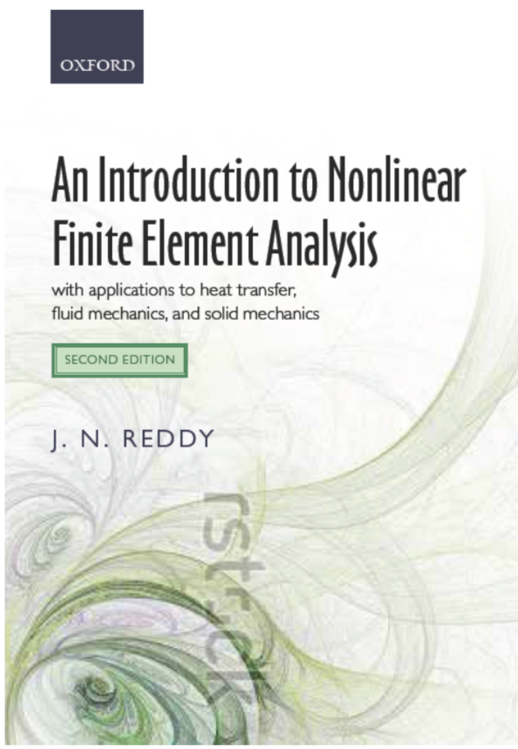
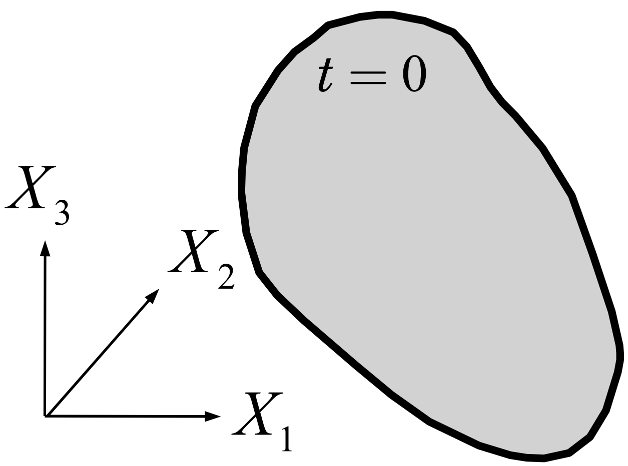
NONLINEAR CONTINUUM FORMULATIONS



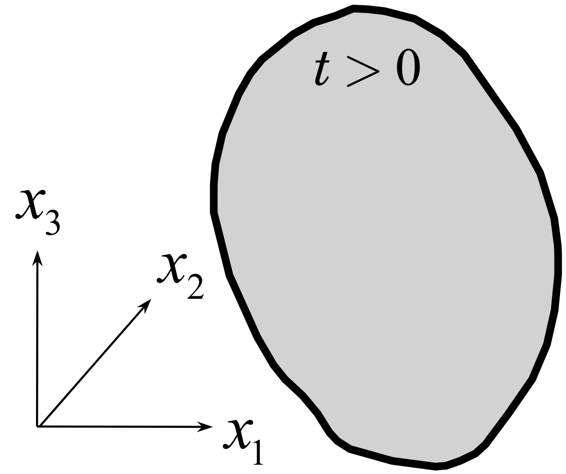
CONTENTS

- Introduction to nonlinear continuum mechanics
- Descriptions of motion
- Measures of stresses and strains
- Updated and Total Lagrangian formulations
- Continuum shell element
- Post-buckling of composite panels

DESCRIPTIONS OF MOTION



Reference configuration



Current configuration

Material description $\mathbf{x} = \chi(\mathbf{X}, t), \qquad \chi(\mathbf{X}, 0) = \mathbf{X}$

$$\phi = \phi(\mathbf{X}, t)$$

Spatial description $\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \qquad \phi = \phi(\mathbf{x}, t)$

MATERIAL TIME DERIVATIVE

Material description

$$\frac{d}{dt}[\phi(\mathbf{X},t)] = \frac{d}{dt}[\phi(\mathbf{X},t)]\Big|_{\mathbf{X}=\text{fixed}} = \frac{\partial \phi}{\partial t}$$

Spatial description

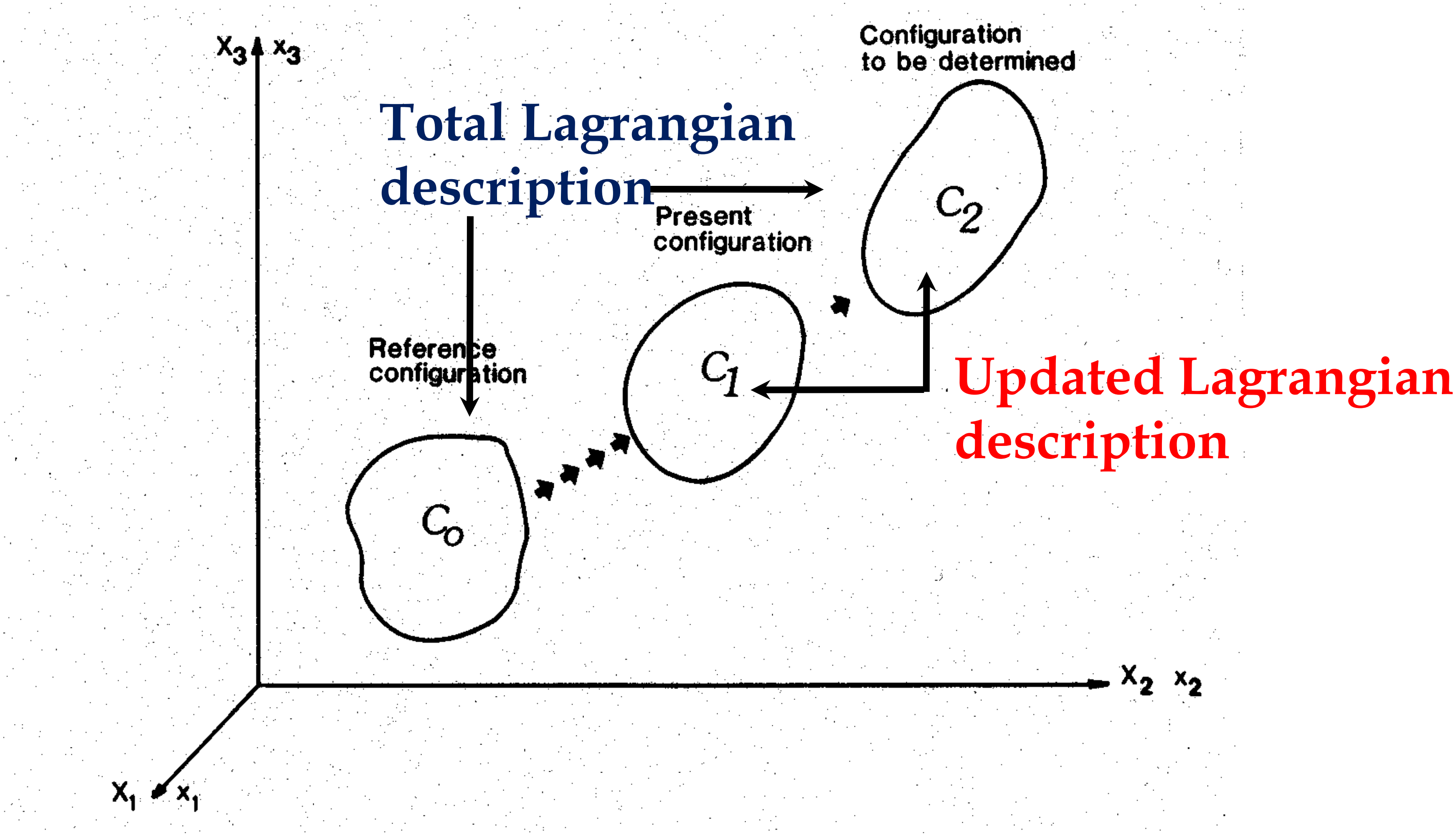
$$\frac{d}{dt}[\phi(\mathbf{x},t)] = \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + \frac{dx_j}{dt}\frac{\partial}{\partial x_j}[\phi(\mathbf{x},t)]$$

$$= \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + v_j \frac{\partial}{\partial x_j}[\phi(\mathbf{x},t)]$$

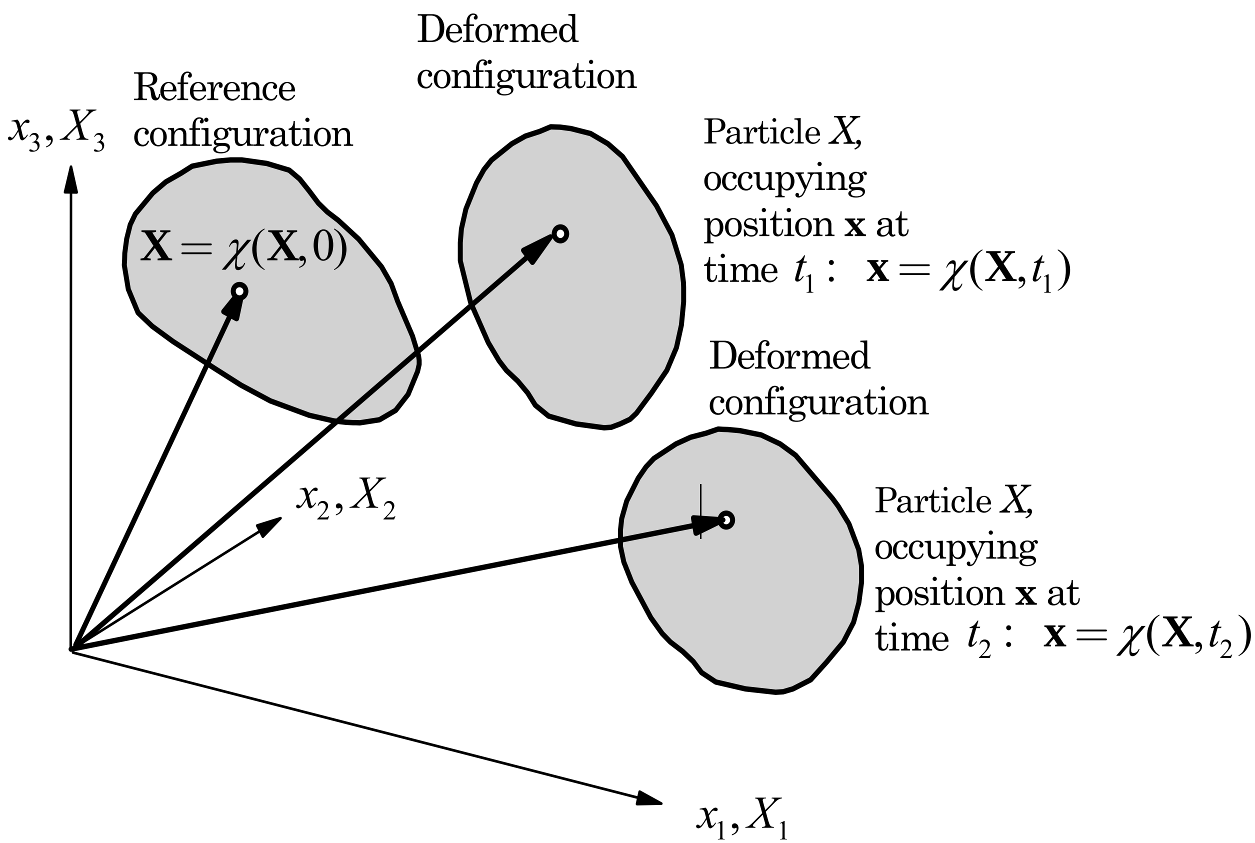
$$= \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + \mathbf{v}\cdot\nabla\phi(\mathbf{x},t)$$

