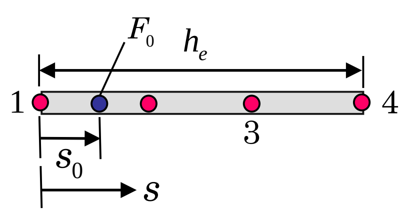
$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$



Representation of Point Sources at points other than nodes

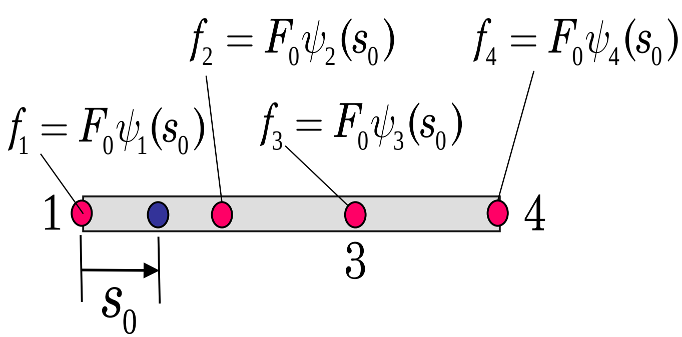


$$f_i^e = \int_0^h f(s)\,\psi_i(s)\,ds$$

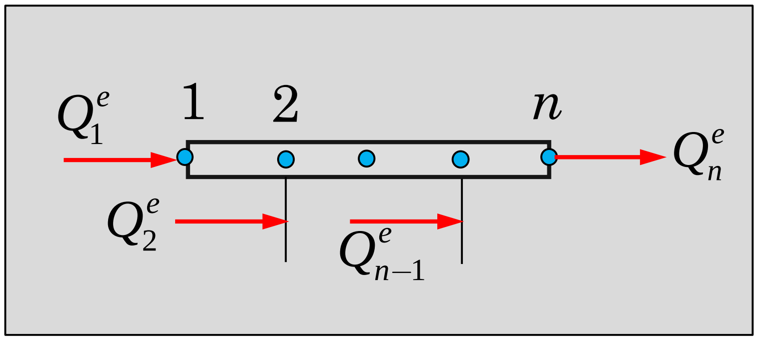


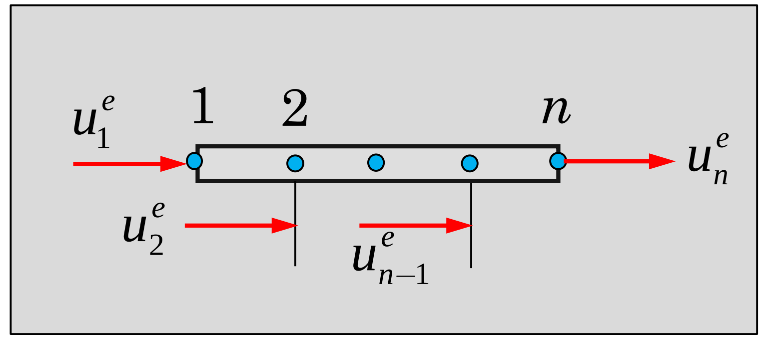
$$f(s) = F_0\,\delta(s - s_0), \qquad f_i^e = \int_0^h f(s)\,\psi_i(s)\,ds$$

$$f_i^e = \int_0^h F_0\,\delta(s - s_0)\,\psi_i(s)\,ds = F_0\psi_i(s_0)$$



Equivalence Between Minimum of a the Total Potential Energy and Weak Form





Strain energy:

$$U = \frac{1}{2}\sum_{i,j=1}^{3}\int_{V^e}\sigma_{ij}\,\varepsilon_{ij}\;dv = \frac{1}{2}\int_{V^e}\sigma_{xx}\,\varepsilon_{xx}\;dv = \frac{1}{2}\int_{V^e}E\left(\varepsilon_{xx}\right)^2\;dv$$

$$= \frac{1}{2}\int_{V^e}E\left(\frac{du}{dx}\right)^2\;dv = \frac{1}{2}\int_{x_a}^{x_b}EA\left(\frac{du}{dx}\right)^2\;dx$$

Work done by external forces:

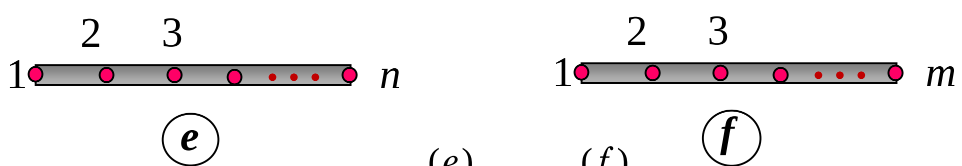
$$W_E = -\left(\int_{x_a}^{x_b}uf\;dx + \sum_{i=1}^{n}u_i^e Q_i^e\right)$$

ASSEMBLY OF ELEMENTS

Assembly of elements is based on two requirements:

- **Continuity** of the primary variables across the element boundaries.
- **Balance** of the secondary variables between the element boundaries.

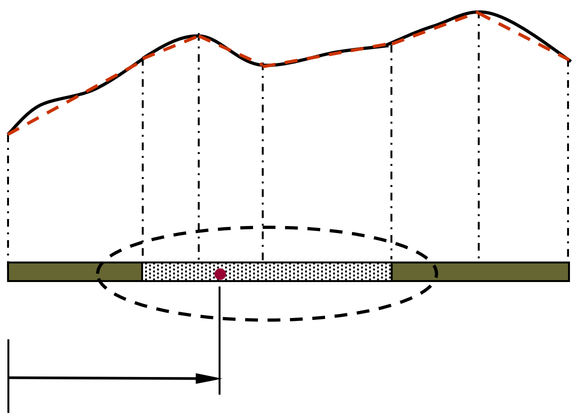
(1)



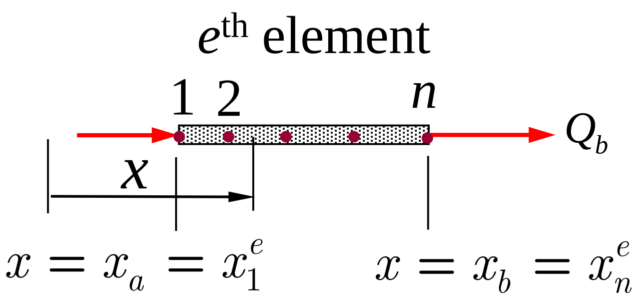
$$u_n^{(e)} = u_1^{(f)}$$

(2) $Q_n^{(e)} + Q_1^{(f)} = 0$ or equal to externally applied source

POST-COMPUTATION OF VARIABLES



$$u_h^e = \begin{cases} \sum_{j=1}^{n} u_j^{(1)} \psi_j^{(1)}(x), & x_1^{(1)} \le x \le x_n^{(1)} \\ \sum_{j=1}^{n} u_j^{(2)} \psi_j^{(2)}(x), & x_1^{(2)} \le x \le x_n^{(2)} \\ \cdots\cdots & \cdots\cdots \quad \cdots\cdots \\ \sum_{j=1}^{n} u_j^{(N)} \psi_j^{(N)}(x), & x_1^{(N)} \le x \le x_n^{(N)} \end{cases}$$



$$\frac{du_h^e}{dx} = \begin{cases} \sum_{j=1}^{n} u_j^{(1)} \frac{d\psi_j^{(1)}}{dx}, & x_1^{(1)} \le x \le x_n^{(1)} \\ \sum_{j=1}^{n} u_j^{(2)} \frac{d\psi_j^{(2)}}{dx}, & x_1^{(2)} \le x \le x_n^{(2)} \\ \cdots\cdots & \cdots\cdots \quad \cdots\cdots \\ \sum_{j=1}^{n} u_j^{(N)} \frac{d\psi_j^{(1)}}{dx}, & x_1^{(N)} \le x \le x_n^{(N)} \end{cases}$$

A DIFFERENTIAL EQUATION

1. **Problem**: Wish to determine the numerical solution of the differential equation

$$-\frac{d^2u}{dx^2} - u = -x^2 \ \text{in} \ 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

FE Solution: We have the following correspondence compared to the model equation:

$$a = 1, \ c = -1, \ f = -x^2, \ f_i^e = -\int_{x_a}^{x_b} x^2 \psi_i \, dx$$

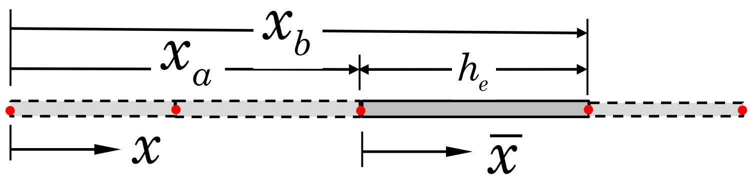
(1) We wish to use a mesh of linear elements to solve the problem. The equations of a typical element are

$$\left(\frac{1}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

Details of the computation of the nodal source vector due to the distributed source

$$f_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx = -\int_{x_a}^{x_b} x^2\psi_i\, dx = -\int_0^{h_e} (\bar{x}+x_a)^2\psi_i(\bar{x})\, d\bar{x}$$

$$\psi_1(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2(\bar{x}) = \frac{\bar{x}}{h_e}$$



$$x = \bar{x} + x_a$$

$$f_i^e = -\int_0^{h_e} (\bar{x}+x_a)^2\psi_i(\bar{x})\, d\bar{x} = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\psi_i(\bar{x})\, d\bar{x}$$

$$f_1^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x}$$

$$f_2^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(\frac{\bar{x}}{h_e}\right) d\bar{x}$$

Details of the computation of the nodal source vector due to the distributed source

$$f_1^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x}$$

$$= -\left[\frac{\bar{x}^3}{3} + \bar{x}^2 x_a + \bar{x}\,x_a^2\right]_0^{h_e} + \frac{1}{h_e}\left[\frac{\bar{x}^4}{4} + \frac{2\bar{x}^3}{3} x_a + \frac{\bar{x}^2}{2} x_a^2\right]_0^{h_e}$$

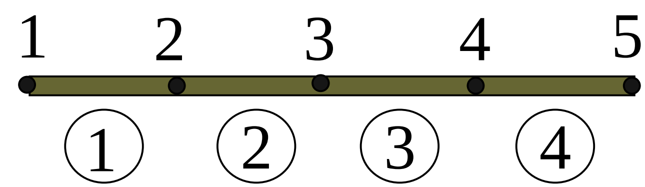
$$= -\left[\frac{h_e^3}{3} + h_e^2 x_a + h_e\,x_a^2\right] + \frac{1}{h_e}\left[\frac{h_e^4}{4} + \frac{2h_e^3}{3} x_a + \frac{h_e^2}{2} x_a^2\right]$$

$$f_2^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(\frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{h_e^4}{4} + \frac{2h_e^3}{3} x_a + \frac{h_e^2}{2} x_a^2\right]$$

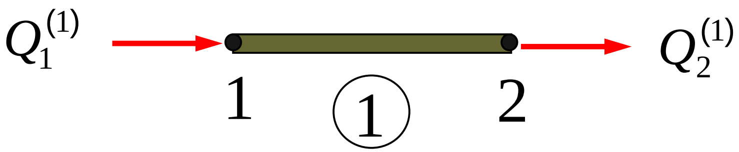
Note that $x_a = 0$ for Element 1, $x_a = h_1$ for Element 2, $x_a = h_1 + h_2$ for Element 3, and $x_a = h_1 + h_2 + h_3$ for Element 4

A DIFFERENTIAL EQUATION (cont.)