LAGRANGIAN DESCRIPTION OF MOTION

Incremental descriptions of deformed configurations

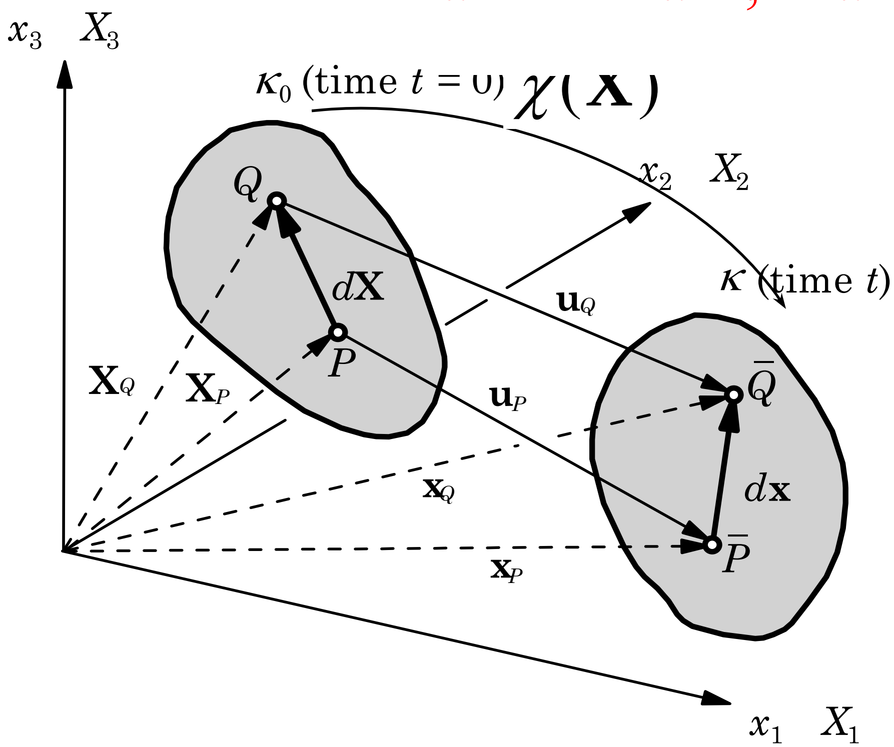


Kinematics of Deformation



Kinematics of Deformation
(continued)

$$d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X}, \quad d\mathbf{X} = \mathbf{F}^{-1}\cdot d\mathbf{x}, \quad \mathbf{F} = \left(\frac{d\mathbf{x}}{d\mathbf{X}}\right)^{\mathrm{T}}$$



$$\mathbf{u} = \mathbf{x} - \mathbf{X}$$

$$d\mathbf{x}\cdot d\mathbf{x} - d\mathbf{X}\cdot d\mathbf{X} = 2\, d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X}$$

$$2\mathbf{E} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} - \mathbf{I}$$

$$2\mathbf{E} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} + (\nabla\mathbf{u})\cdot(\nabla\mathbf{u})^{\mathrm{T}}$$

$$2E_{ij} = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j}$$

E Green strain tensor

MEASURES OF STRAIN

Definitions

$$ {}^1u = {}^1x - {}^0x \quad \text{or} \quad {}^1u_i = {}^1x_i - {}^0x_i $$

$$ {}^2u = {}^2x - {}^0x \quad \text{or} \quad {}^2u_i = {}^2x_i - {}^0x_i $$

$$ u = {}^2u - {}^1u \quad \text{or} \quad u_i = {}^2u_i - {}^1u_i $$

$$ {}^1_0\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial\, {}^1u_i}{\partial\, {}^0x_j} + \frac{\partial\, {}^1u_j}{\partial\, {}^0x_i} + \frac{\partial\, {}^1u_m}{\partial\, {}^0x_i}\frac{\partial\, {}^1u_m}{\partial\, {}^0x_j}\right] $$

$$ {}^2_0\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial\, {}^2u_i}{\partial\, {}^0x_j} + \frac{\partial\, {}^2u_j}{\partial\, {}^0x_i} + \frac{\partial\, {}^2u_m}{\partial\, {}^0x_i}\frac{\partial\, {}^2u_m}{\partial\, {}^0x_j}\right] $$

Green *Incremental* Strain Tensor

$$ 2\,{}_0E_{ij}({}^0x_i, {}^0x_j) = ({}^2F^T\cdot{}^2F) - ({}^1F^T\cdot{}^1F) $$

$$ = 2\left[{}^2_0\varepsilon_{ij} - {}^1_0\varepsilon_{ij}\right]({}^0x_i, {}^0x_j) $$

$$ \equiv 2\left({}_0e_{ij} + {}_0\eta_{ij}\right)({}^0x_i, {}^0x_j) $$

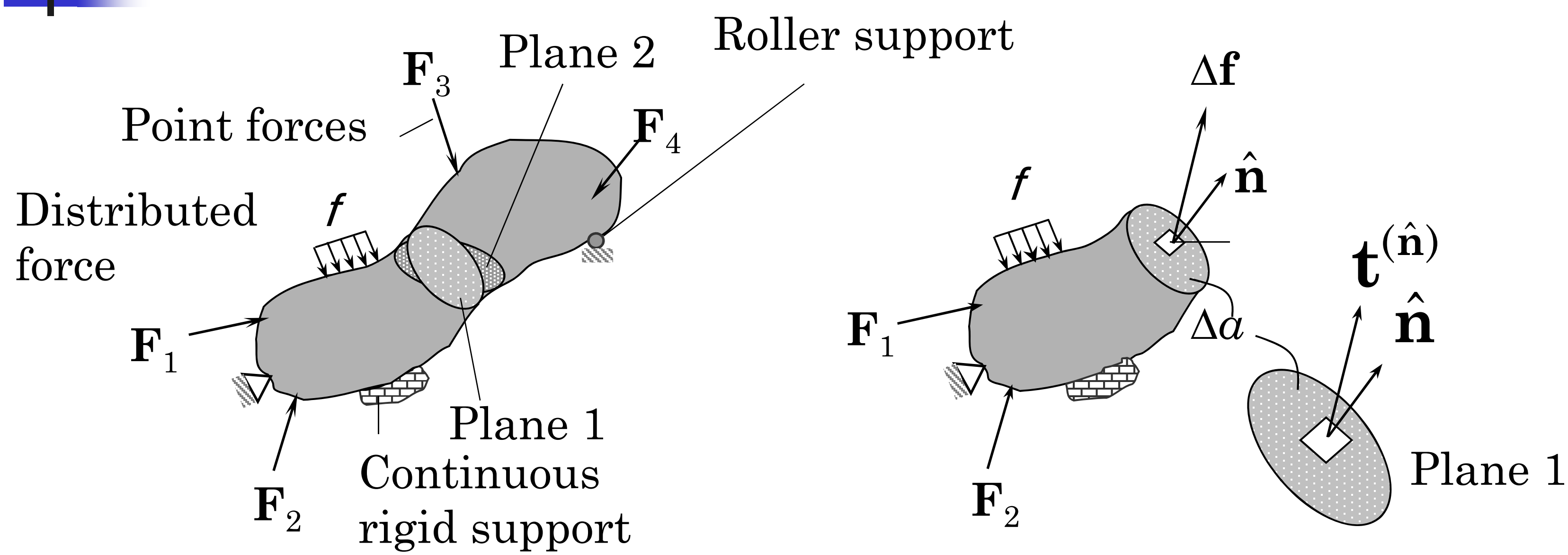
$$ {}_0e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial\, {}^0x_j} + \frac{\partial u_j}{\partial\, {}^0x_i} + \frac{\partial\, {}^1u_m}{\partial\, {}^0x_i}\frac{\partial u_m}{\partial\, {}^0x_j} + \frac{\partial u_m}{\partial\, {}^0x_i}\frac{\partial\, {}^1u_m}{\partial\, {}^0x_j}\right] \qquad {}_0\eta_{ij} = \frac{1}{2}\frac{\partial u_m}{\partial\, {}^0x_i}\frac{\partial u_m}{\partial\, {}^0x_j} $$

Linear Green Incremental Strain Tensor   *Nonlinear* Incremental Strain Tensor

Infinitesmal Green-Lagrange Incremental Strain Tensor

$$ e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial\, {}^0x_j} + \frac{\partial u_j}{\partial\, {}^0x_i}\right] $$

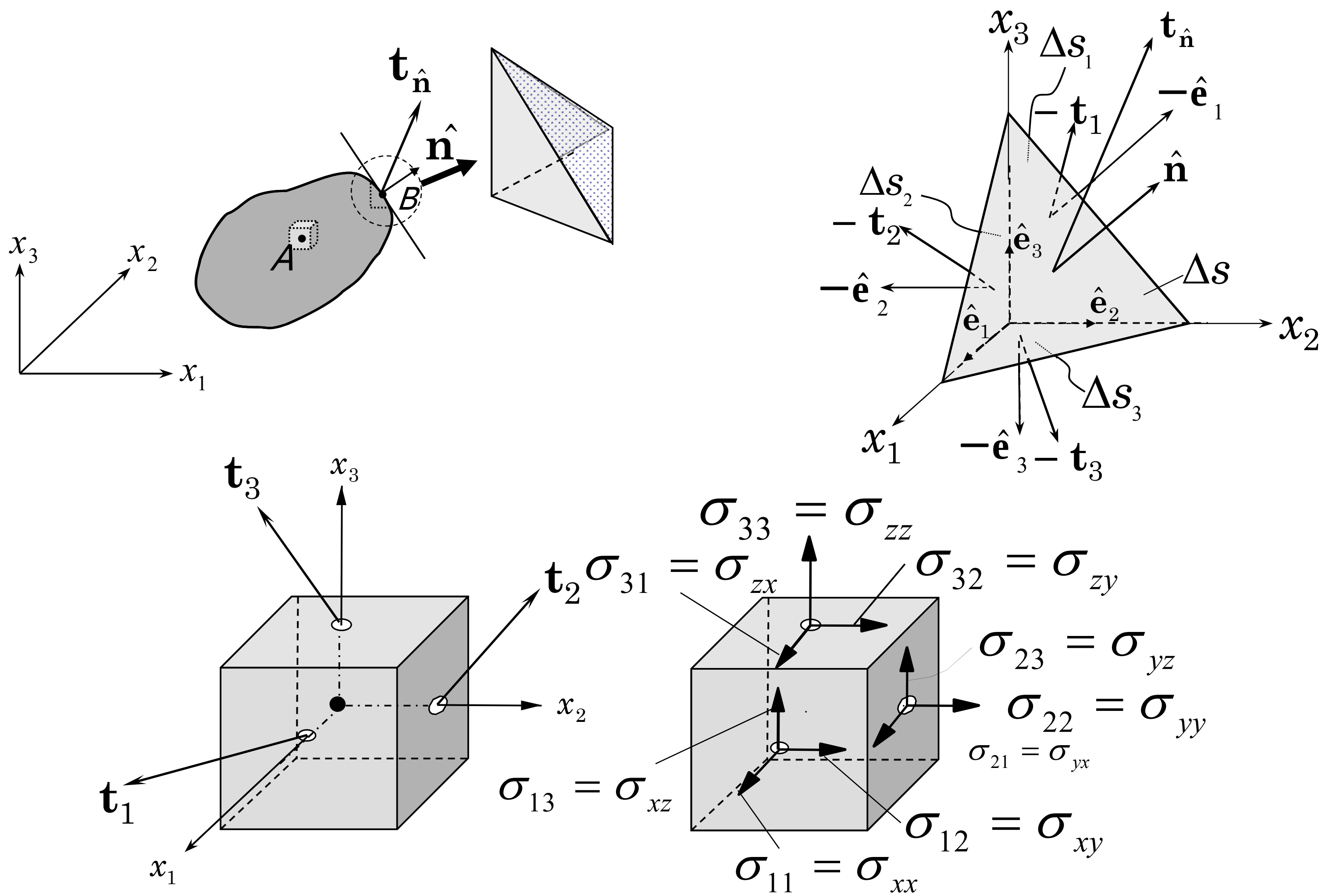
STRESS VECTOR



Stress vector

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}}{\Delta a} \qquad\qquad \mathbf{t}^{(-\hat{\mathbf{n}})} = -\mathbf{t}^{(\hat{\mathbf{n}})}$$

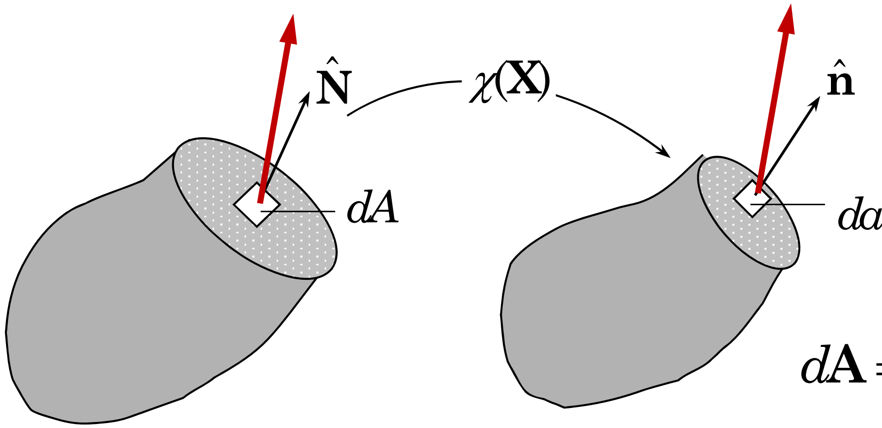
CAUCHY STRESS TENSOR



$$\mathbf{t}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \quad \text{or} \quad t_i^{(\hat{\mathbf{n}})} = \sigma_{ij} n_j; \;\; \mathbf{t}_i = \sigma_{ji}\,\hat{\mathbf{e}}_j, \quad \boldsymbol{\sigma} = \mathbf{t}_j\hat{\mathbf{e}}_j = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

FIRST AND SECOND PIOLA-KIRCHHOFF STRESS TENSORS

$$d\mathbf{f} = \mathbf{T}\,dA = \mathbf{P}\cdot\hat{\mathbf{N}}\,dA = \mathbf{P}\cdot d\mathbf{A} \qquad d\mathbf{f} = \mathbf{t}\,da = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\,da = \boldsymbol{\sigma}\cdot d\mathbf{a}$$



$$d\mathbf{f} = \mathbf{P}\cdot d\mathbf{A} = \boldsymbol{\sigma}\cdot d\mathbf{a}$$

$$d\mathbf{A} = \hat{\mathbf{N}}\,da, \qquad d\mathbf{a} = \hat{\mathbf{n}}\,da$$

Reference configuration **Current configuration**

$$d\mathbf{a} = J\,\mathbf{F}^{-1}\cdot d\mathbf{A} \qquad d\mathfrak{I} = \mathbf{F}^{-1}\cdot d\mathbf{f} \qquad d\mathfrak{I} = \mathbf{S}\cdot d\mathbf{A}$$

$$d\mathfrak{I} = \mathbf{F}^{-1}\cdot d\mathbf{f} = \mathbf{F}^{-1}\cdot\left(\mathbf{P}\cdot d\mathbf{A}\right) = \mathbf{S}\cdot d\mathbf{A}$$

$$\mathbf{F}^{-1}\cdot\mathbf{P} = \mathbf{S}, \qquad \mathbf{S} = J\,\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}}$$

P First Piola-Kirchhoff stress tensor

S Second Piola-Kirchhoff stress tensor

CONSERVATION PRICIPLES

Conservation of Mass

$$\int_{\Omega} \rho \, dv = \int_{\Omega_0} \rho_0 \, dV \Rightarrow \int_{\Omega_o} \rho J dV = \int_{\Omega_0} \rho_0 \, dV$$

$$\rho_0 = \rho \boldsymbol{J}$$

Conservation of Linear Momentum

$$\frac{d}{dt} \int_{\Omega} \rho \, \mathbf{v} dv = \int_{\Omega} \rho \mathbf{f} \, dv + \int_{\Gamma} \mathbf{t} \, da$$

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \sigma^{\mathrm{T}} + \rho \mathbf{f}$$

Stress Measures (continued)

Cauchy Stress Tensor

Configuration in which it occurs

T >ﬃ ' T T >ﬃ ^ U >ﬃ ' U U >ﬃ

Configuration in which it is measured

First Piola-Kirchhoff Stress Tensor

U » ' U U > $ U £ ^ U w ' U s $ S £

' U « ' U » ' U d ' U w ' S d

K U U > $ U £ L ' U d ' K U s $ S £ L ' S d

Stress Measures (continued)

Second Piola-Kirchhoff Stress Tensor

KS £ $vL' Sd ` 'S«

' S« ` U S m# T $' U«

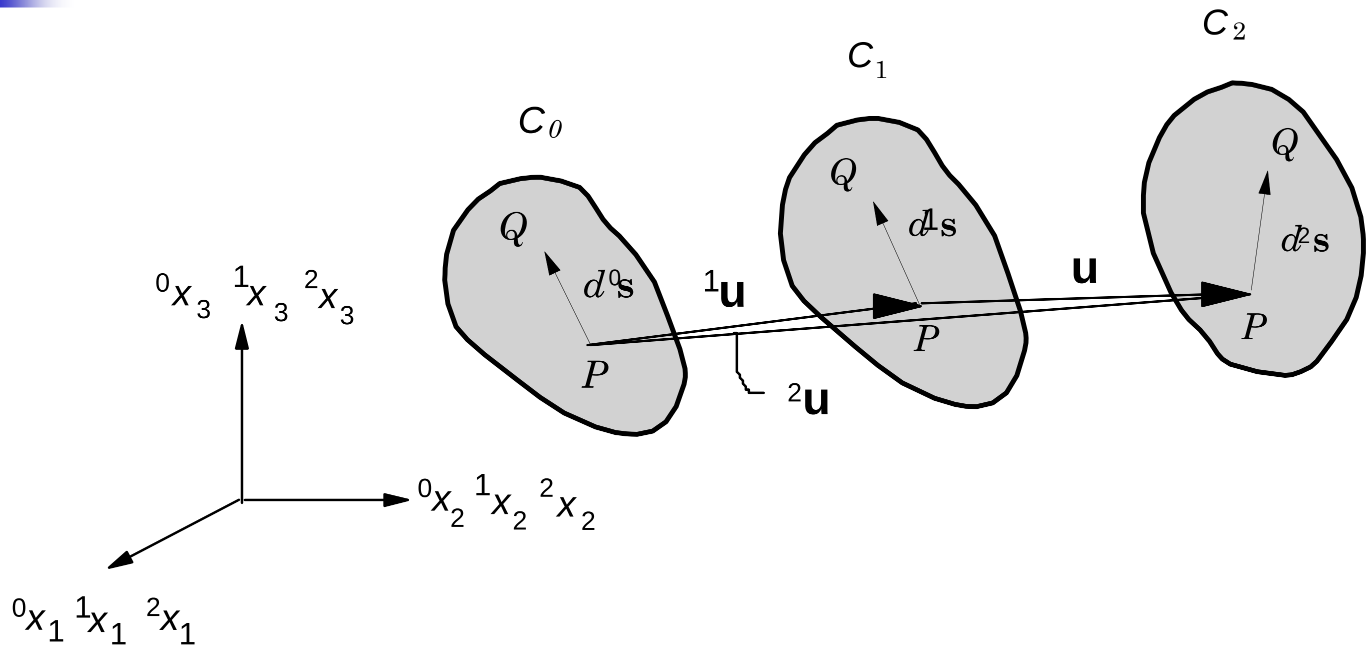
U S m ` c U` / c S` 4 i ^ U S m ` '"» 5 c U` / c S` 6

U S v ` i # T KU s L

$$\boxed{\mathbf{S} = J\, \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}}}$$

$$\mathbf{F} = \frac{\partial({}^{2}\mathbf{x})}{\partial({}^{0}\mathbf{x})} \quad \left(= \frac{\partial\,\mathbf{x}}{\partial\mathbf{X}} \right), \text{Deformation gradient tensor}$$

DESCRIPTION OF MOTION



Motion: ` ' `K{ T^{ U^{ V^»L

Notation: ¤¶¶„'fi· ƒ»‴ ¶«ƒ •¶fi·»] S ` ^ T ` ^ U `

‰¶ŧ‥‡ ‴] Sy^ Ty^ Uy

ƒ„"ƒ"] Sd^ Td^ Ud

' ".″fi»´] S=^ T=^ U=

»¶»ƒ† 'f″•†ƒ¢"‡ "·»″ ¶« ƒ •¶f·»] T… U…

Various Strain Components

Displacements:

T… \ T` # S` ¶„ T…fi \ T`fi # S`fi
U… \ U` # S` ¶„ U…fi \ U`fi # S`fi
… \ U… # T… ¶„ …fi \ U…fi # T…fi

Green-Lagrange Strain Tensor

KS ' "LU \ 'S` $'S` \ 'S`fi'S` i.
KT ' "LU \ 'T` $'T` \ 'T`fi'T`fi
KU ' "LU \ 'U` $'U` \ 'U`fi'U`fi

U T S h ffl 'S`fi'S`fl \ KT ' "LU # KS ' "LU
U U S h ffl 'S`fi'S`fl \ KU ' "LU # KS ' "LU

' T`fi \ C T`fi / C S`fl ' S`fl^ ' U`fi \ C U`fi / C S`fl ' S`fl

Strain Measures (continued)

Green-Lagrange Strain Tensor

$$
{}^1_0\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial\,{}^1u_i}{\partial\,{}^0x_j} + \frac{\partial\,{}^1u_j}{\partial\,{}^0x_i} + \frac{\partial\,{}^1u_k}{\partial\,{}^0x_i}\frac{\partial\,{}^1u_k}{\partial\,{}^0x_j}\right]
$$

$$
{}^2_0\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial\,{}^2u_i}{\partial\,{}^0x_j} + \frac{\partial\,{}^2u_j}{\partial\,{}^0x_i} + \frac{\partial\,{}^2u_k}{\partial\,{}^0x_i}\frac{\partial\,{}^2u_k}{\partial\,{}^0x_j}\right]
$$

Green-Lagrange *Incremental* Strain Tensor

$$
\begin{aligned}
2\,{}_0E_{ij}\,d\,{}^0x_i\,d\,{}^0x_j &= ({}^2ds)^2 - ({}^1ds)^2 \\
&= 2\left({}^2_0\varepsilon_{ij} - {}^1_0\varepsilon_{ij}\right)d\,{}^0x_i\,d\,{}^0x_j \\
&= 2\left({}_0e_{ij} + {}_0\eta_{ij}\right)d\,{}^0x_i\,d\,{}^0x_j
\end{aligned}
$$

Linear Green-Lagrange Incremental Strain Tensor

$$
{}_0e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial\,{}^0x_j} + \frac{\partial u_j}{\partial\,{}^0x_i} + \frac{\partial\,{}^1u_k}{\partial\,{}^0x_i}\frac{\partial u_k}{\partial\,{}^0x_j} + \frac{\partial u_k}{\partial\,{}^0x_i}\frac{\partial\,{}^1u_k}{\partial\,{}^0x_j}\right]
$$

Strain Measures (continued)

Nonlinear Green-Lagrange Incremental StrainTensor

$${}_0\eta_{ij} = \frac{1}{2}\frac{\partial u_k}{\partial\, {}^0x_i}\frac{\partial u_k}{\partial\, {}^0x_j}$$

Infinitesimal Green-Lagrange Incremental Strain Tensor

$${}_0e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^0x_j} + \frac{\partial u_j}{\partial\, {}^0x_i}\right)$$

Updated Green-Lagrange Strain (increment) Tensor

$${}^{2}_{1}\varepsilon_{ij}({}^1x_i, {}^1x_j) = \text{[illegible]}$$

$${}^{2}_{1}\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, {}^2x_k}{\partial\, {}^1x_i}\frac{\partial\, {}^2x_k}{\partial\, {}^1x_j} - \delta_{ij}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^1x_j} + \frac{\partial u_j}{\partial\, {}^1x_i} + \frac{\partial u_k}{\partial\, {}^1x_i}\frac{\partial u_k}{\partial\, {}^1x_j}\right)$$

$$= {}_1e_{ij} + {}_1\eta_{ij}$$

$${}_1\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_k}{\partial\, {}^1x_i}\frac{\partial u_k}{\partial\, {}^1x_j}\right)$$

$${}_1e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^1x_j} + \frac{\partial u_j}{\partial\, {}^1x_i}\right)$$

CONSERVATION OF MASS AND CONSTITUTIVE RELATIONS

Conservation of Mass

S= ‛ U= U S m, T= ‛ U= U T m

Constitutive Relations

T V ffl ‛ T f ffl– ¥ T E ¥, S V ffl ‛ S f ffl– ¥ S E ¥

S f ffl– ¥ ‛ S= / T= C S ˋ fi / C T ˋ • C S ˋ fl / C T ˋ , C S ˋ – / C T ˋ „ C S ˋ ¥ / C T ˋ „ T f • , „ ″

T f ffl– † ‛ T= / S= C T ˋ fi / C S ˋ = C T ˋ fl / C S ˋ , C T ˋ – / C S ˋ „ C T ˋ ¥ / C S ˋ „ S f • , „ ″

Virtual Work Statements

General Virtual Work Statement
(in configuration 2 to be determined)

}
1z 4 U>] 1K U "L ' U y # 1 U u ' S
} U y
' U >ffl 1K U "ffl L ' U y # 1 U u ' S
U y
} }
1 U u ' U « $1… ' U y N U » $1… ' U v
} U y } U v
' U «fi 1…fi ' U y N U »fi 1…fi ' U v
U y U v

Infinitesimal Green-Lagrange incremental strain tensor components

$$ {}_{2}e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^{2}x_j} + \frac{\partial u_j}{\partial\, {}^{2}x_i} \right) $$

Virtual Work Statement for the *Total Lagrangian Formulation*

Write various integrals over configuration 2 as equivalent integrals over the reference configuration

$$\int_{{}^2V} {}^2\sigma_{ij}\, \delta({}^2\varepsilon_{ij})\, d\,{}^2V = \int_{{}^0V} {}^2_0S_{ij}\, \delta({}^2_0E_{ij})\, d\,{}^0V$$

$$\int_{{}^2V} {}^2f_i\, \delta u_i\, d\,{}^2V = \int_{{}^0V} {}^2_0f_i\, \delta u_i\, d\,{}^0V$$

$$\int_{{}^2S} {}^2t_i\, \delta u_i\, d\,{}^2S = \int_{{}^0S} {}^2_0t_i\, \delta u_i\, d\,{}^0S$$

$$\int_{{}^0V} {}^2_0S_{ij}\, \delta({}^2_0E_{ij})\, d\,{}^0V - \delta({}^2_0R) = 0$$

$$\delta({}^2_0R) = \int_{{}^0V} {}^2_0f_i\, \delta u_i\, d\,{}^0V + \int_{{}^0S} {}^2_0t_i\, \delta u_i\, d\,{}^0S$$

$$\boxed{\int_{{}^0V} {}^2_0S_{ij}\, \delta({}^2_0E_{ij})\, d\,{}^0V - \delta({}^2_0R) = 0}$$

Total Lagrangian Formulation

1. Virtual Work Statement:

2. Incremental Decompositions:

Total Lagrangian Formulation (continued)

3. Virtual Work Statement with Incremental Decompositions:

$$\delta({}^2_0E_{ij}) = \delta({}^1_0E_{ij}) + \delta({}_0\varepsilon_{ij}) = \delta({}_0\varepsilon_{ij}),\quad \delta({}_0\varepsilon_{ij}) = \delta({}_0e_{ij}) + \delta({}_0\eta_{ij})$$