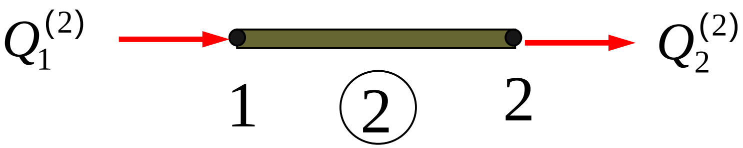
(2) We consider a mesh of 4 linear elements (h =0.25). The element equations are



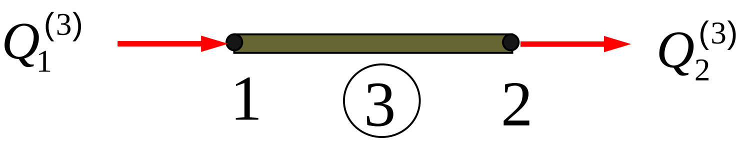
$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = -\begin{Bmatrix} 0.00130 \\ 0.00391 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$



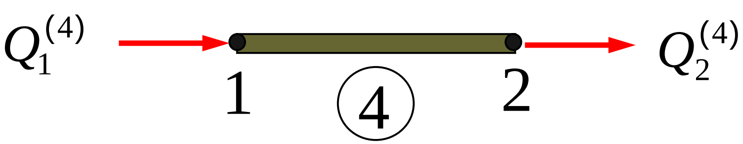
$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = -\begin{Bmatrix} 0.01432 \\ 0.02232 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$



$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = -\begin{Bmatrix} 0.04297 \\ 0.05599 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$



$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix} = -\begin{Bmatrix} 0.08724 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

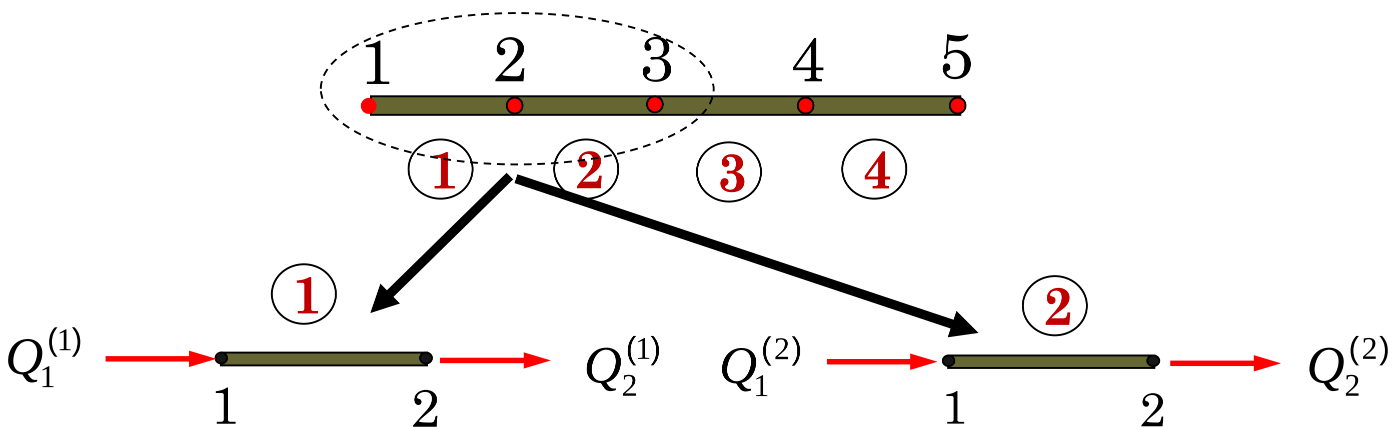


A DIFFERENTIAL EQUATION (cont.)

(3) The boundary conditions on the primary variables are

$$U_1 = 0, \quad U_5 = 0$$

The equilibrium conditions are



$$Q_2^{(1)} + Q_1^{(2)} = 0, \; Q_2^{(2)} + Q_1^{(3)} = 0, \quad Q_2^{(3)} + Q_1^{(4)} = 0$$

A DIFFERENTIAL EQUATION (cont.)

(4) The assembled equations are

$$\frac{1}{24}\begin{bmatrix} 94 & -97 & 0 & 0 & 0 \\ -97 & 188 & -97 & 0 & 0 \\ 0 & -97 & 188 & -97 & 0 \\ 0 & 0 & -97 & 188 & -97 \\ 0 & 0 & 0 & -97 & 94 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = -\begin{Bmatrix} 0.001302 \\ 0.018229 \\ 0.065104 \\ 0.143230 \\ 0.105470 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

$\mathbf{0}$ $\mathbf{0}$ $\mathbf{0}$ $\mathbf{0}$ $\mathbf{0}$

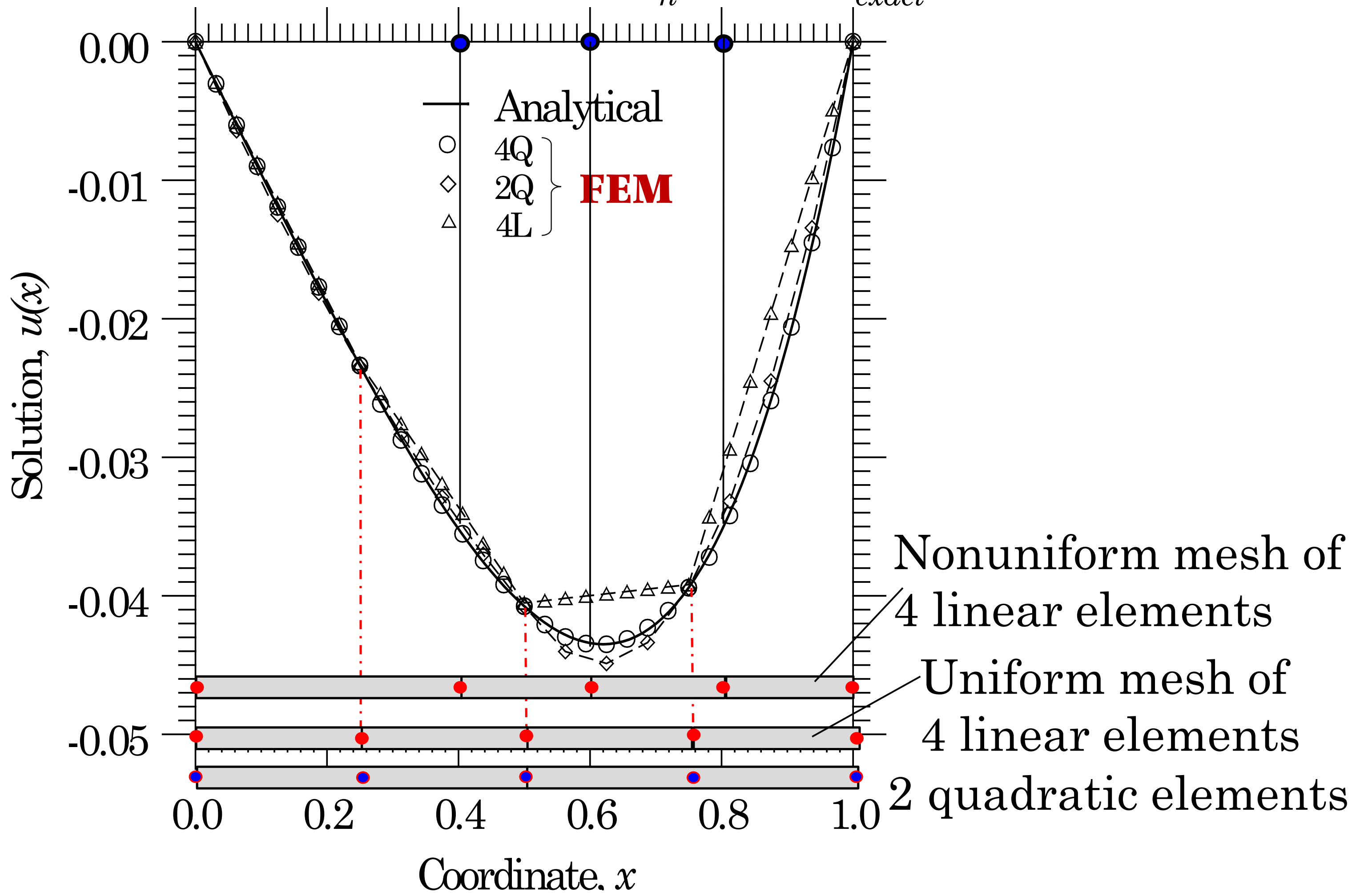
(5) The condensed equations for the unknown *U*'s and *Q*'s are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^1 \\ Q_2^4 \end{Bmatrix} = \begin{bmatrix} -4.0417 & 0 & 0 \\ 0 & 0 & -4.0417 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} 0.00130 \\ 0.10547 \end{Bmatrix} = \begin{Bmatrix} 0.09520 \\ 0.26386 \end{Bmatrix}$$

A DIFFERENTIAL EQUATION (cont.)

Plot of the solution, u_h and u_{exact}



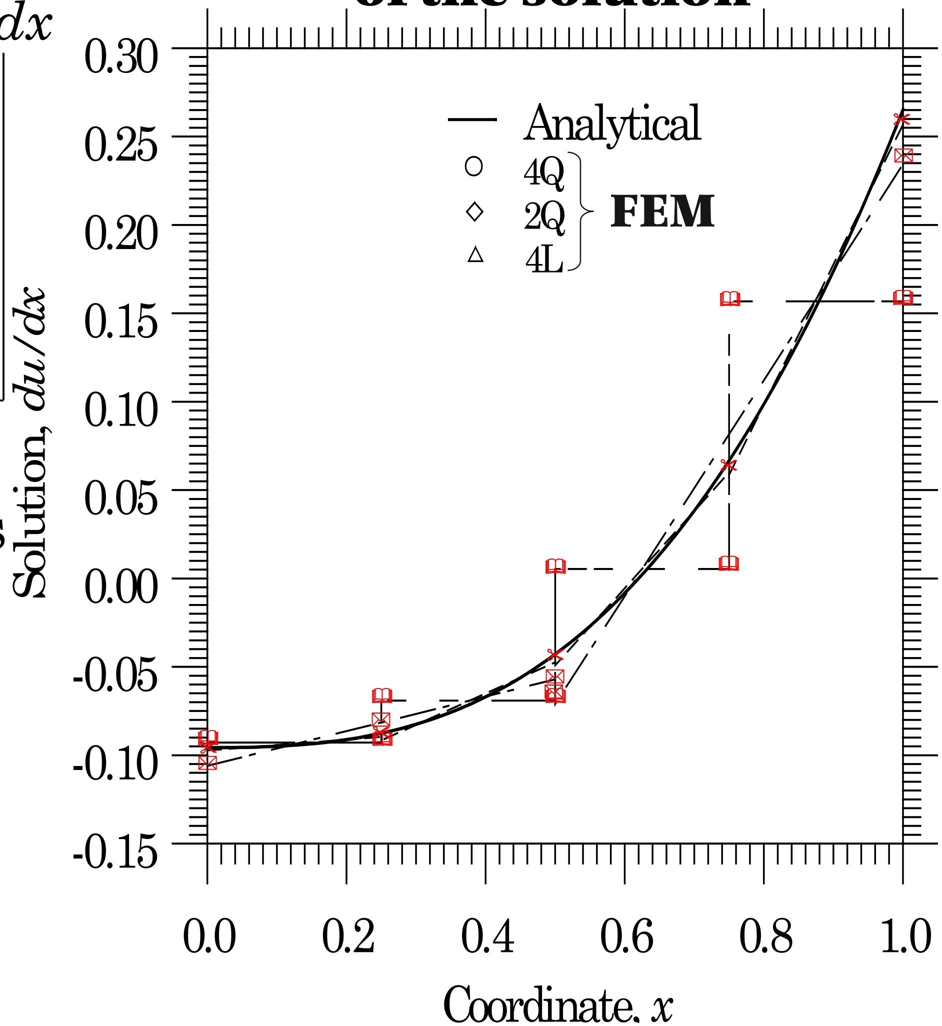
A DIFFERENTIAL EQUATION (cont.)

$$\frac{du_h^e}{dx} = \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \neq \frac{du_h^{e+1}}{dx} = \sum_{j=1}^{n} u_j^{e+1} \frac{d\psi_j^{e+1}}{dx}$$

Same at the nodes common to elements

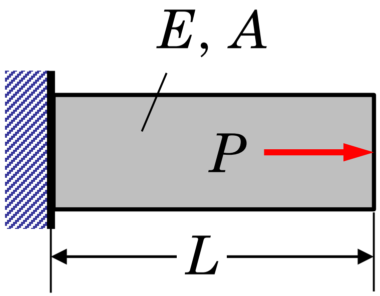
Not the same at the nodes common to elements

Plot of the derivative of the solution



Examples of Uniaxially-Loaded Members

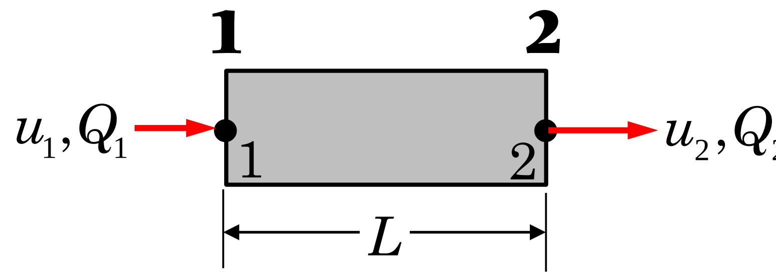
3



Problem

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=0$$

GDE

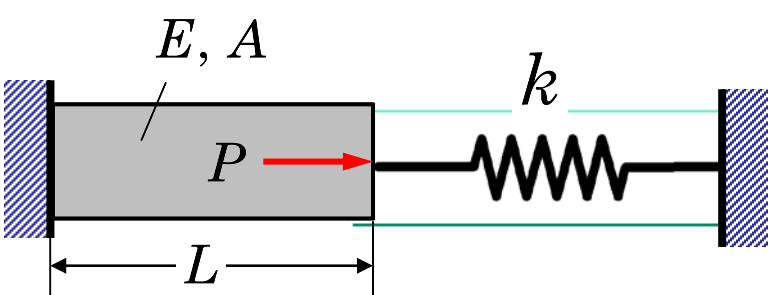


Finite element mesh

Solution:

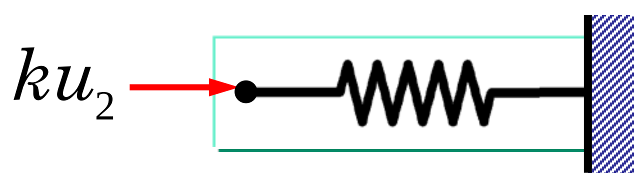
$$\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\cancel{u_1}^{\,0}\\ u_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}+\begin{Bmatrix}Q_1\\ \cancel{Q_2}^{\,P}\end{Bmatrix} \Rightarrow u_2=\frac{PL}{EA},\quad Q_1=-\frac{EA}{L}u_2=-P$$

4



Problem

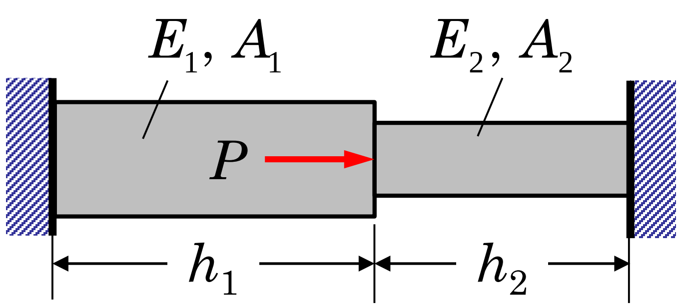
$$\boxed{Q_2=P-ku_2}$$



Spring force

UNIAXIALLY-LOADED MEMBERS (cont.)

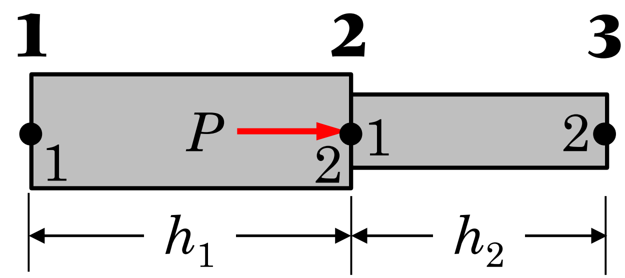
5



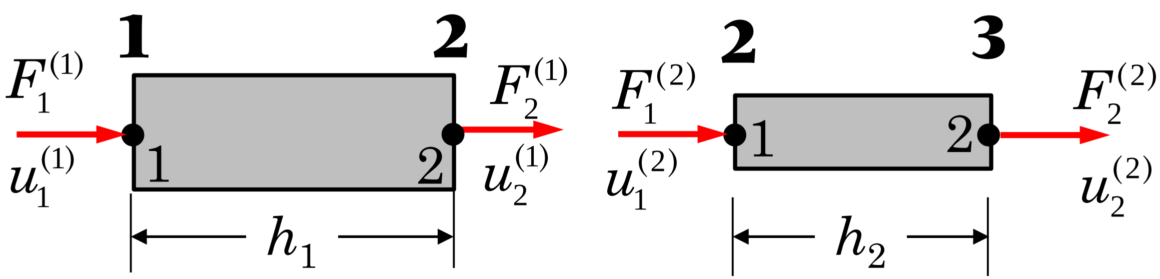
Problem

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=0$$

GDE



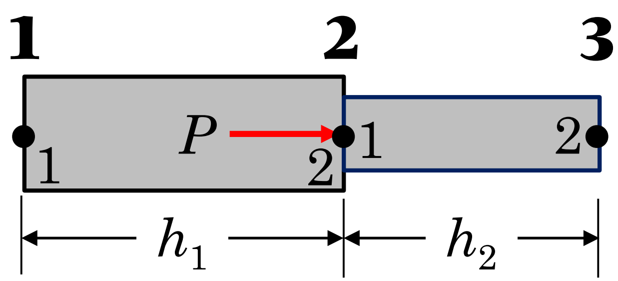
Finite element mesh



$$\begin{cases} F_2^{(1)}+F_1^{(2)}=P \\ u_1^{(1)}=U_1 \\ u_2^{(1)}=u_1^{(2)}=U_2 \\ u_2^{(2)}=U_3 \end{cases}$$

Inter-element compatibility

UNIAXIALLY-LOADED MEMBERS (cont.)



Finite element mesh

$$\begin{bmatrix} \frac{E_1A_1}{h_1} & -\frac{E_1A_1}{h_1} & 0 \\ -\frac{E_1A_1}{h_1} & \frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} \\ 0 & -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix}$$

Boundary conditions: $U_1 = U_3 = 0 \qquad F_2^{(1)} + F_1^{(2)} = P$

Solution:

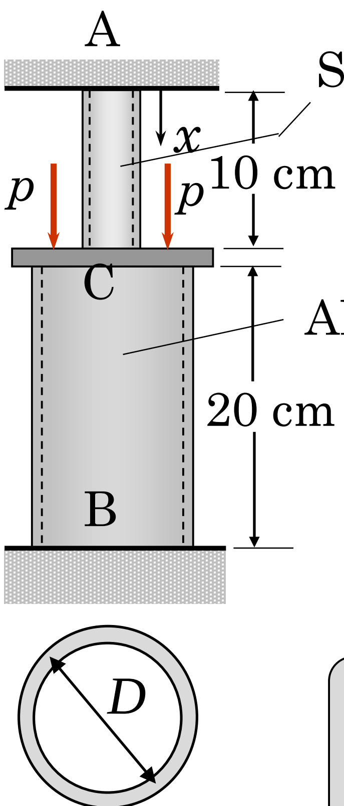
$$U_2 = \frac{P}{\frac{E_1A_1}{h_1} + \frac{E_2A_2}{h_2}}$$

Post-computation:

$$F_1^{(1)} = -\frac{E_1A_1}{h_1}U_2, \quad F_2^{(2)} = -\frac{E_2A_2}{h_2}U_2$$

EXERCISE PROBLEMS FOR DISCUSSION

Problem: Find stresses in each member



Steel ($E_s = 200$ GPa, $A_s = 60$ mm^2)

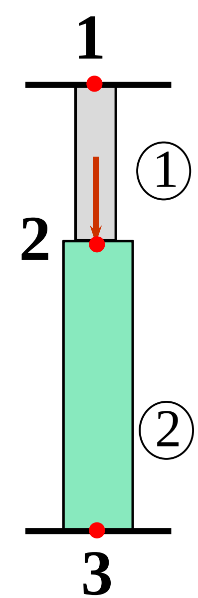
Aluminum ($E_a = 70$ GPa, $A_a = 600$ mm^2)

$$P = \pi D p = 10^5 \text{ N}$$

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

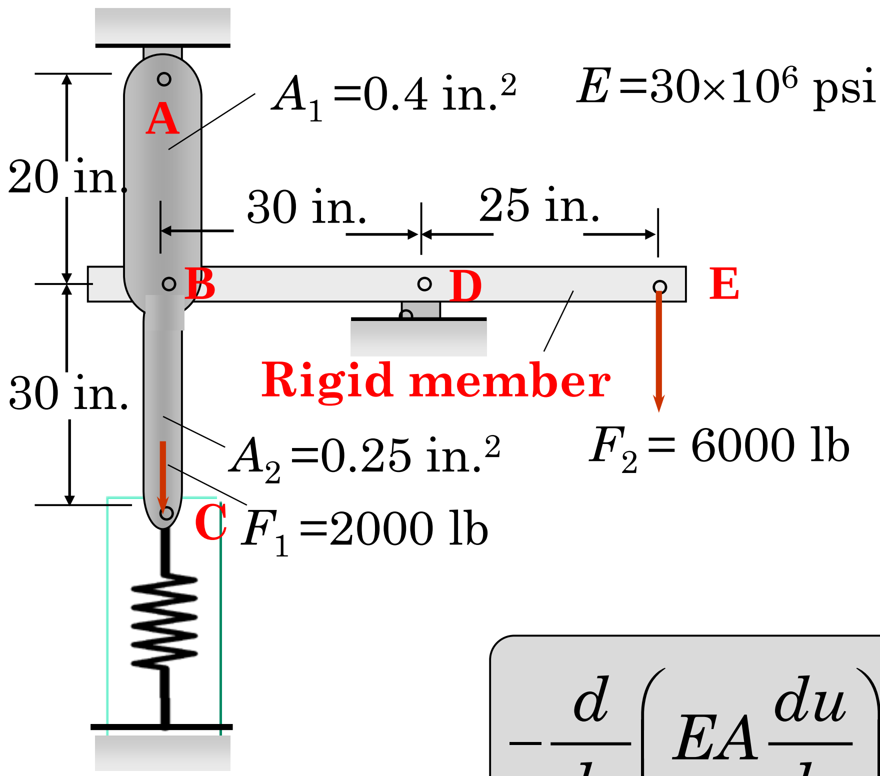
GDE

Problem set up and FEA

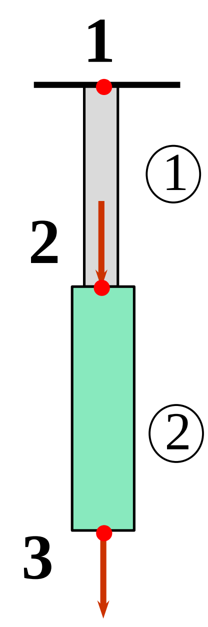


EXERCISE PROBLEMS FOR DISCUSSION (continued)

Problem: Find stresses in each member



Problem set up and FEA



$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

GDE

SUMMARY

Beginning with a model second-order differential equation that arises, for example, in connection with axial deformation of bars, 1-D heat transfer in fins of a heat exchanger, or 1-D flow through pipes and channels, the following steps are used to in the finite element analysis of the problem:

1. Divided the domain into subdomains, called *finite elements*.

2. Over each element, an integral statement, called *weak form*, is developed. The weak form is equivalent to the differential equation as well as specified natural boundary conditions on the boundary of the element.

SUMMARY (continued)

3. Using polynomial approximation of the variables, a system of algebraic equations, called **finite element model**, is developed. The model relates the nodal values of the PVs and the SVs.

4. The element equations are then **assembled** to eliminate excessive unknown SVs by requiring continuity of PVs and balance of SVs at the nodes.

5. The assembled system of equations are then **solved** for the unknown PVs at the nodes by using the known boundary conditions.

6. Post-computation may be used to compute SVs and PVs at points other than nodes. **The SVs are discontinuous between elements.**

FINITE ELEMENT ANALYSIS OF Euler-Bernoulli and Timoshenko Beams

CONTENTS

Euler-Bernoulli beam theory

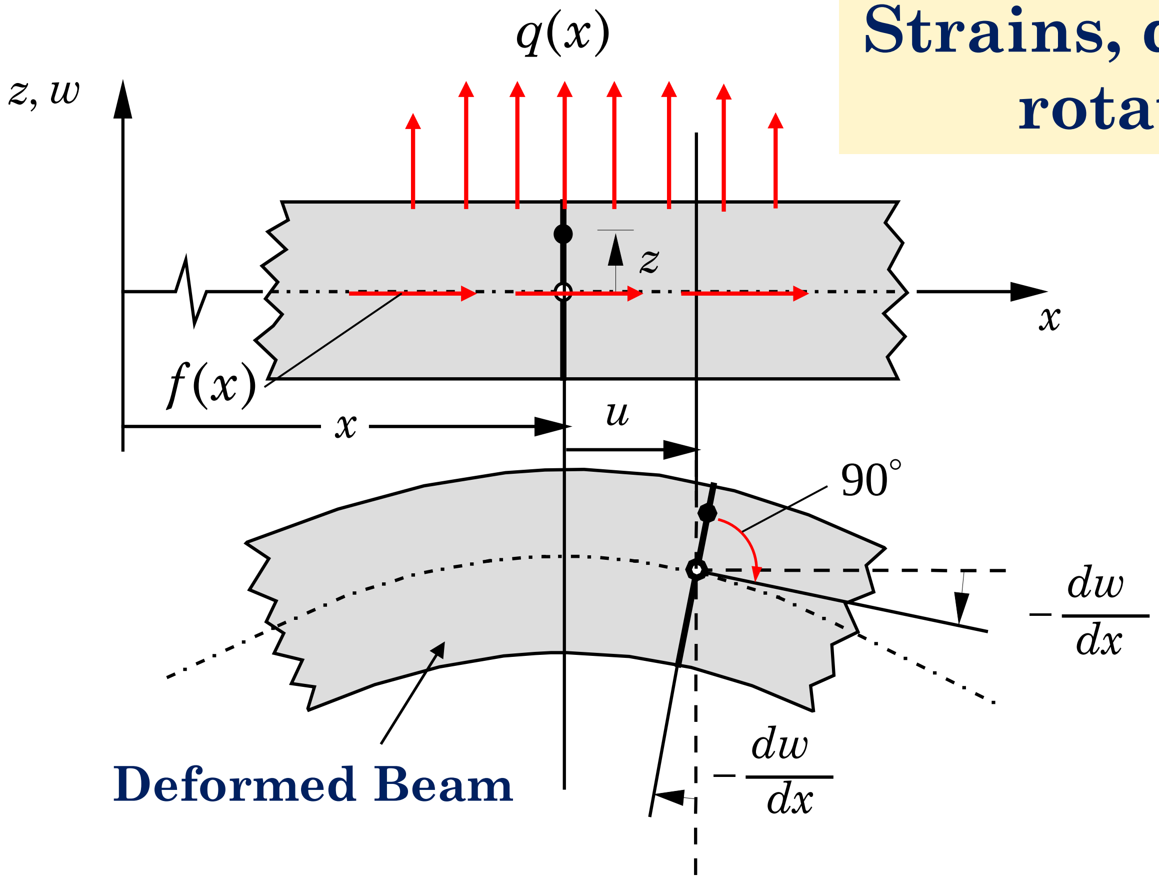
- Governing Equations
- Finite element model
- Numerical examples

Timoshenko beam theory

- Governing Equations
- Finite element model
- Shear locking
- Numerical example

KINEMATICS OF THE LINEARIZED EULER-BERNOULLI BEAM THEORY

Strains, displacements, and rotations are small

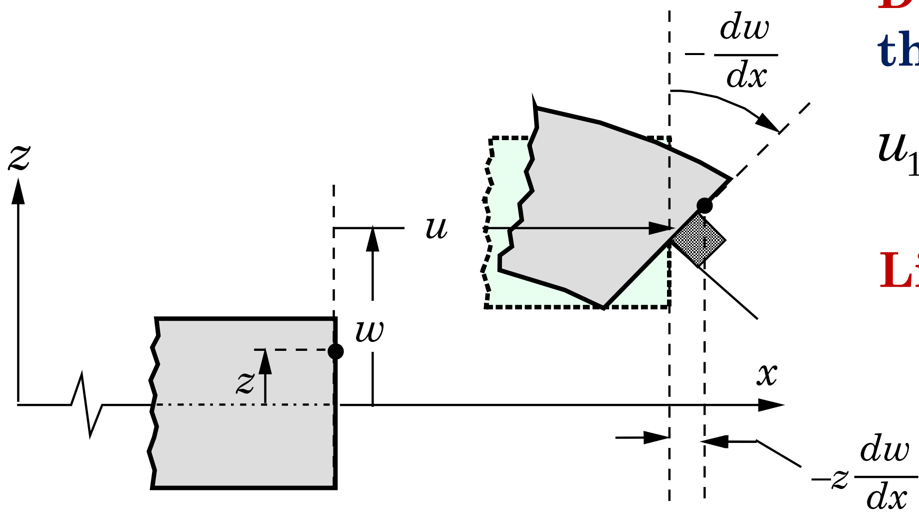


Undeformed Beam

Deformed Beam

Euler-Bernoulli Beam Theory (EBT) is based on the assumptions of *(1) straightness, (2) inextensibility,* and *(3) normality*

Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



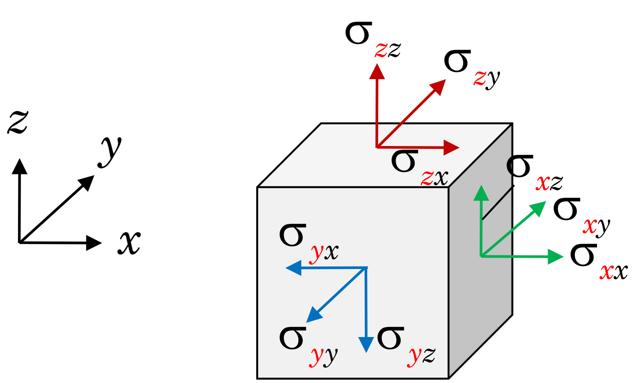
Displacement field (constructed using the hypothesis)

$$u_1(x,z) = u - z\frac{dw}{dx}, \qquad u_2 = 0, \quad u_3 = w(x)$$

Linear strains

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} - z\frac{d^2w}{dx^2},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\frac{dw}{dx} + \frac{dw}{dx} = 0.$$



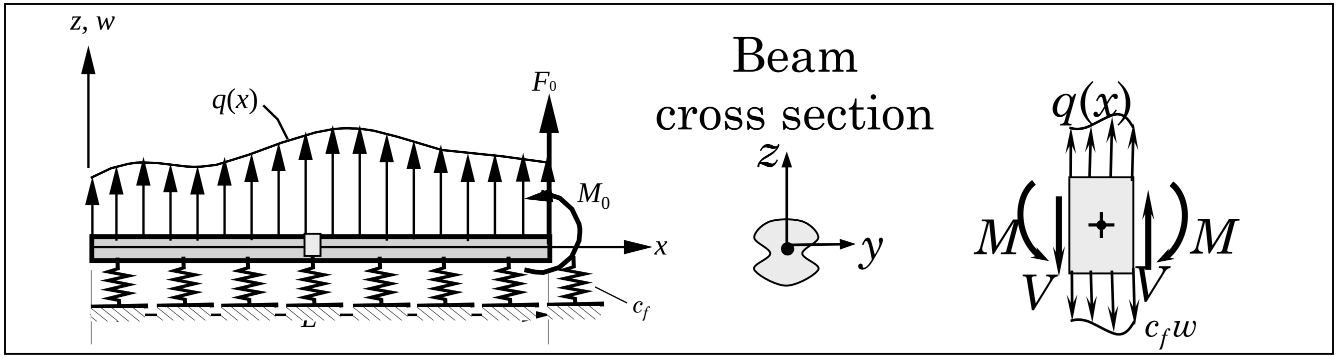
Notation for stress components

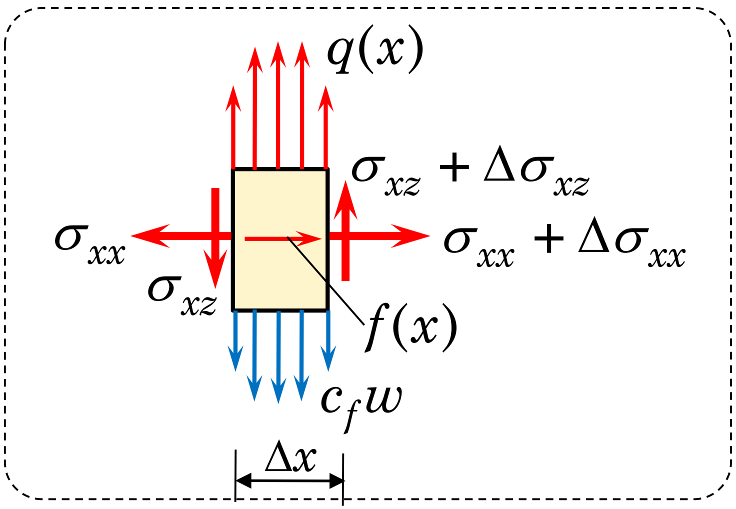
Constitutive relations

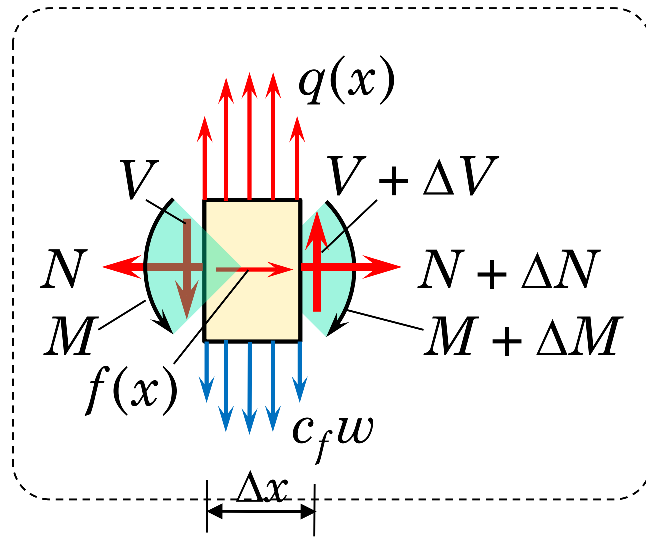
$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx} - Ez\frac{d^2w}{dx^2},$$

$$\sigma_{xz} = G\gamma_{xz} = 0$$

Euler-Bernoulli Beam Theory







Definition of stress resultants

$$N = \int_A \sigma_{xx}\, dA, \;\; M = \int_A \sigma_{xx} \cdot z\, dA, \;\; V = \int_A \sigma_{xz}\, dA.$$

Euler-Bernoulli Beam Theory (Continued)

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + q - c_f w = 0$$

$$\int_A 1 \cdot dA = A, \qquad \int_A z \cdot dA = 0, \qquad \int_A z^2 \cdot dA = I$$

Stress resultants in terms of deflection

$$N = \int_A \sigma_{xx} \, dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) dA = EA\frac{du}{dx}$$

$$M = \int_A \sigma_{xx} \times z \, dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) z \, dA = -EI\frac{d^2w}{dx^2}$$

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2} \right)$$

Euler-Bernoulli Beam Theory (Continued)

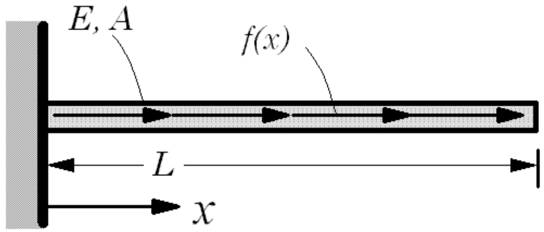
Governing equations in terms of the displacements

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-f=0,\quad 0<x<L$$

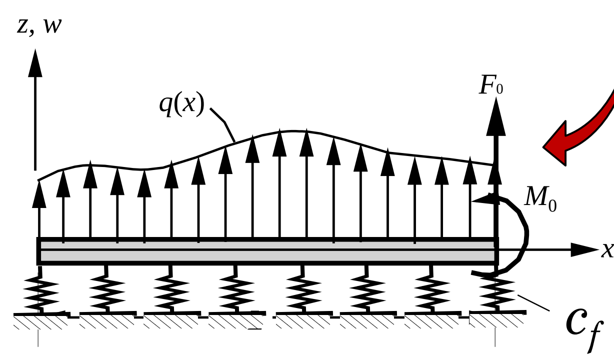
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q=0,\quad 0<x<L$$

Bars u

Beams w



Axial deformation of a bar



Bending of a beam

Axial displacement is uncoupled from transverse displacement

Weak Form of the EB Beam Theory

Governing equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q=0,\quad 0<x<L$$

Weak form $\{v_i\}$ – set of weight functions

$$0=\int_{x_a}^{x_b} v_i\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w_h}{dx^2}\right)+c_f w_h-q\right]dx$$

$$=\int_{x_a}^{x_b}\left[-\frac{dv_i}{dx}\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)+c_f v_i w_h-v_i q\right]dx+\left[v_i\,\boxed{\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)}\right]_{x_a}^{x_b}$$

Implies that the primary variable is *w* (displacement)

Secondary variable (shear force)

$$\boxed{0=\int_{x_a}^{x_b}\left[-\frac{dv}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)+c_f vw-vq\right]dx-v(x_a)Q_1-v(x_b)Q_3}$$

Weak Form (Continued)

$$0 = \int_{x_a}^{x_b} \left[EI \frac{d^2 v_i}{dx^2} \frac{d^2 w_h}{dx^2} + c_f v_i w_h - v_i q \right] dx - v_i(x_a) Q_1 - v_i(x_b) Q_3$$

Primary Variable, θ Slope/rotation

$$+ \left[\left(-\frac{dv_i}{dx} \right) \cdot EI \frac{d^2 w_h}{dx^2} \right]_{x_a}^{x_b}$$