} }

S V ffl 1K S E ffl L ' S Y N T S V ffl 1K S 4 ffl L ' S Y

S Y S Y

}

' 1K U S u L# T S V ffl 1K S " ffl L ' S Y

S Y

4. Linearized Virtual Work Statement with Incremental Decompositions:

S V ffl ' S f ffl-¥ S E ¥ < S f ffl-¥ S "-¥ ^ 1K S E ffl L < 1K S " ffl L

ʃ ʃ

S f ffl-¥ S "-¥ 1K S " ffl L ' S Y N T S V ffl 1K S 4 ffl L ' S Y

S Y S Y

}

' 1K U S u L# T S V ffl 1K S " ffl L ' S Y

S Y

FINITE ELEMENT MODEL OF THE *TOTAL LAGRANGIAN* FORMULATION

[illegible]

FINITE ELEMENT MODEL (continued)

$[D]$

$[D_u]$

FINITE ELEMENT MODEL (continued)

}

«1S "‹W ~S f ¢«S "‹ ' S y

S y -ʃ

' «1.9‹W K-g ¢N -g ... ȻW ~S f ¢K-g ¢N -g ... Ȼ«.9‹ ' S V

S y

U c/c V S ... U c.../c c/c c‰/c c/c V

-g ¢' W S c/c X ^ -g ... ¢' W c.../c c/c c‰/c c/c X

c/c c/c c.../c c/c N c.../c c/c c‰/c c/c N c‰/c c/c

U c/c S V U V

Y c/c S Z S f TT S f TU S

-g9 ¢' Y Z ^ ~S f ¢' W S f TU S f UU S X

W S c/c X S S S f YY

S c/c

--l \ --

1 S "- - '

«1 S "‹ ' 1 S "- - ^ ' K-g ¢N -g ... Ȼ«1.9 ‹

] U1 S "- -

FINITE ELEMENT MODEL (continued)

FINITE ELEMENT MODEL (continued)

U T S V ` ` T S V ` ´ S S V

T S V ¢ ' Y W T S V ` ´ T S V ´ ´ S S Z X

S S T S V ` ` T S V ` ´

S S T S V ` ´ T S V ´ ´

= > = S . >

«… < ' … ‰ ' S fl' T …fl C fl K ` L ; fl' T ‰fl C fl K ` L ' ~, ¢«$ <

= > = S . >

«9 < ' 9 ‰ ' S fl' T 9fl C fl K ` L ; fl' T ‰fl C fl K ` L ' ~, ¢«$ <

~, ¢ ' C T S C U S $$$ C . S ; S C T S C U $$$ S C .

«$ < W ' «… T ^ ‰ T ^ … U ^ ‰ U ^ $$$ ^ … . ^ ‰ . <

«$ < W ' «9 T ^ ‰ T ^ 9 U ^ ‰ U ^ $$$ ^ 9 . ^ ‰ . <

FINITE ELEMENT MODEL (continued)

}

s«1s "<ʷ ~sf ¢«s "< ' ˢy

ˢy }

‘ «1.9 <ʷ K-g ¢N -g … ȼLʷ ~sf ¢K-g ¢N -g … ȼL«.9 < ' ˢy

}ˢy

‘ «1$ <ʷ «K-g ¢N -g … ȼL~, ¢«ʷ ~sf ¢K-g ¢N -g … ȼL~, ¢«$ < ' ˢy

}ˢy

‘ «1$ <ʷ -e o ¢ʷ ~sf ¢-e o ¢«$ < ' ˢy

ˢy

}

«1 s4 <ʷ «ᵀsv < ' ˢy

ˢy }

‘ «1.9 <ʷ -g ¢ʷ ~ᵀsv ¢-g ¢«.9 < ' ˢy

}ˢy

‘ «1$ <ʷ -e q o ¢ʷ ~ᵀsv ¢-e q o ¢«$ < ' ˢy

}ˢy

FINITE ELEMENT MODEL (continued)

ʃ
1K$_S^T$u L ' ^T_Sv ffl 1K$_S$ " ffl L ' ^{S}y
} ^{S}y
' «1 $_S$ "< ^{w} «$_S^T$v < ' ^{S}y
} ^{S}y
' «1$ < ^{w} ~e ○ ¢ ^{w} «$_S^T$v < ' ^{S}y
ˎ ^{S}y ˎ

ʃ ʃ
1K$_S^U$u L ' ^U_S«fi1…fi ' ^{S}y N ^U_S»fi1…fi ' ^{S}v
} ^{S}y ^{S}v }
' «1$ < ^{w} ~, ¢ ^{w} «$_S^U$«< ' ^{S}y N «1$ < ^{w} ~, ¢ ^{w} «$_S^U$»< ']
^{S}y ^{S}v

FINITE ELEMENT MODEL (continued)

K ~n ₒ ¢N ~n q ₒ ¢L«$ < ' «U S i < # «T S i <

} }

~n ₒ ¢' ~e ₒ ¢W ~S f ¢~e ₒ ¢' S y ^ «T S i < ' ~e ₒ ¢W «T S v < ' S y

}S y S y

~n q ₒ ¢' ~e q ₒ ¢W ~T S v ¢~e q ₒ ¢' S y

}S y }

«U S i < ' ~, ¢W «U S «< ' S y N ~, ¢W «U S »< ' S v

S y S v

= > = >

«U S «< ' U S «` ^ «U S »< ' U S »`

U S «- U S »-

U c/c` S V U c…/c` c/c` c‰/c` c/c` V

~g ¢' W S c/c` X ^ ~g … ¢' W c…/c` c/c` c‰/c` c/c` X

c/c` c/c` c…/c` c/c` N c…/c` c/c` c‰/c` c/c` N c‰/c` c/c`

FINITE ELEMENT MODEL (continued)

FINITE ELEMENT MODEL (continued)

FINITE ELEMENT MODEL (continued)

7 8 = > = >

ꝴn TTo ¢N ꝴn TTq ¢ ꝴn TUo ¢ «9< \ «U S i T < # «T S i T <

ꝴn UTo ¢ ꝴn UUo ¢N ꝴn UUq ¢ «8< «U S i U < # «T S i U <

} = 5 6 U 5 6 U

n TTo ffl \ >" -" S f TT TN c… / c` cCfi / c` cCfl / c` N S f UU c… / c´ cCfi / c´ cCfl / c´

5 6 5 6

N S f TU TN c… / c` c… / c´ cCfi / c` cCfl / c´ N cCfi / c´ cCfl / c`

75 6 8

N S f YY TN c… / c` cCfi / c´ N c… / c´ cCfi / c`

75 6 8>

% TN c… / c` cCfl / c´ N c… / c´ cCfl / c` '` '´

} = 5 6 5 6

n TUo ffl \ >" -" T f TT TN c… / c` c‰ / c` cCfi / c` cCfl / c` N S f UU TN c‰ / c´ c… / c´ cCfi / c´ cCfl / c´

75 6 5 6 8

N S f TU TN c… / c` TN c‰ / c´ cCfi / c` cCfl / c´ N c… / c´ c‰ / c` cCfi / c´ cCfl / c`

75 6 8

N S f YY TN c… / c` cCfi / c´ N c… / c´ cCfi / c`

75 6 8>

% TN c‰ / c´ cCfl / c` N c‰ / c` cCfl / c´ '` '´ \ n UTo flfi

FINITE ELEMENT MODEL (continued)

} = 5 6 U 5 6 U

n ﬀl UUo ‛ >„ -„ s f TT c‰/c` 5 cCﬁ/c` cCﬂ/c` 6 N s f UU T N c‰/c´ cCﬁ/c´ cCﬂ/c´ 6

5 5 N s f TU T N c‰/c´ c‰/c` cCﬁ/c` cCﬂ/c´ N cCﬁ/c´ cCﬂ/c` 6

75 6 8 N s f YY T N c‰/c´ cCﬁ/c` N c‰/c` cCﬁ/c´

75 6 8> % T N c‰/c´ cCﬂ/c` N c‰/c` cCﬂ/c´ '` '´

} 7 5 6

n ﬀl TTq ‛ >„ -„ T/S v`` cCﬁ/c` cCﬂ/c` N T/S v`´ cCﬁ/c´ cCﬂ/c` N cCﬁ/c` cCﬂ/c´

8 N T/S v´´ cCﬁ/c´ cCﬂ/c´ '` '´ ‛ n ﬀl UUq K\]Y]VSƒL

} 1

U/S i T/ﬁ ‛ >„ U/S «` Cﬁ '` '´ N >„ U/S » Cﬁ '″

}-„ 1#„

U/S i U/ﬁ ‛ >„ U/S «´ Cﬁ '` '´ N >„ U/S » Cﬁ '″

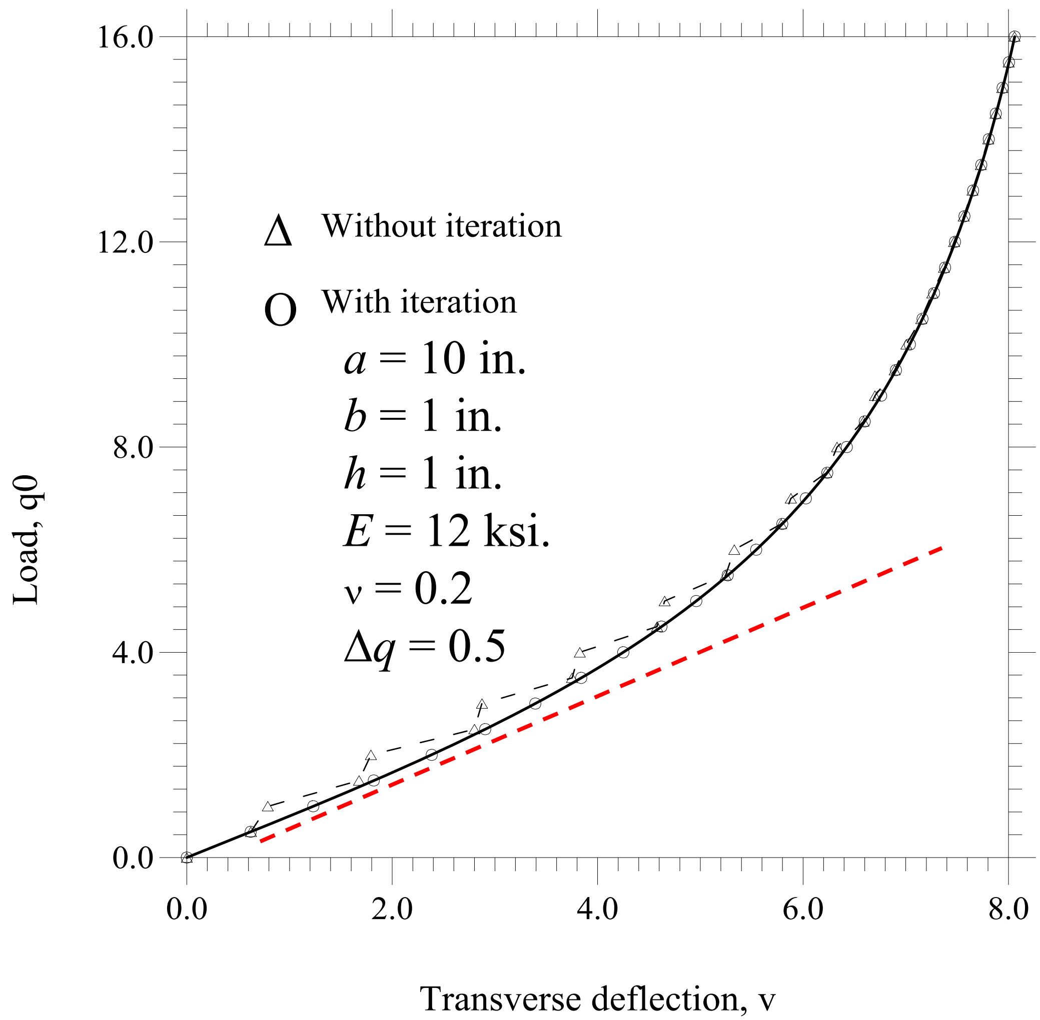
}-„ = 5 6 #„

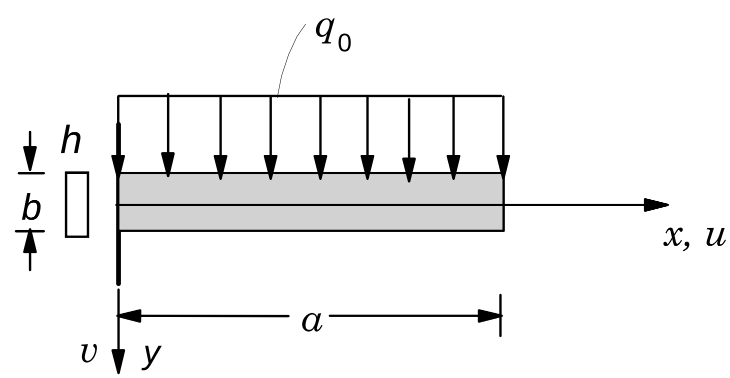
T/S i T/ﬁ ‛ >„ -„ T N c…/c` cCﬁ/c` T/S v`` N c…/c´ cCﬁ/c´ T/S v`´

75 6 8 > N T N c…/c` cCﬁ/c´ N c…/c´ cCﬁ/c` T/S v`´ '` '´

NUMERICAL RESULTS

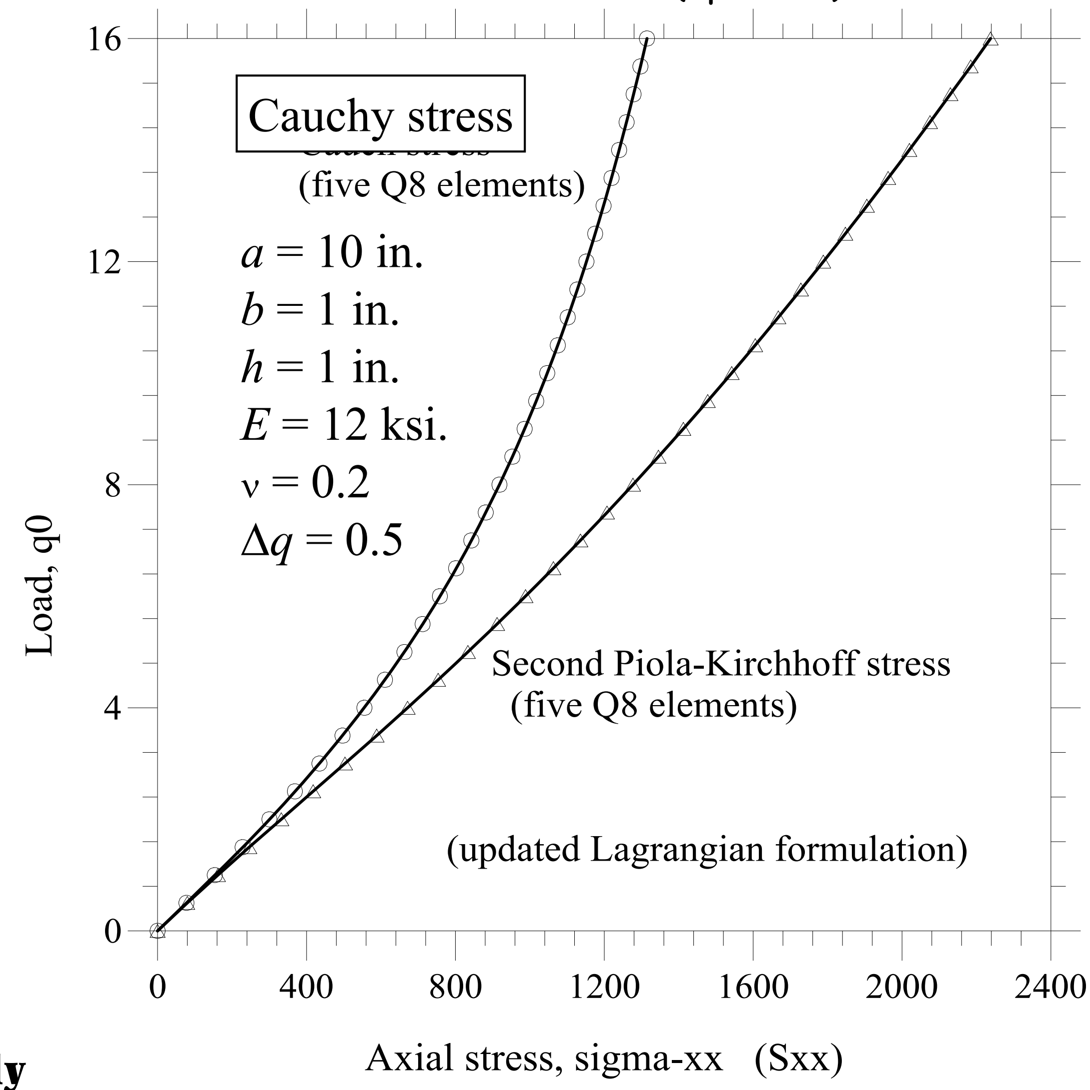
Transverse deflection versus load (cantilever beam under uniformly distributed load).





NUMERICAL RESULTS

Normal stresses vs. load for a cantilever beam under uniform load (updated).



SHELL FINITE ELEMENTS

Degenerate Continuum Shell Elements

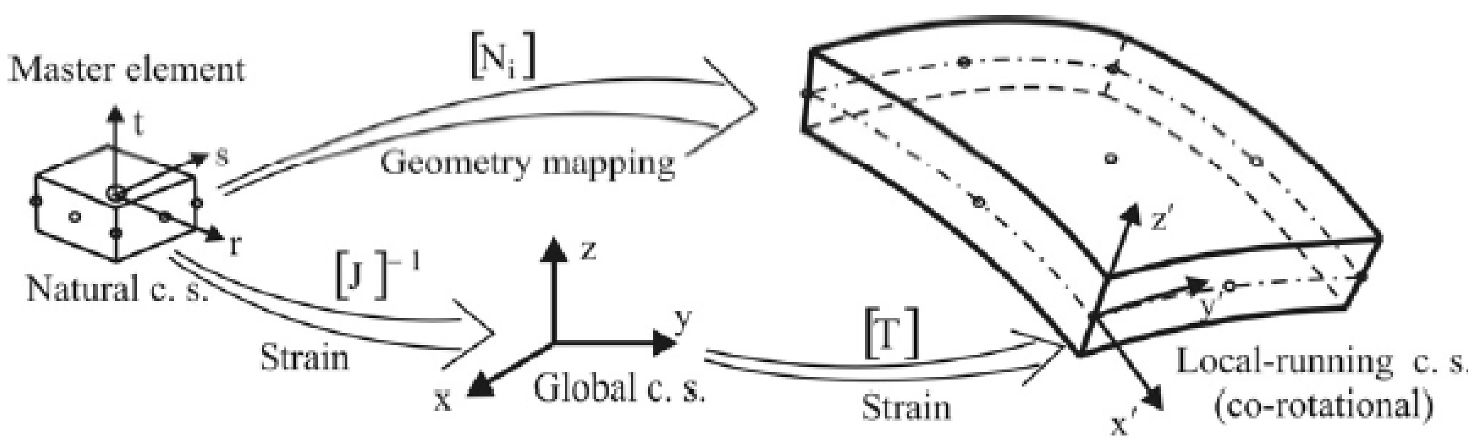
It is obtained by a degeneration process from the 3D continuum equations. Less expensive than the solid element but more expensive than the shell-theory element. Also exhibit locking problems.

Shell Theory Elements

Based on a curvilinear description of the continuum using an specific shell theory. Analytical integration of energy terms over the thickness. Also exhibit locking problems.

CONTINUUM SHELL ELEMENTS

The degenerate continuum shell element approach was first developed by Ahmad et al. from a three-dimensional solid element by a process which the 3D elasticity equations are expressed in terms of mid-surface nodal variables. It uses displacements and rotational DOFs.



CONTINUUM SHELL ELEMENTS

Assumptions are:

1. Fibers remain straight
2. The stress normal to the midsurface vanishes (the plane stress condition)

Discretization of the reference surface by using Cartesian coordinates

$$X_i = \sum_{\text{k}=1}^{\text{n}} \psi_k(\xi,\eta)\left[\frac{1+\zeta}{2}(X_i^k)_{\text{TOP}} + \frac{1-\zeta}{2}(X_i^k)_{\text{BOTTOM}}\right]$$

where

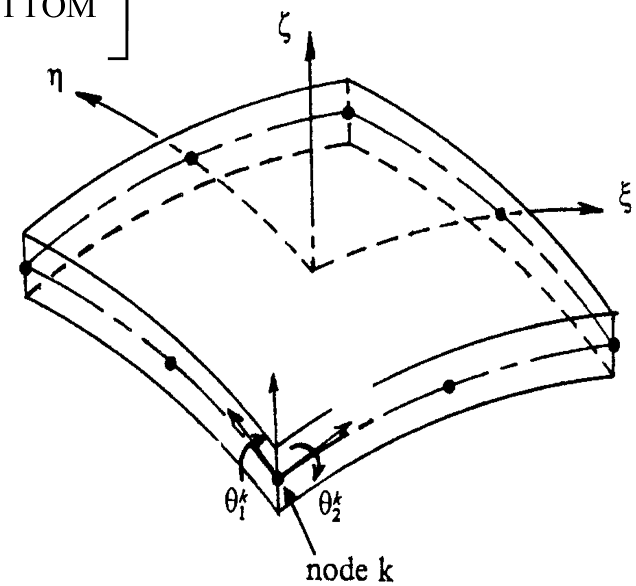
$\psi_{\text{k}}(\xi,\eta)$: Lagrangian interpolation functions

(ξ,η,ζ): Natural coordinates

CONTINUUM SHELL ELEMENTS

$$x_i = \sum_{k=1}^{n} \psi_k(\xi,\eta)\left[\frac{1+\zeta}{2}(x_i^k)_{\mathrm{TOP}} + \frac{1-\zeta}{2}(x_i^k)_{\mathrm{BOTTOM}}\right]$$

Discretization of the final configuration of the reference surface



Equivalent to the discretization of the displacement of the reference surface as

$$u_i = \sum_{k=1}^{n} \psi_{\mathrm{k}}(\xi,\eta)\left[u_i^k + \frac{\zeta}{2} h_k\left(\mathrm{e}_{3i}^k\right)\right]$$

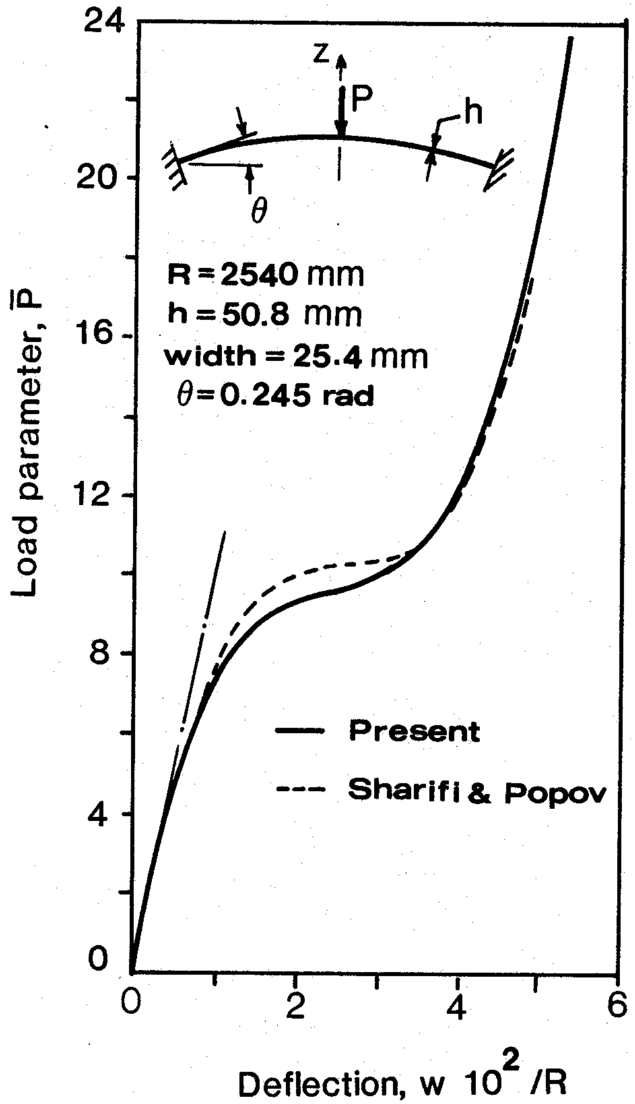
Continuum Shell Element

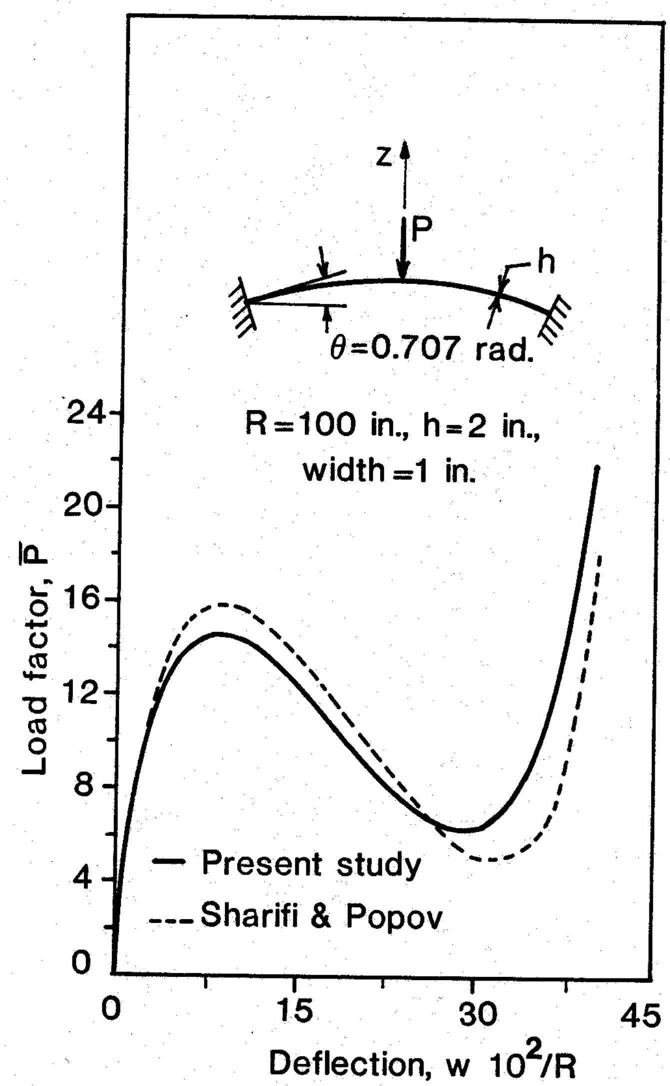
(continued)