Secondary variable (Bending Moment)

$$0 = \int_{x_a}^{x_b} \left[EI \frac{d^2 v_i}{dx^2} \frac{d^2 w_h}{dx^2} + c_f v_i w_h - v_i q \right] dx - v_i(x_a) Q_1 - v_i(x_b) Q_3$$

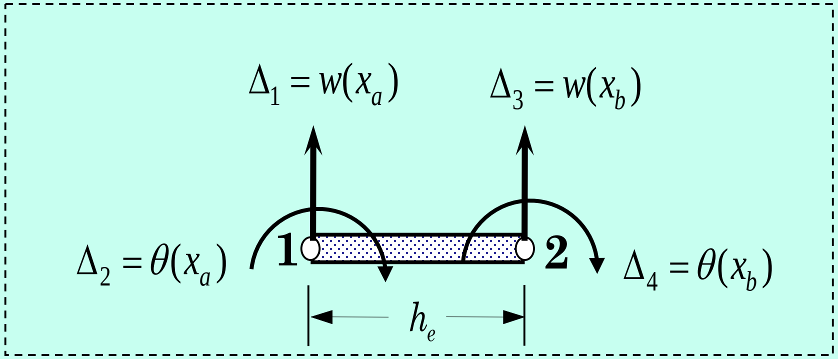
$$- \left(-\frac{dv_i}{dx} \right)_{x_a} \cdot Q_2 - \left(-\frac{dv_i}{dx} \right)_{x_b} \cdot Q_4$$

$$Q_1 = \left[\frac{d}{dx} \left(EI \frac{d^2 w_h}{dx^2} \right) \right]_{x_a} = -V_h(x_a), \quad Q_3 = \left[-\frac{d}{dx} \left(EI \frac{d^2 w_h}{dx^2} \right) \right]_{x_b} = V_h(x_b)$$

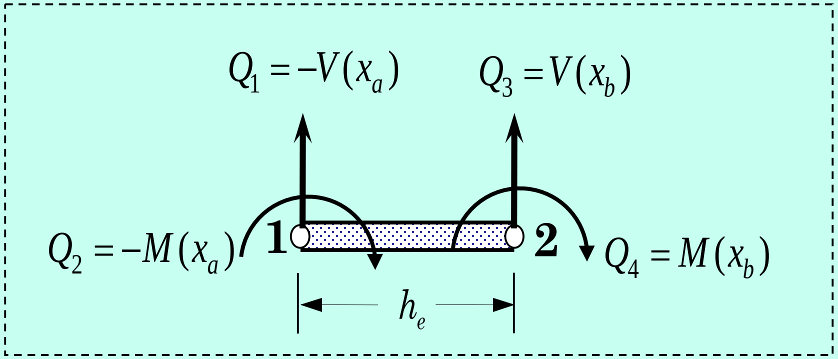
$$Q_2 = \left(EI \frac{d^2 w_h}{dx^2} \right)_{x_a} = -M_h(x_a), \qquad Q_4 = \left(-EI \frac{d^2 w_h}{dx^2} \right)_{x_b} = M_h(x_b)$$

Beam Element Degrees of Freedom

Generalized displacements



Generalized forces



FINITE ELEMENT APPROXIMATION: Some Remarks

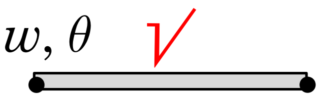
- Continuity requirement based on the weak form, which requires that the second derivative of w exists and square-integrable.
- Continuity based on the primary variables, which requires carrying w and its first derivative as the nodal variables, requires cubic approximation w.
- Post-computation of secondary variables (bending moment and shear force) requires the third derivative of w to exist.

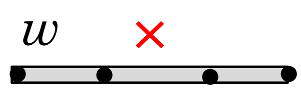
FINITE ELEMENT APPROXIMATION

Primary variables (serve as the nodal variables that must be continuous across elements)

$$w, \ \theta = -\frac{dw}{dx}$$

$$w(x) \approx c_0 + c_1 x + c_2 x^2 + c_3 x^3$$





$$w(x_a) \approx c_0 + c_1 x_a + c_2 x_a^2 + c_3 x_a^3 \equiv \Delta_1$$

$$w(x_b) \approx c_0 + c_1 x_b + c_2 x_b^2 + c_3 x_b^3 \equiv \Delta_3$$

$$\theta(x_a) \approx -c_1 - 2c_2 x_a - 3c_3 x_a^2 \equiv \Delta_2$$

$$\theta(x_b) \approx -c_1 - 2c_2 x_b - 3c_3 x_b^2 \equiv \Delta_4$$

Hermite cubic polynomials

$$\phi_1^e = 1 - 3\left(\frac{x - x_a}{h_e}\right)^2 + 2\left(\frac{x - x_a}{h_e}\right)^3$$

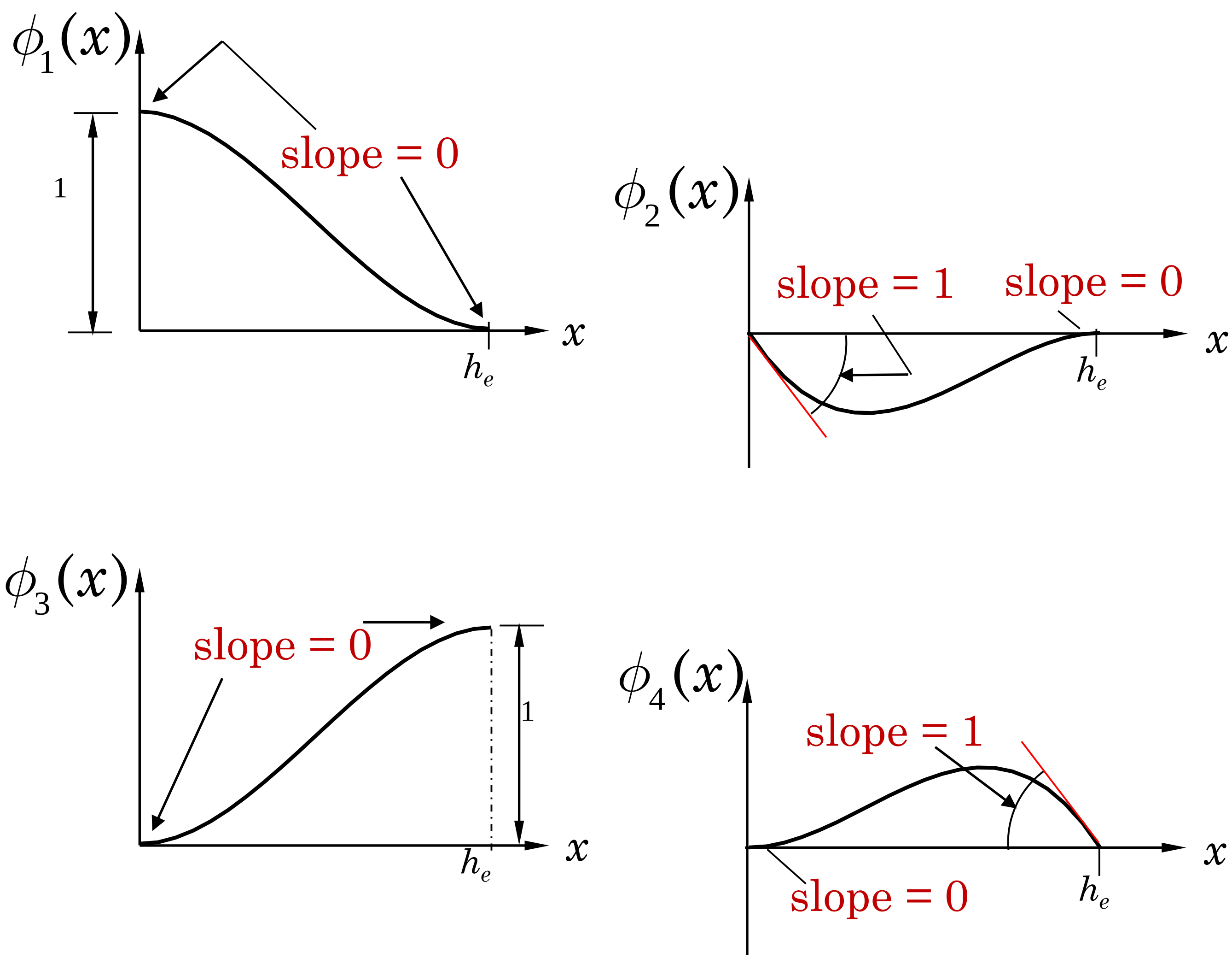
$$\phi_2^e = -(x - x_a)\left(1 - \frac{x - x_a}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_a}{h_e}\right)^2 - 2\left(\frac{x - x_a}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_a)\left[\left(\frac{x - x_a}{h_e}\right)^2 - \frac{x - x_a}{h_e}\right]$$

$$\boxed{w(x) \approx c_0 + c_1 x + c_2 x^2 + c_3 x^3 = \sum_{j=1}^{4} \Delta_j \phi_j(x)}$$

HERMITE CUBIC INTERPOLATION FUNCTIONS $\phi_i(x)$



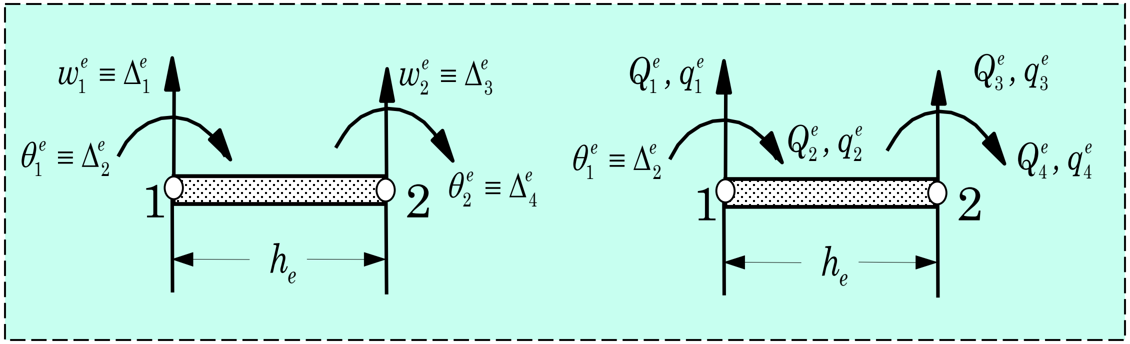
FINITE ELEMENT MODEL

$$\boxed{w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x)}$$

$$\sum_{j=1}^{4} K_{ij}^e \Delta_j^e - F_i^e = 0 \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\}$$

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} \boxed{+ c_f \, \phi_i^e \phi_j^e} \right) dx \qquad F_i^e = \int_{x_a}^{x_b} \phi_i^e q \, dx + Q_i^e$$



Finite Element Model (Continued)

For element-wise constant values of E_eI_e and q_e (and $c_f = 0$):

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} \quad \{F^e\} = \frac{q_eh_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Postprocessing

$$M(x) = -EI\frac{d^2w}{dx^2} = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e}{dx^2}$$

$$V(x) = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^3\phi_j^e}{dx^3}$$

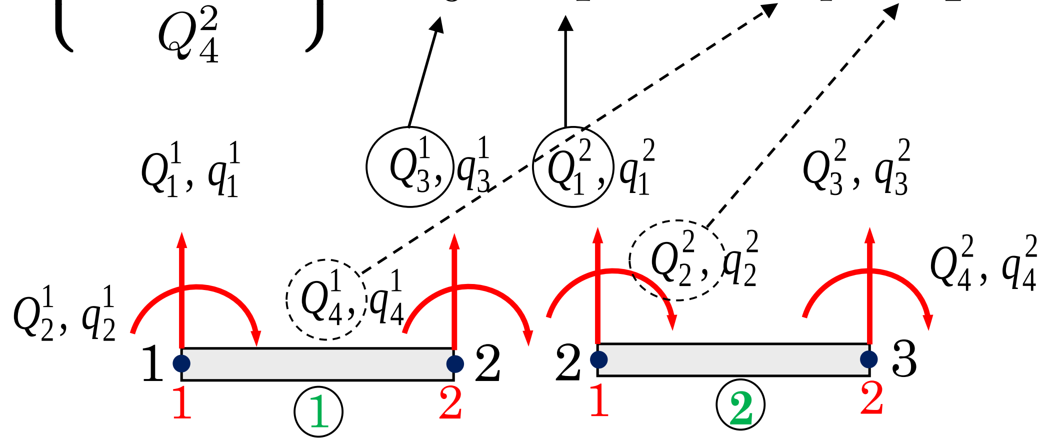
$$\sigma_x(x,z) = -\frac{M(x)z}{I} = Ez\frac{d^2w}{dx^2} = Ez\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e(x)}{dx^2}$$