Virtual Work Statement for the Dynamic Case

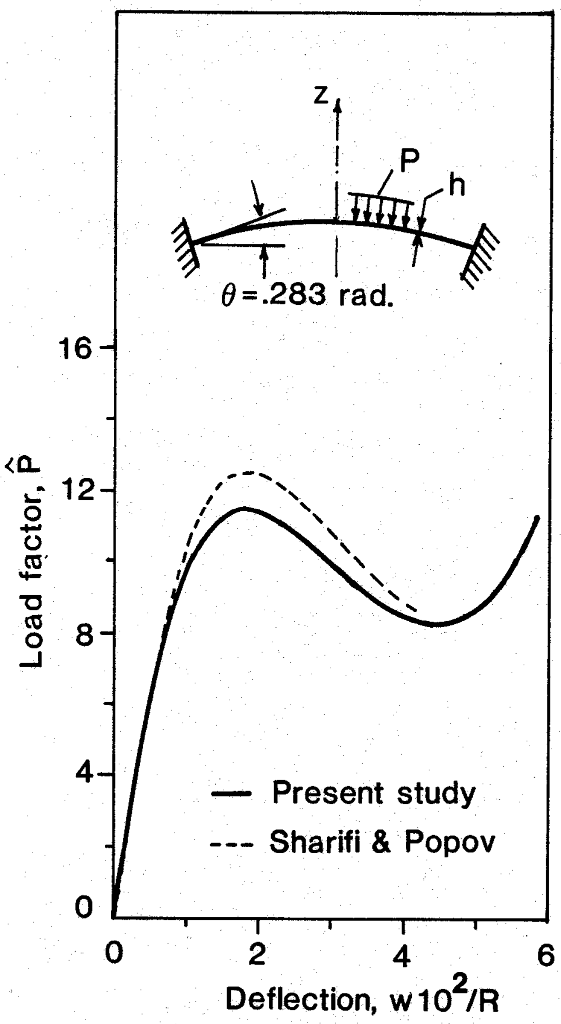
Finite Element Model

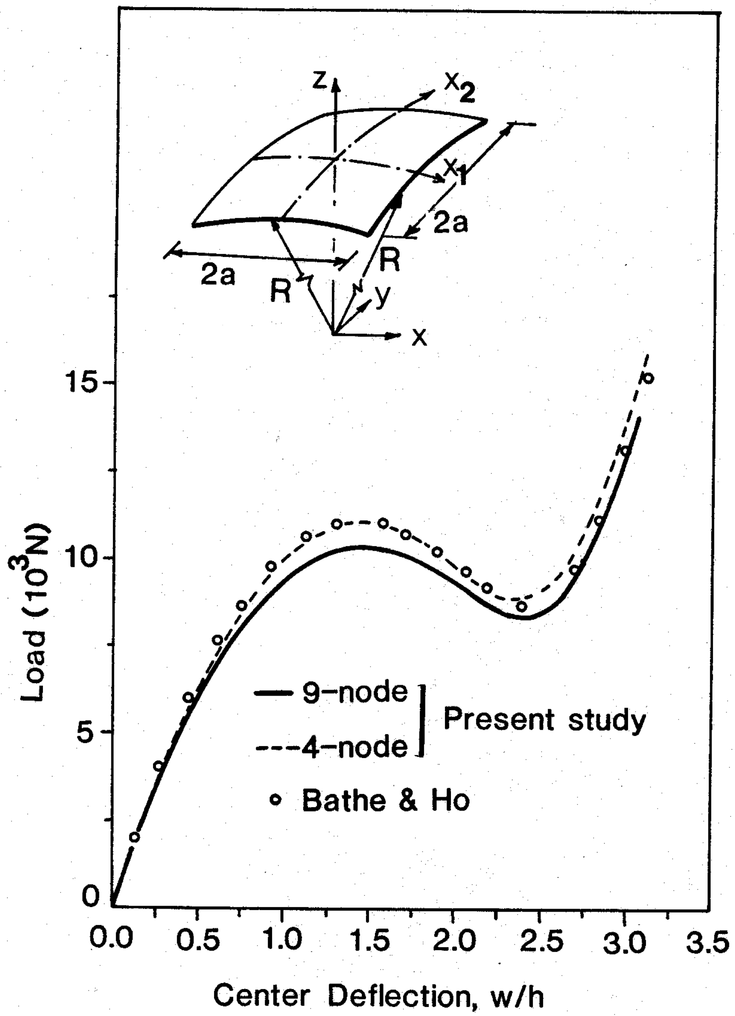
Numerical Examples



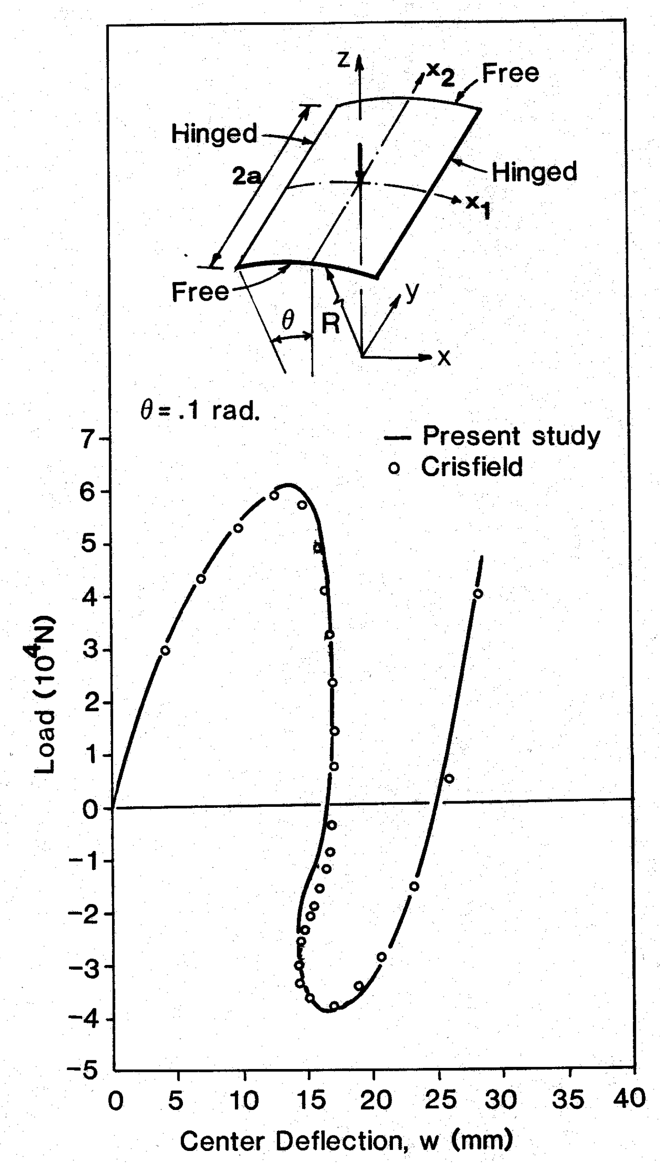


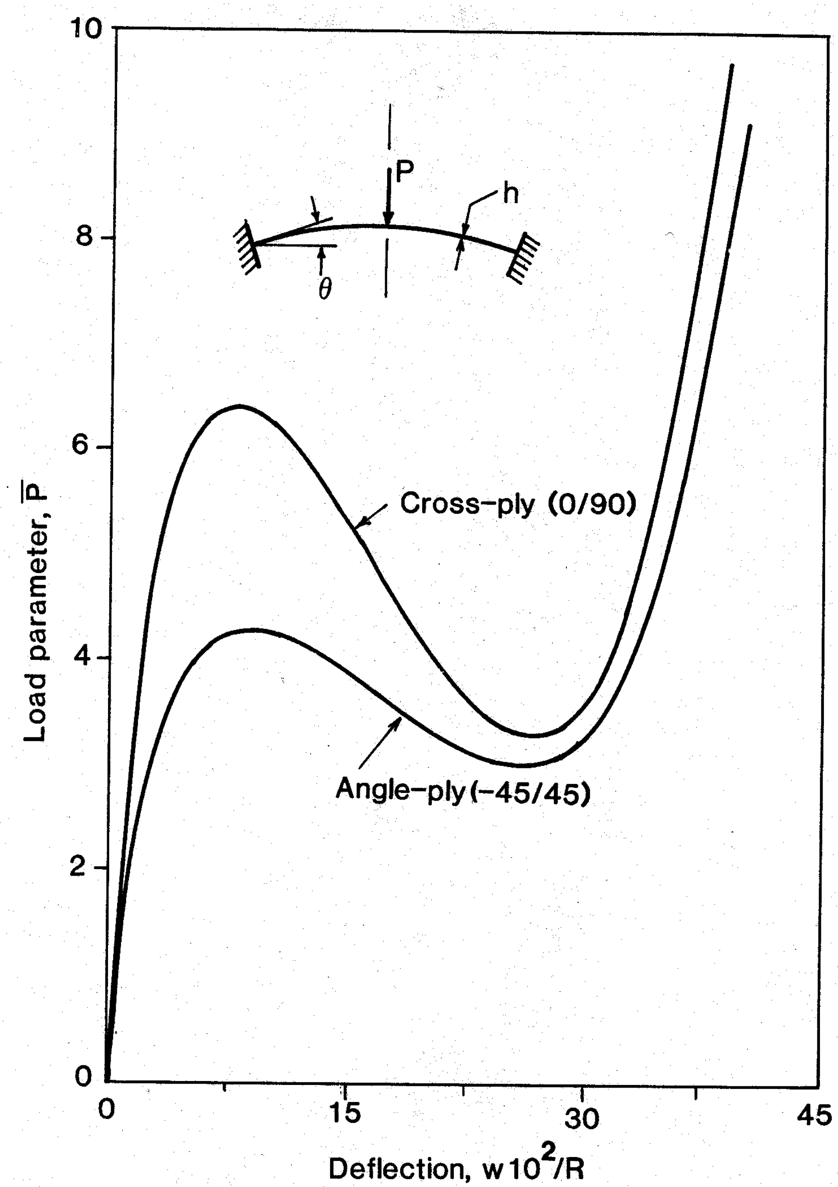
Numerical Examples



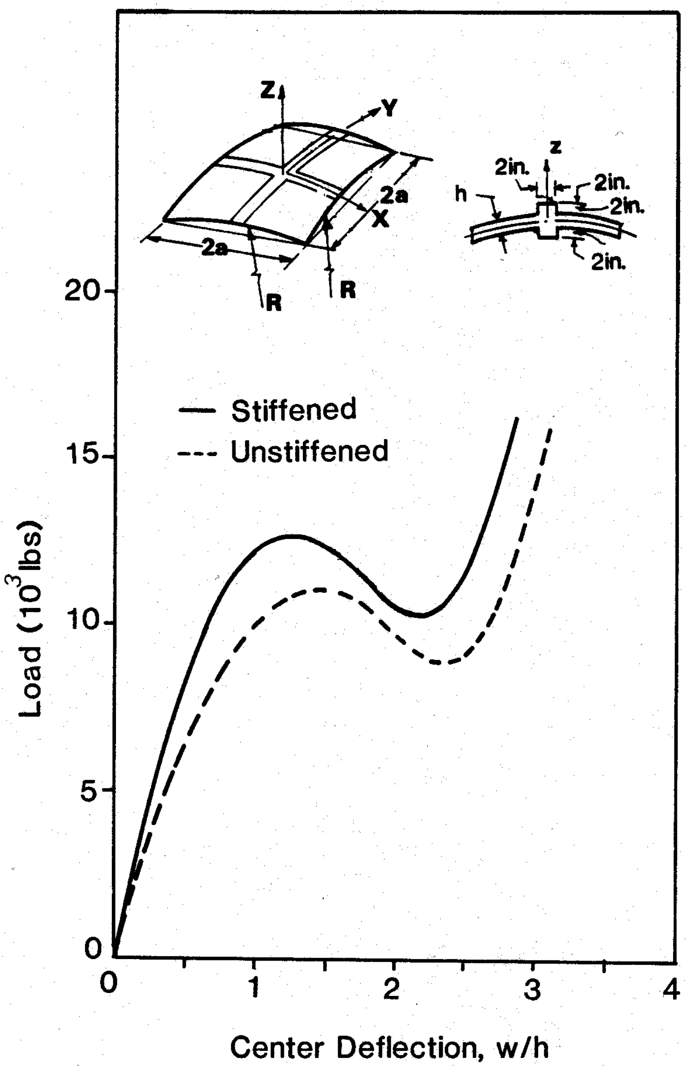


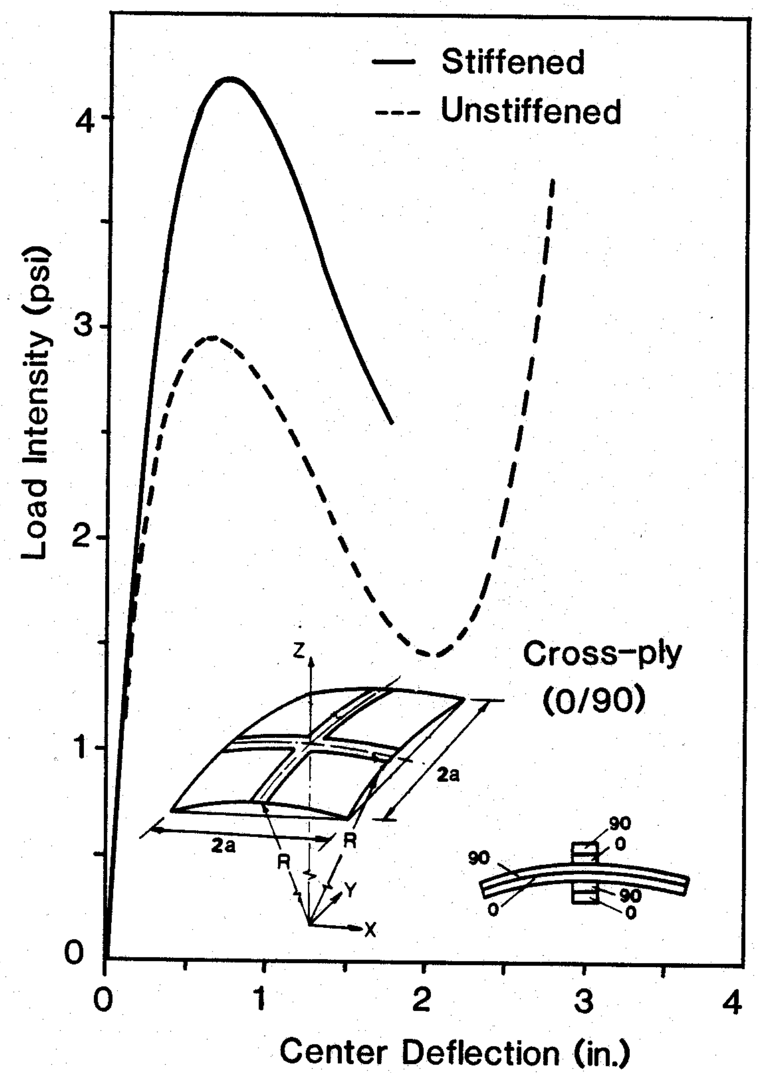
Numerical Examples





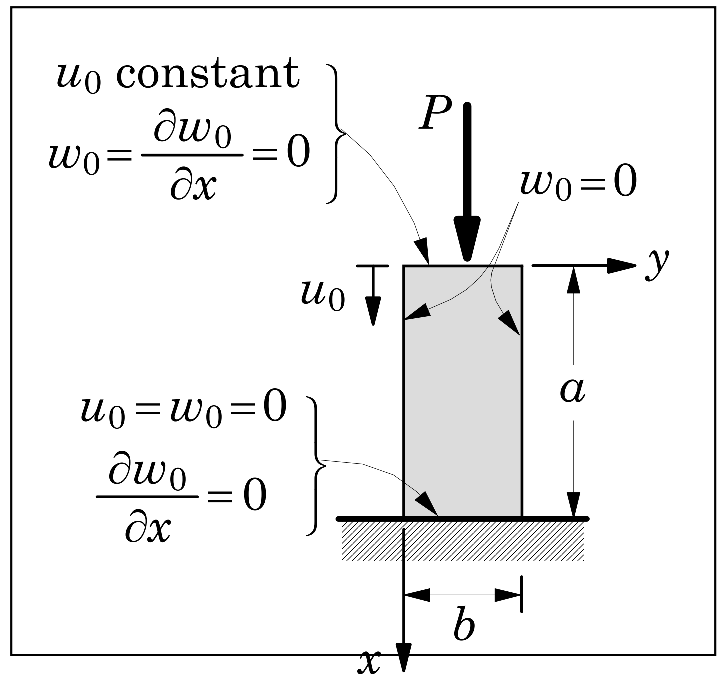
Numerical Examples





POSTBUCKLING AND FAILURE ANALYSIS

- Nonlinear FE analysis
- Comparison with experimental results of Starnes and Rouse
- Progressive failure analysis



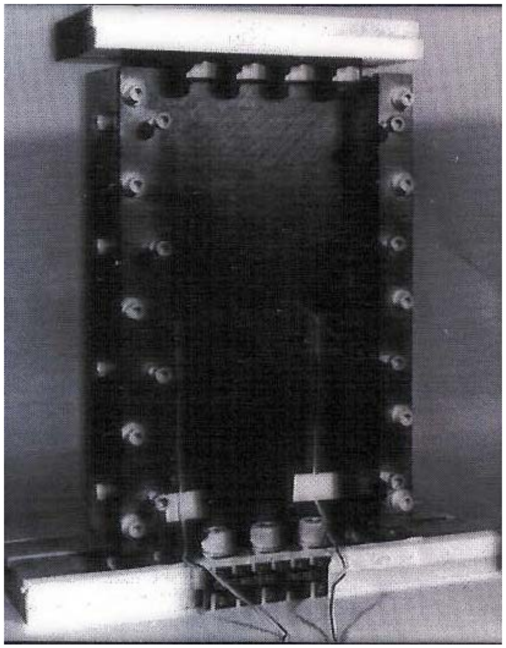
a = 50.8 cm (20 in.), b = 17.8 cm (7 in.), h_k = 0.14 mm (0.0055 in.)

24-ply laminate: $[45/-45/0_2/45/-45/0_2/45/-45/0/90]_s$

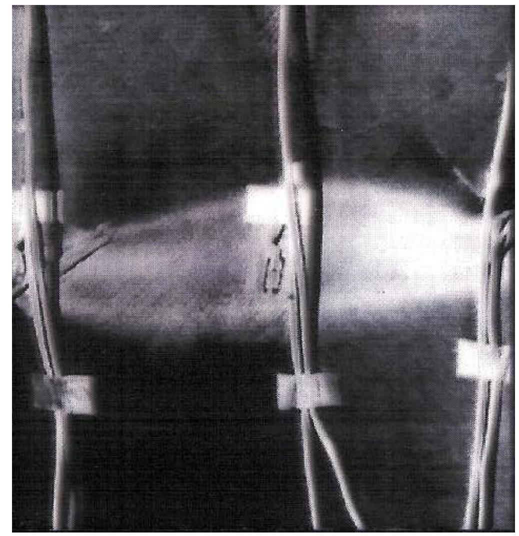
E_1 = 131 Gpa (19,000 ksi), E_2 = 13 Gpa (1,890 ksi),

G_{12} = 6.4 Gpa (930 ksi), ν_{12} = 0.38 (graphite-epoxy)