ASSEMBLY OF TWO BEAM ELEMENTS connected end-to-end

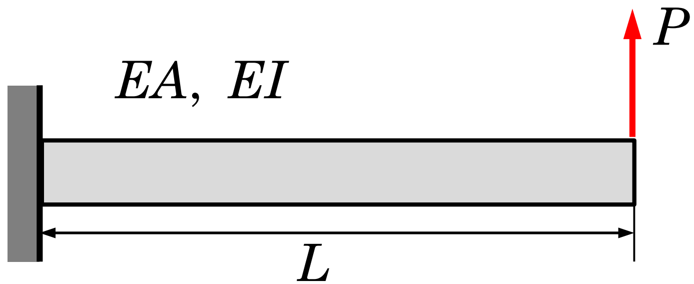
$$
\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \qquad h = L/2
$$

$$
= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}
$$

$$
Q_3^1 + Q_1^2 = 0, \qquad Q_4^1 + Q_2^2 = 0
$$



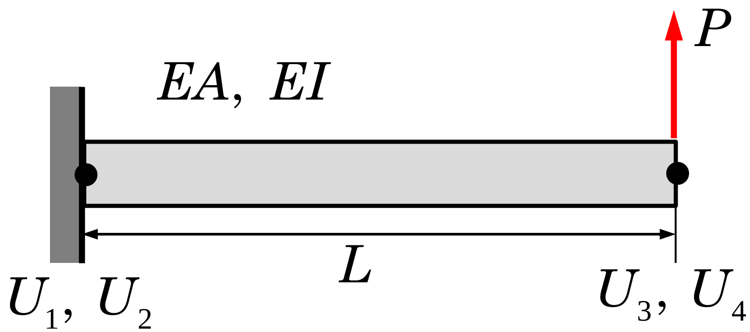
A SIMPLE EXAMPLE - 1



Given problem

Exact solution (according to the Euler-Bernoulli beam theory)

$$w(L) = \frac{PL^3}{3EI}$$



One element discretization

$$[K^e]\{\Delta^e\} = \{q^e\} + \{Q^e\} \qquad h_e = L$$

$$\{F^e\} = \frac{q_e h_e}{12}\left\{\begin{array}{c} 6 \\ -h_e \\ 6 \\ h_e \end{array}\right\} + \left\{\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{array}\right\}$$

0

Boundary conditions:

$$U_1 = U_2 = 0, \quad Q_3 = P, \ Q_4 = 0$$

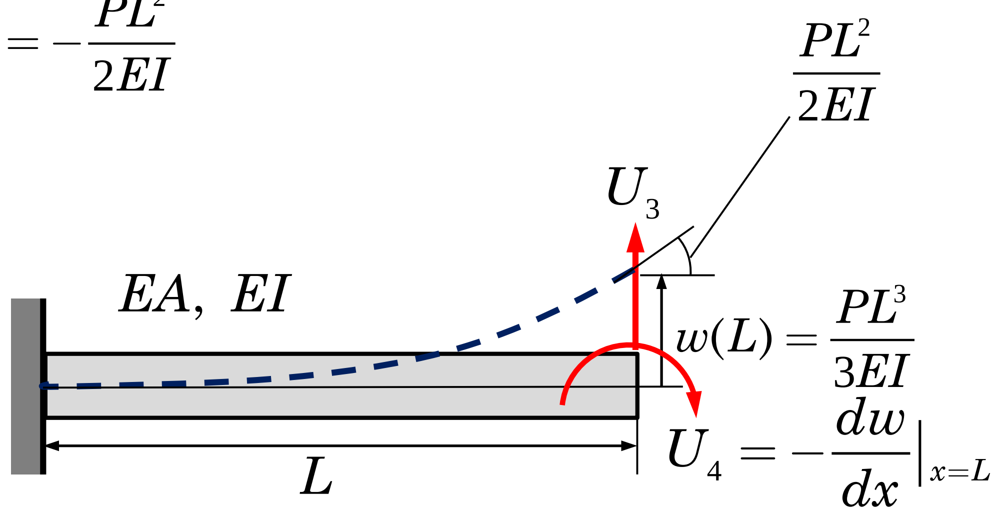
$$[K^e] = \frac{2E_e I_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

A SIMPLE EXAMPLE – 1 (continued)

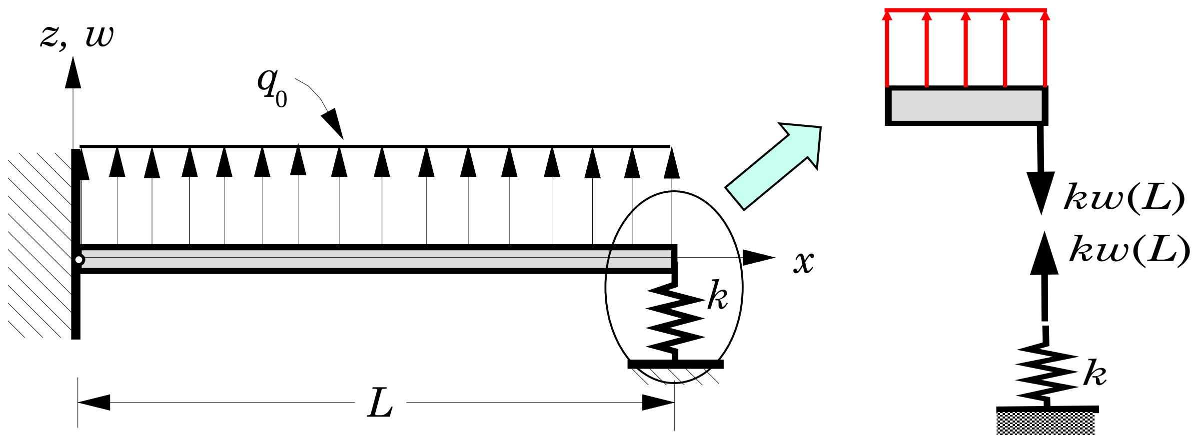
Solution using Cramer's rule

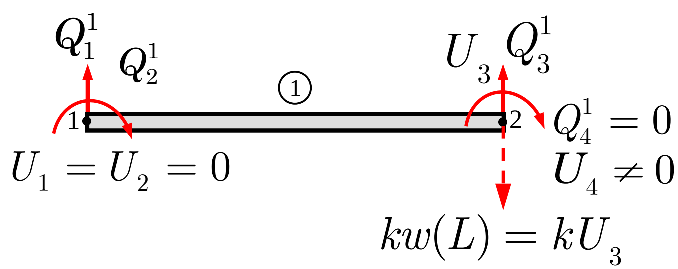
$$\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\begin{vmatrix} P & \frac{6EI}{L^2} \\ 0 & \frac{4EI}{L} \end{vmatrix}}{\begin{vmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{vmatrix}} = \frac{(4PEI / L)}{\left[12(EI)^2 / L^4\right]} = \frac{PL^3}{3EI}$$

$$U_4 = \frac{\begin{vmatrix} \frac{12EI}{L^3} & P \\ \frac{6EI}{L^2} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{vmatrix}} = \frac{-(6PEI / L^2)}{\left[12(EI)^2 / L^4\right]} = -\frac{PL^2}{2EI}$$



EXAMPLE – 3: Handling of a vertical spring





$$Q_4^1 = 0 \;\Rightarrow\; Q_3^1 = -kw(L) = -kU_3$$

Alternatively,

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^s \\ u_2^s \end{Bmatrix} = \begin{Bmatrix} Q_1^s \\ Q_2^s \end{Bmatrix}, \qquad u_1^s = 0,\ u_2^s = U_3 \;\Rightarrow\; Q_2^s = kU_3$$

SOLUTION TO THE SPRING-SUPPORTED BEAM

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} \cancelto{0}{U_1 = w_1} \\ \cancelto{0}{U_2 = \theta_1} \\ U_3 = w_2 \\ U_4 = \theta_2 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ \cancelto{-kU_3}{Q_3} \\ \cancelto{0}{Q_4} \end{Bmatrix}$$

Boundary conditions

$$w_1 = 0,\ \theta_1 = 0,\ Q_3 = -kU_3,\ Q_4 = 0$$

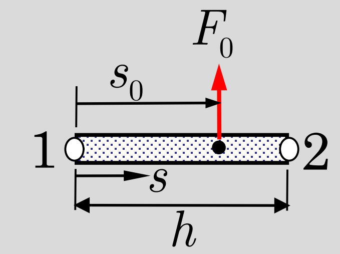
Condensed equations for the unknown generalized nodal displacements

$$\begin{bmatrix} \dfrac{12EI}{L^3} + k & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \end{Bmatrix}$$

HANDLING OF A POINT SOURCES INSIDE AN ELEMENT

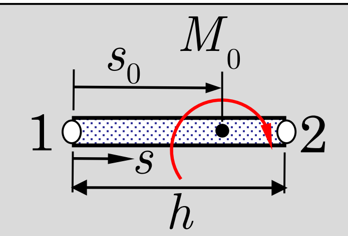
$$q_i = \int_0^h q(s)\phi_i(s)\,ds$$

$$q_i = \int_0^h q(s)\phi_i(s)\,ds = F_0\,\phi_i(s_0),\; i = 1,2,3,4$$



$$q(s) = F_0\delta(s - s_0)$$

$$q_i = \int_0^h q(s)\phi_i(s)\,ds = -M_0\left.\frac{d\phi_i}{ds}\right|_{s=s_0},\; i = 1,2,3,4$$



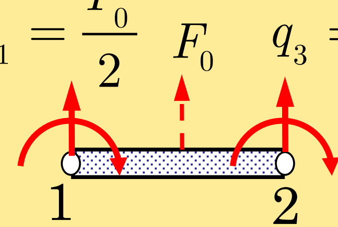
$$q(s) = M_0\delta'(s - s_0)$$

$$\phi_1(s) = 1 - 3\left(\frac{s}{h}\right)^2 + 2\left(\frac{s}{h}\right)^3,\quad \phi_2(s) = -s\left(1 - \frac{s}{h}\right)^2$$

$$\phi_3(s) = 3\left(\frac{s}{h}\right)^2 - 2\left(\frac{s}{h}\right)^3,\quad \phi_4(s) = -s\left[\left(\frac{s}{h}\right)^2 - \frac{s}{h}\right]$$

for F_0 placed at $s_0 = 0.5h$

$$q_1 = \frac{F_0}{2}\qquad q_3 = \frac{F_0}{2}$$

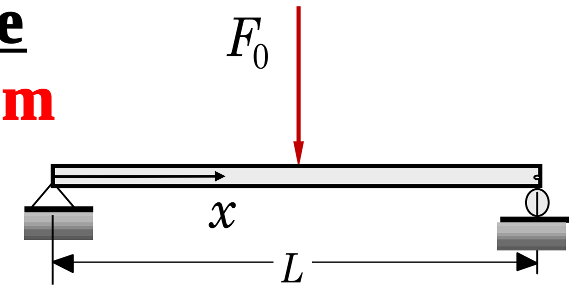


$$q_2 = -\frac{F_0 h}{8}\qquad q_4 = \frac{F_0 h}{8}$$

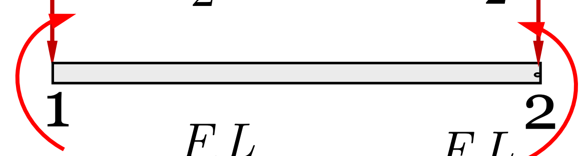
EXAMPLE – 4: A simply-supported beam

(a) Find the center deflection using <u>one</u> Euler-Bernoulli element in full beam

$$\frac{2EI}{L^3}\begin{bmatrix}6&-3L&-6&-3L\\-3L&2L^2&3L&L^2\\-6&3L&6&3L\\-3L&L^2&3L&2L^2\end{bmatrix}\begin{Bmatrix}\cancelto{0}{U_1=w_1}\\U_2=\theta_1\\\cancelto{0}{U_3=w_2}\\U_4=\theta_2\end{Bmatrix}=\begin{Bmatrix}q_1\\\cancelto{\frac{F_0L}{8}}{q_2}\\q_3\\\cancelto{-\frac{F_0L}{8}}{q_4}\end{Bmatrix}+\begin{Bmatrix}Q_1\\\cancelto{0}{Q_2}\\Q_3\\\cancelto{0}{Q_4}\end{Bmatrix}$$



$$q_1=\frac{F_0}{2}\qquad q_3=\frac{F_0}{2}$$



$$q_2=\frac{F_0L}{8}\qquad q_4=\frac{F_0L}{8}$$

<u>Condensed equations</u>

$$\frac{2EI}{L^3}\begin{bmatrix}2L^2&L^2\\L^2&2L^2\end{bmatrix}\begin{Bmatrix}U_2\\U_4\end{Bmatrix}=\frac{F_0L}{8}\begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$U_2=\frac{F_0L^2}{16EI},\quad U_4=-\frac{F_0L^2}{16EI}$$

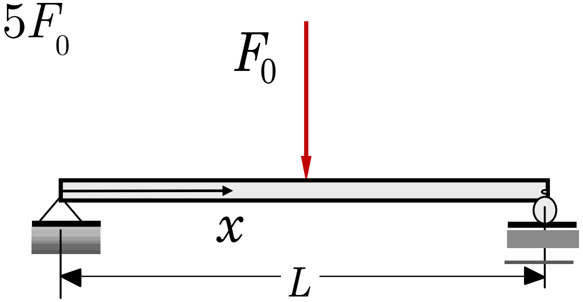
$$\begin{aligned}w(x)&=U_1\phi_1(x)+U_2\phi_2(x)+U_3\phi_3(x)+U_4\phi_4(x)\\&=U_2\phi_2(x)+U_4\phi_4(x)\\&=\frac{F_0L^2}{16EI}\left\{\left[-x\left(1-\frac{x}{L}\right)^2\right]+x\left[\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)\right]\right\}\end{aligned}$$

$$w(0.5L)=\frac{F_0L^2}{16EI}\left(-\frac{L}{8}-\frac{L}{8}\right)=-\frac{F_0L^3}{64EI}$$

EXAMPLE – 4: A simply-supported beam

(b) Find the center deflection using one Euler-Bernoulli element in half beam

$$\frac{16EI}{L^3}\begin{bmatrix} 6 & -1.5L & -6 & -1.5L \\ -1.5L & 0.5L^2 & 1.5L & 0.25L^2 \\ -6 & 1.5L & 6 & 1.5L \\ -3L & 0.25L^2 & 1.5L & 0.5L^2 \end{bmatrix}\begin{Bmatrix} U_1 = w_1 \to 0 \\ U_2 = \theta_1 \\ U_3 = w_2 \\ U_4 = \theta_2 \to 0 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \to 0 \\ Q_3 \to -0.5F_0 \\ Q_4 \end{Bmatrix}$$

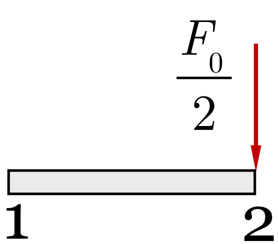


Condensed equations

$$\frac{16EI}{L^3}\begin{bmatrix} 0.5L^2 & 1.5L \\ 1.5L & 6 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 0.5F_0\begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$U_2 = \frac{F_0L^3}{32EI}\frac{4}{3L^2}\frac{1.5L}{1} = \frac{F_0L^2}{16EI},$$

$$U_3 = \frac{F_0L^3}{32EI}\frac{4}{3L^2}\frac{0.5L^2}{1} = \frac{F_0L^3}{48EI}$$

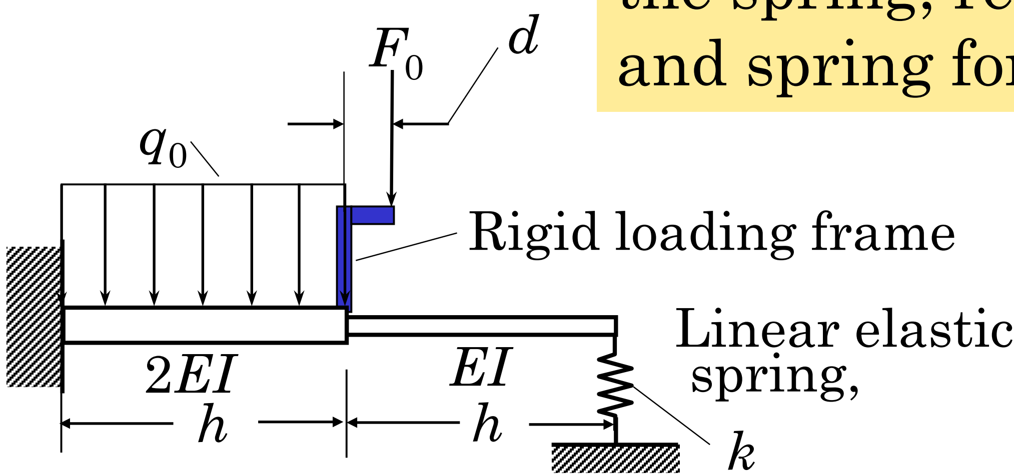


$U_1 = 0, \quad U_4 = 0,$

$Q_2 = 0 \quad Q_3 = -0.5F_0$

EXERCISE PROBLEM

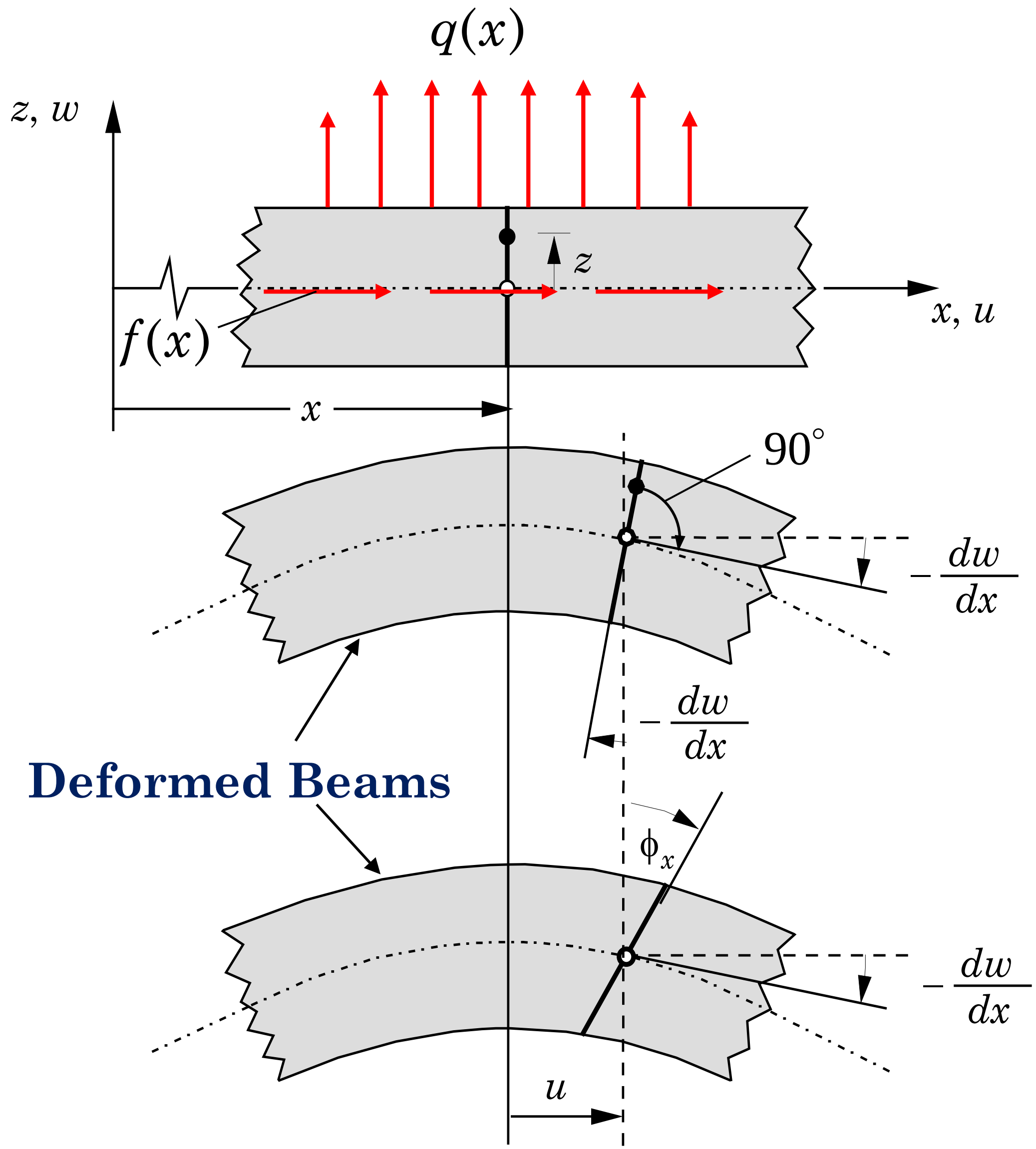
Problem: Use the minimum number of EBT elements to find the compression in the spring, reactions at the fixed support, and spring force.



TIMOSHENKO BEAM THEORY and its Finite Element Model

- Governing Equations
- Finite element model
- Shear locking
- Numerical example

Kinematics of Timoshenko Beam Theory



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)
Straightness and inextensibility

Timoshenko Beam Theory

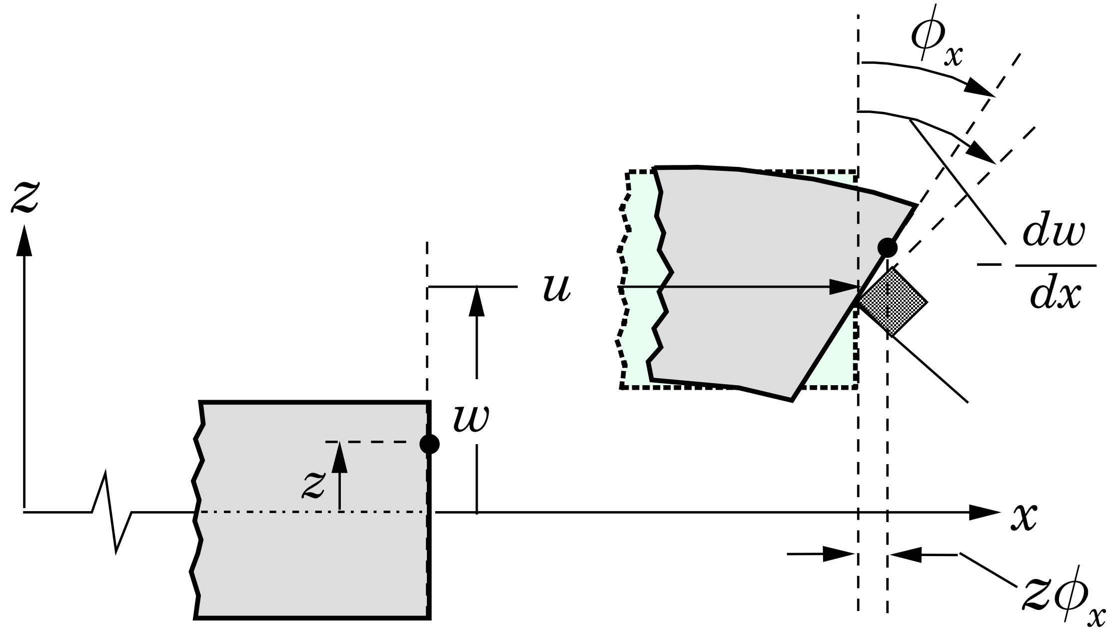
Kinematic Relations

$$u_1(x,z) = u(x) + z\phi_x(x),$$

$$u_2 = 0, \quad u_3(x,z) = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} + z\frac{d\phi_x}{dx},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \phi_x + \frac{dw}{dx}$$



Constitutive Equations

$$\sigma_{xx} = E\,\varepsilon_{xx} = E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right)$$

$$\sigma_{xz} = G\,\gamma_{xz} = G\left(\phi_x + \frac{dw}{dx}\right)$$

Timoshenko Beam Theory (Continued)

Equilibrium Equations $\frac{dN}{dx} + f = 0$, $\boxed{-\frac{dV}{dx} - q + c_f w = 0}$, $\boxed{-\frac{dM}{dx} + V = 0.}$

Beam Constitutive Equations

$$N = \int_A \sigma_{xx}\, dA = \int_A E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right) dA = EA\frac{du}{dx}$$

$$M = \int_A \sigma_{xx} z\, dA = \int_A E\left(\frac{du}{dx} + z\frac{d\phi}{dx}\right) z\, dA = EI\frac{d\phi}{dx}$$

$$V = K_s \int_A \sigma_{xz}\, dA = GK_s\left(\phi + \frac{dw}{dx}\right)\int_A dA = GAK_s\left(\phi + \frac{dw}{dx}\right)$$

Governing Equations in terms of the displacements

$$\boxed{-\frac{d}{dx}\left[GAK_s\left(\phi + \frac{dw}{dx}\right)\right] + c_f w = q} \qquad (1)$$

$$\boxed{-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GAK_s\left(\phi + \frac{dw}{dx}\right) = 0} \qquad (2)$$