Experimental Setup and Failure Region (Starnes & Rouse, NASA Langley)



(a) Typical panel with test fixture



(b) A transverse shear failure mode

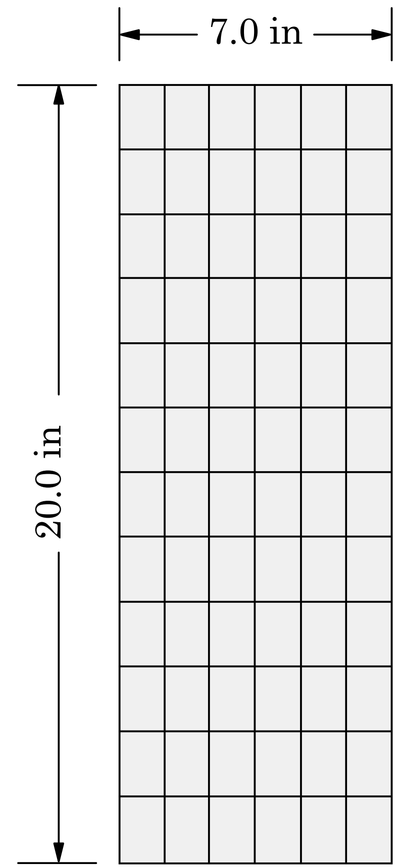
POST-BUCKLING OF A COMPOSITE PANEL UNDER IN-PLANE LOADING

Finite Element Models

- Mesh has six elements per buckling mode half wave in each direction

- Elements used:
  (1) Four-node C^1 -based flat shell element (**STAGS**)
  (2) Nine-node continuum shell element (**Chao & Reddy**)
  (3) For-node and nine-node ANS shell elements (**Park & Stanley**)

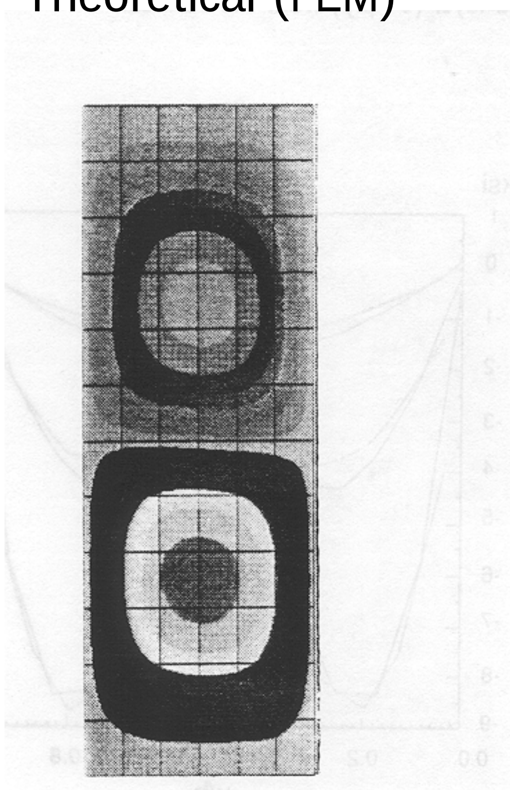
- All meshes have 72 elements (91 nodes for the meshes with four-node elements and 325 nodes for meshes with nine-node elements).



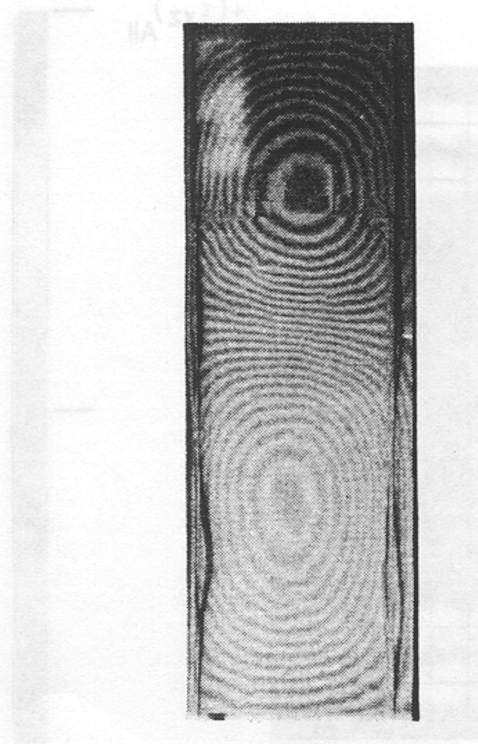
Continuum Shell Element

Comparison of the Experimental (Moire) and Analytical Out-of-Plane Deflection Patterns

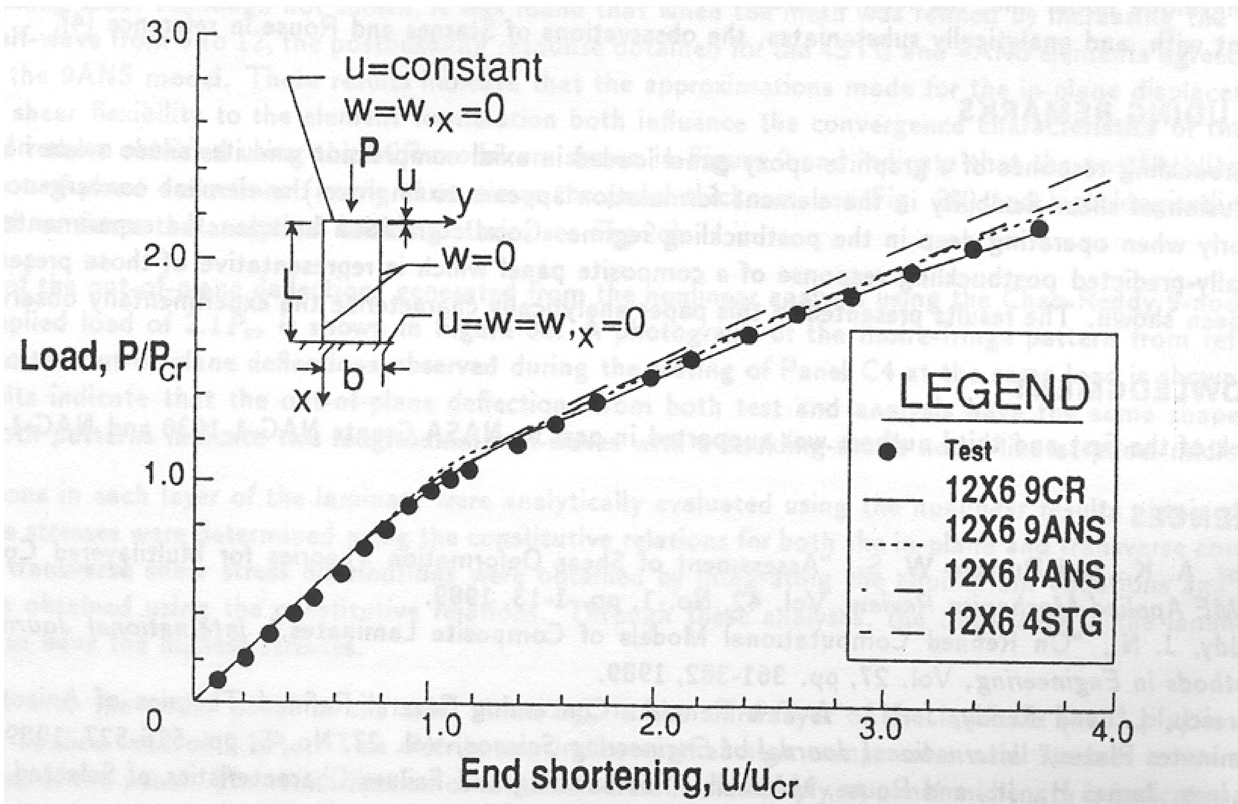
Theoretical (FEM)