WEAK FORMS OF TBT

Weak Form of Eq. (1)

$$0=\int_{x_a}^{x_b} v_1\left\{-\frac{d}{dx}\left[GAK_s\left(\phi+\frac{dw}{dx}\right)\right]+c_f w-q\right\}dx$$

$$=\int_{x_a}^{x_b}\left\{\frac{dv_1}{dx}\left[GAK_s\left(\phi+\frac{dw}{dx}\right)\right]+c_f v_1 w-v_1 q\right\}dx-\left[v_1\cdot GAK_s\left(\phi+\frac{dw}{dx}\right)\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left\{\frac{dv_1}{dx}\left[GAK_s\left(\phi+\frac{dw}{dx}\right)\right]+c_f v_1 w-v_1 q\right\}dx$$

$$-v_1(x_a)\cdot\left[-GAK_s\left(\phi+\frac{dw}{dx}\right)\right]_{x_a}-v_1(x_b)\cdot\left[GAK_s\left(\phi+\frac{dw}{dx}\right)\right]_{x_b}$$

$$\boxed{=\int_{x_a}^{x_b}\left\{GAK_s\frac{dv_1}{dx}\left(\phi+\frac{dw}{dx}\right)+c_f v_1 w-v_1 q\right\}dx-v_1(x_a)\cdot Q_1-v_1(x_b)\cdot Q_3}$$

Weak Forms of TBT (continued)

Weak Form of Eq. (2)

$$0=\int_{x_a}^{x_b} v_2\left[-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s\left(\phi+\frac{dw}{dx}\right)\right]dx$$

$$=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-\left[v_2\cdot EI\frac{d\phi}{dx}\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-v_2(x_a)\cdot\left(-EI\frac{d\phi}{dx}\right)_{x_a}-v_2(x_b)\cdot\left(EI\frac{d\phi}{dx}\right)_{x_b}$$

$$\boxed{0=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-v_2(x_a)\cdot Q_2-v_2(x_b)\cdot Q_4}$$

Total Potential Energy

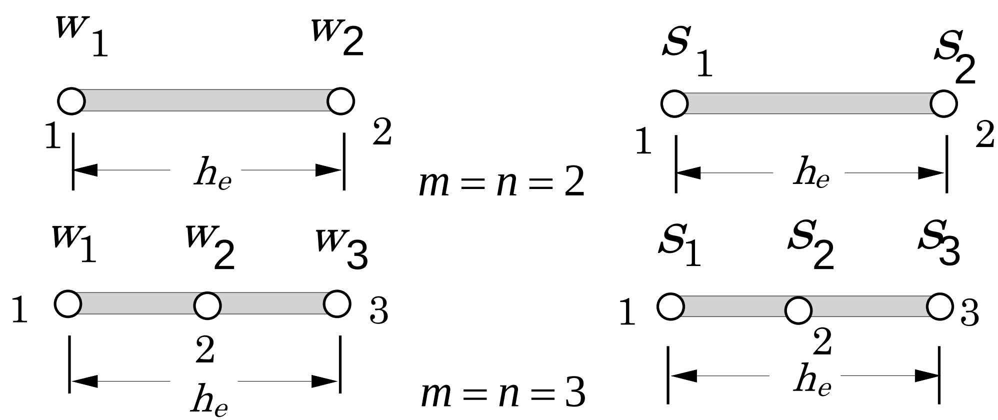
$$\Pi(w,\phi_x)=\int_{x_a}^{x_b}\left[\frac{EI}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{GAK_s}{2}\left(\phi+\frac{dw}{dx}\right)^2+\frac{c_f}{2}w^2\right]dx$$

$$-\int_{x_a}^{x_b} wq\;dx+w(x_a)Q_1+w(x_b)Q_3+\phi(x_a)Q_2+\phi(x_b)Q_4$$

FINITE ELEMENT MODELS OF TIMOSHENKO BEAMS

Finite Element Approximation

$$w \approx \sum_{j=1}^{m} w_j \psi_j(x), \quad \phi \approx \sum_{j=1}^{n} S_j \varphi_j(x)$$



$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c_f \psi_i \psi_j \right) dx, \quad K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i}{dx} \varphi_j \; dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left[EI \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + GAK_s \varphi_i \varphi_j \right] dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} GAK_s \; \varphi_i \frac{d\psi_j}{dx} \, dx$$

$$F_i^1 = \int_{x_a}^{x_b} q\psi_i \; dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_3, \qquad F_i^2 = \varphi_i(x_a) Q_2 + \varphi_i(x_b) Q_4$$

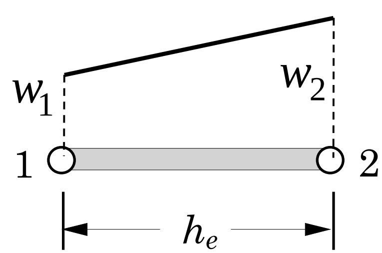
Shear Locking in Timoshenko Beams

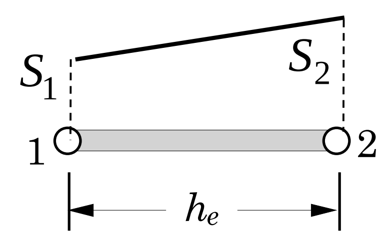
(1) Thick beam experiences shear deformation, $\phi_x \neq -\dfrac{dw}{dx}$

(2) Shear deformation is negligible in thin beams, $\phi_x = -\dfrac{dw}{dx}$

Linear interpolation of both w, ϕ_x: $\boxed{w \approx \sum_{j=1}^{2} w_j^e \psi_j^e(x), \quad \phi_x \approx \sum_{j=1}^{2} S_j^e \psi_j^e(x)}$

$$w(x) \approx w_1\psi_1(x) + w_2\psi_2(x), \quad \phi_x(x) \approx S_1\psi_1(x) + S_2\psi_2(x)$$





Thus, in the **thin beam limit** it is not possible for the element to realize the requirement

$$\phi_x = -\frac{dw}{dx}$$

SHEAR LOCKING - REMEDY

In the thin beam limit, ϕ should become constant so that it matches dw/dx. However, if ϕ is a constant then the bending energy becomes zero. If we can mimic the two states (constant and linear) in the formulation, we can overcome the problem. Numerical integration of the coefficients allows us to evaluate both ϕ and $d\phi/dx$ as constants. The terms highlighted should be evaluated using "reduced integration".

$$K_{ij}^{11} = \int_{x_a}^{x_b} \left(\boxed{GAK_s \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx}} + c_f \psi_i^{(1)} \psi_j^{(1)} \right) dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} \boxed{GAK_s \frac{d\psi_i^{(1)}}{dx} \psi_j^{(2)}} \, dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left[EI \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + \boxed{GAK_s \psi_i^{(2)} \psi_j^{(2)}} \right] dx$$

STIFFNESS MATRICES OF TIMOSHENKO BEAM ELEMENT

(for constant *EI* and *GA*)

Reduced integration linear element (RIE) Linear approximation of both w and ϕ

$$\frac{2E_eI_e}{\mu_0^e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2\xi_e & 3h_e & h_e^2\zeta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\zeta_e & 3h_e & h_e^2\xi_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

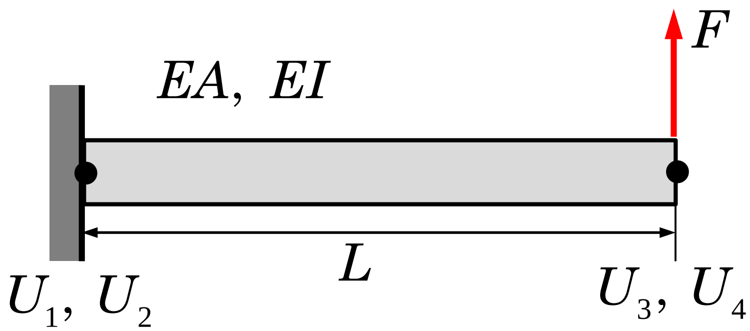
$$\xi_e = 1.5 + 6\Lambda_e,\ \zeta_e = 1.5 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_0^e = 12\Lambda_e$$

Consistent interelement element (CIE) Hermite cubic approximation of w and dependent quadratic approximation of ϕ

$$\frac{2E_eI_e}{\mu_e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2\Sigma_e & 3h_e & h_e^2\Theta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\Theta_e & 3h_e & 2h_e^2\Sigma_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\Sigma_e = 1.0 + 3\Lambda_e,\ \Theta_e = 1.0 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_e = 1 + 12\Lambda_e$$

AN EXAMPLE of TBT



Exact solution (according to the E-B beam theory)

$$w(L) = \frac{FL^3}{3EI}$$

One element discretization using the RIE element

$$\frac{2E_eI_e}{\mu_0^e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2\xi_e & 3h_e & h_e^2\zeta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\zeta_e & 3h_e & h_e^2\xi_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\xi_e = 1.5 + 6\Lambda_e,\ \zeta_e = 1.5 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_0^e = 12\Lambda_e$$

Boundary conditions:

$$U_1 = U_2 = 0,\quad Q_3 = F,\ Q_4 = 0$$

AN EXAMPLE (TBT) (continued)

$$\frac{2EI}{\mu_0 L^3}\begin{bmatrix} 6 & 3L \\ 3L & \xi L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\mu_0 L^3}{2EI}\frac{FL^2\xi}{\left(6L^2\xi - 9L^2\right)} = \frac{12\Lambda FL^3(1.5+6\Lambda)}{6EI(12\Lambda)}$$

$$\text{When} \quad \Lambda = \frac{EI}{GAK_s L^2} = 0 \Rightarrow U_3 = \frac{1.5FL^3}{6EI} = \frac{FL^3}{4EI} \ (\text{too stiff})$$

$$\text{When} \ \ \Lambda \neq 0, \quad \text{then} \ \ U_3 = \frac{FL^3(1.5+6\Lambda)}{6EI} = (0.75+3\Lambda)\frac{FL^3}{3EI}$$

$$\Lambda = \frac{EI}{GAK_s L^2} = \frac{2(1+\nu)H^2}{12L^2 K_s} = \frac{(1+\nu)}{6K_s}\left(\frac{H}{L}\right)^2 = \frac{1.3}{5}\left(\frac{H}{L}\right)^2 = 0.26\left(\frac{H}{L}\right)^2$$

AN EXAMPLE of TBT

One element discretization using the CIE element

$$\frac{2EI}{\mu L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2\Sigma & 3L & L^2\Theta \\ -6 & 3L & 6 & 3L \\ -3L & L^2\Theta & 3L & 2L^2\Sigma \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\Sigma = 1.0 + 3\Lambda\ ,\ \Theta = 1.0 - 6\Lambda\ ,\ \Lambda = \frac{E\ I}{G\ A\ K_s L^2},\ \mu = 1 + 12\Lambda$$

Condensed equations for the unknown displacements

$$\frac{2EI}{\mu L^3}\begin{bmatrix} 6 & 3L \\ 3L & 2\Sigma L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\mu L^3}{2EI}\frac{2FL^2\Sigma}{\left(12L^2\Sigma - 9L^2\right)} = \frac{\mu FL^3\Sigma}{EI(12\Sigma - 9)}$$

AN EXAMPLE (TBT) (continued)

When $\Lambda = 0 \Rightarrow \Sigma = 1$ and $\mu = 1$; then $U_3 = \dfrac{\mu F L^3 \Sigma}{EI(12\Sigma - 9)} = \dfrac{FL^3}{3EI}$

When $\Lambda \neq 0$, $\quad U_3 = \dfrac{\mu F L^3 \Sigma}{EI(12\Sigma - 9)} = \dfrac{FL^3}{3EI}\dfrac{(1+3\Lambda)(1+12\Lambda)}{(1+12\Lambda)} = (1+3\Lambda)\dfrac{FL^3}{3EI}$

$$\Lambda = \frac{EI}{GAK_sL^2} = \frac{2(1+\nu)H^2}{12L^2K_s} = \frac{(1+\nu)}{6K_s}\left(\frac{H}{L}\right)^2 = \frac{1.3}{5}\left(\frac{H}{L}\right)^2 = 0.26\left(\frac{H}{L}\right)^2$$

SUMMARY

In this lecture we have covered the following topics:

- **Derived the governing equations of the Euler-Bernoulli beam theory**
- **Derived the governing equations of the Timoshenko beam theory**
- **Developed Weak forms of EBT and TBT**
- **Developed Finite element models of EBT and TBT**
- **Discussed shear locking in Timoshenko beam finite element**
- **Discussed assembly of beam elements**
- **Discussed examples**