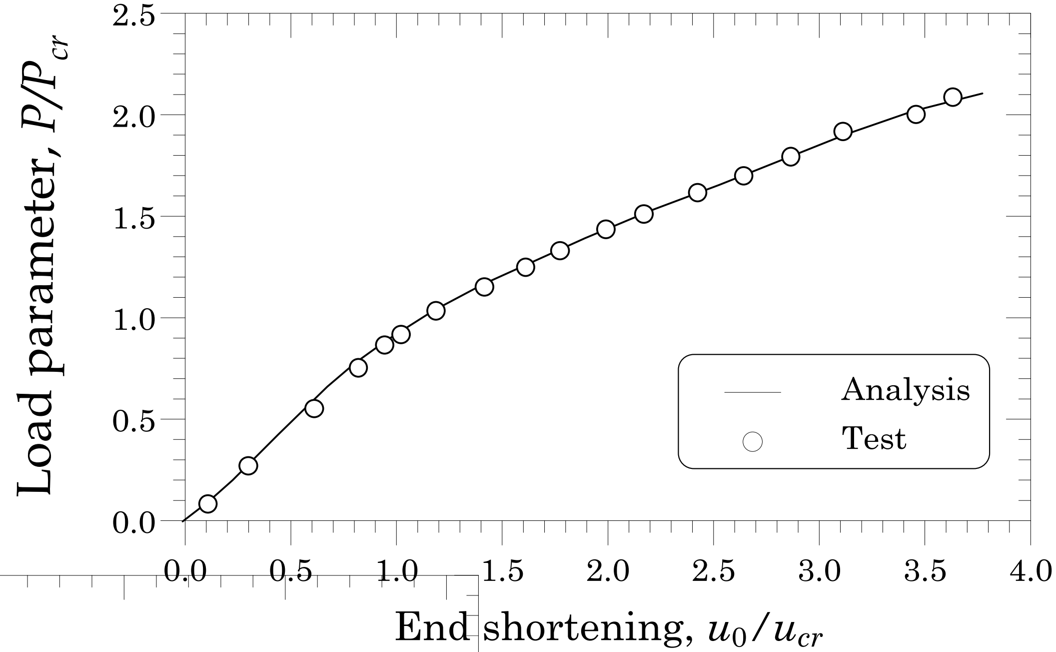
Experimental

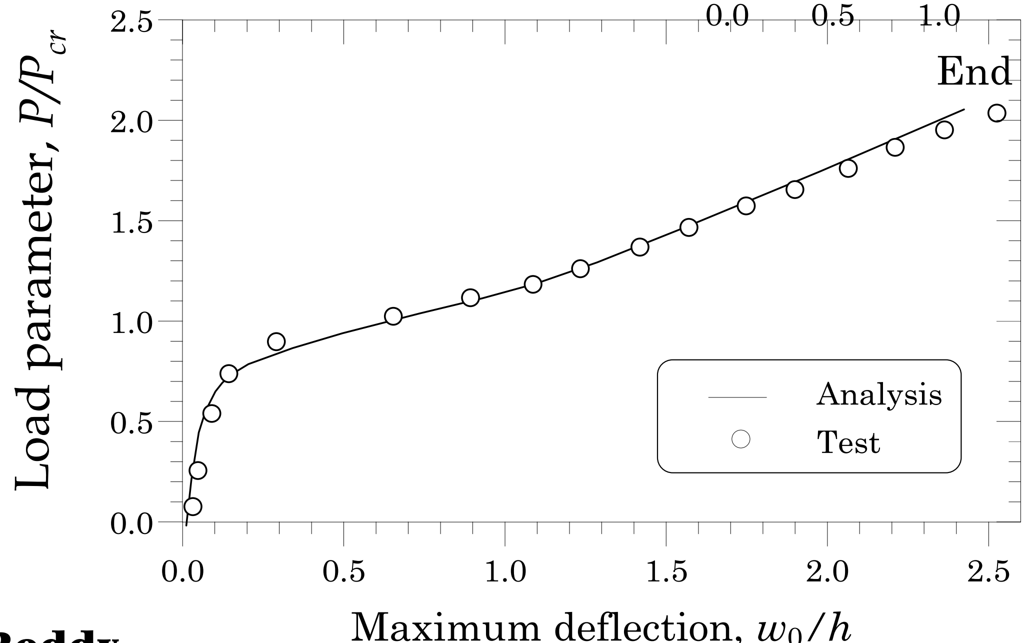


Comparison of the Experimental and Analytical Solutions for End Shortening

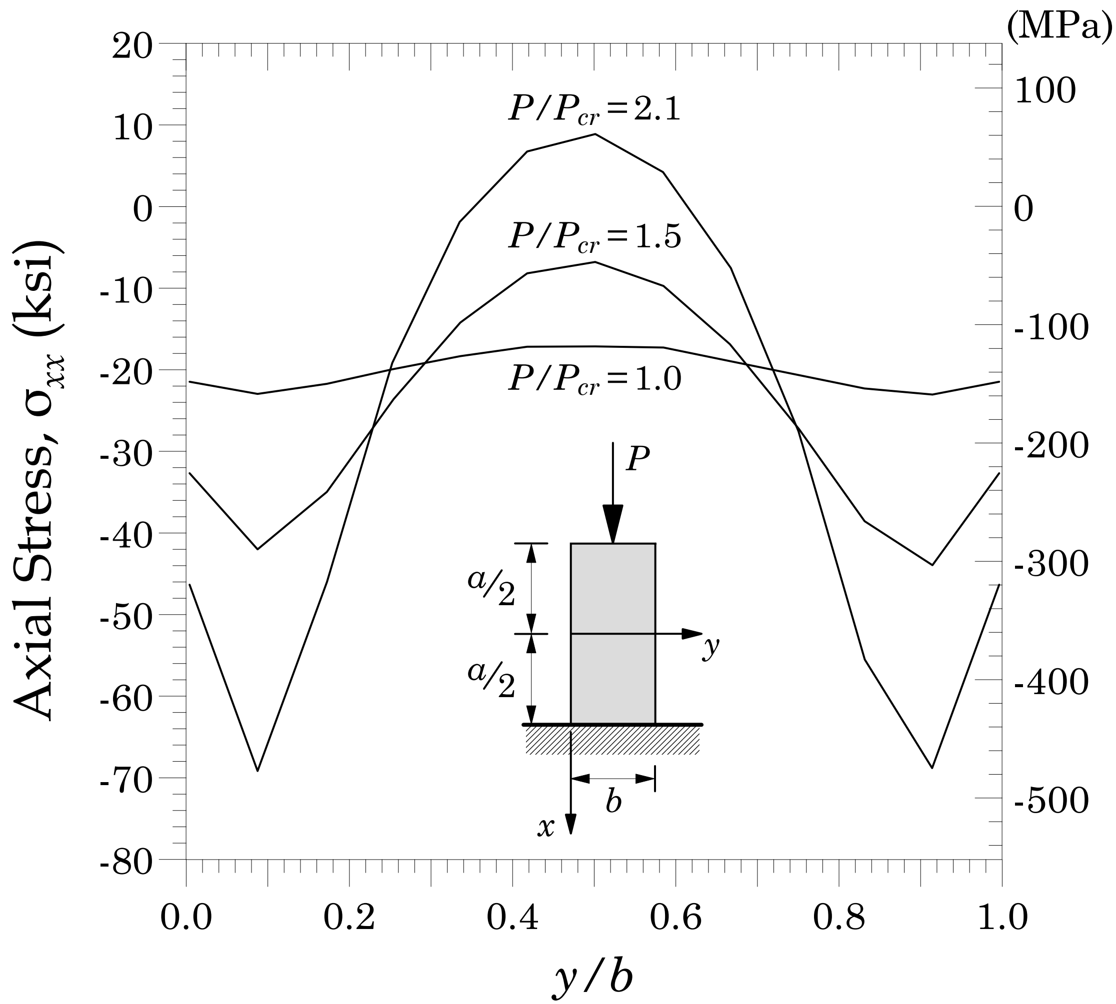


Postbuckling of a Composite Panel

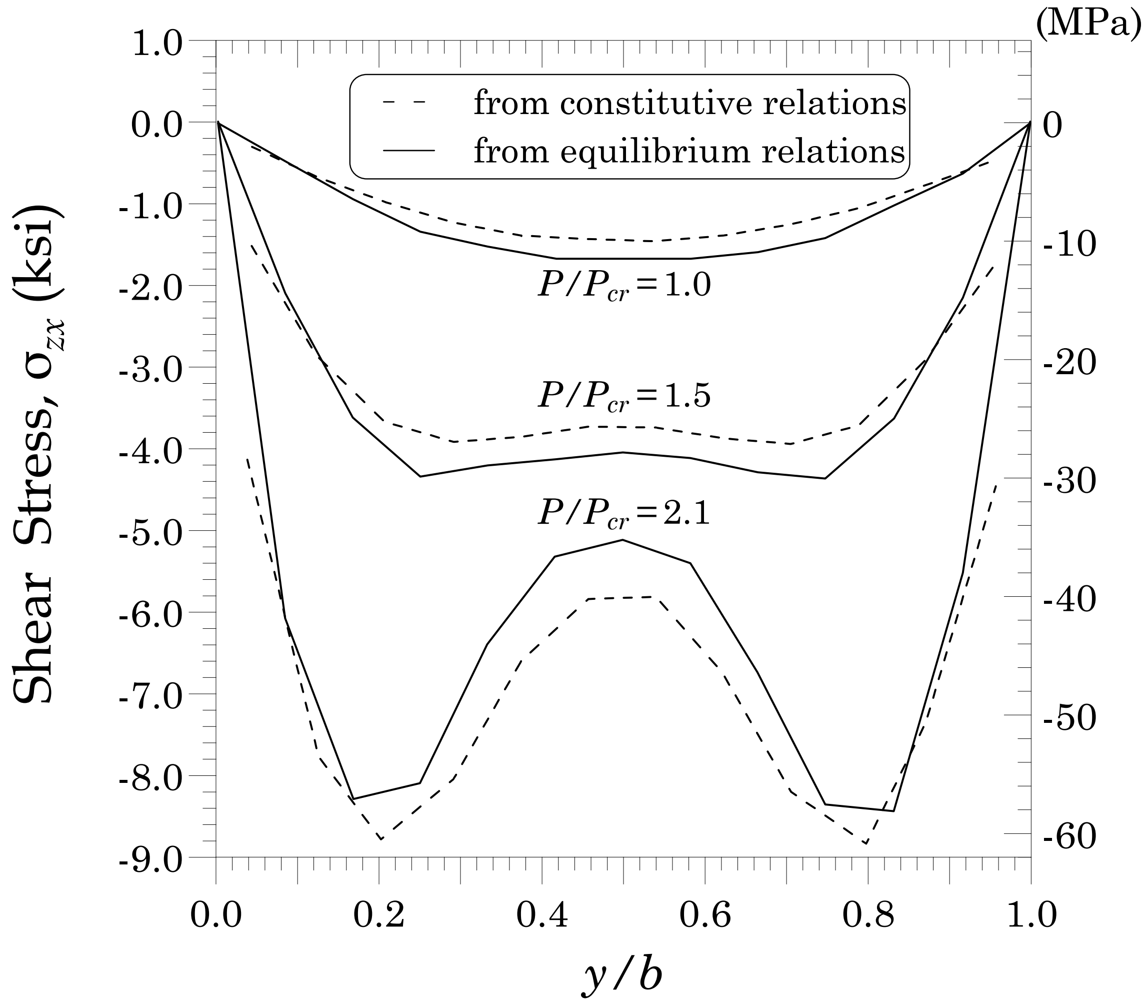




Postbuckling of a Composite Panel (Stress Distributions)



Postbuckling of a Composite Panel (Stress Distributions)



Summary

- Introduction to nonlinear continuum mechanics
- Descriptions of motion
- Measures of stresses and strains
- Updated and Total Lagrangian formulations
- Continuum shell element
- Post-buckling of composite panels