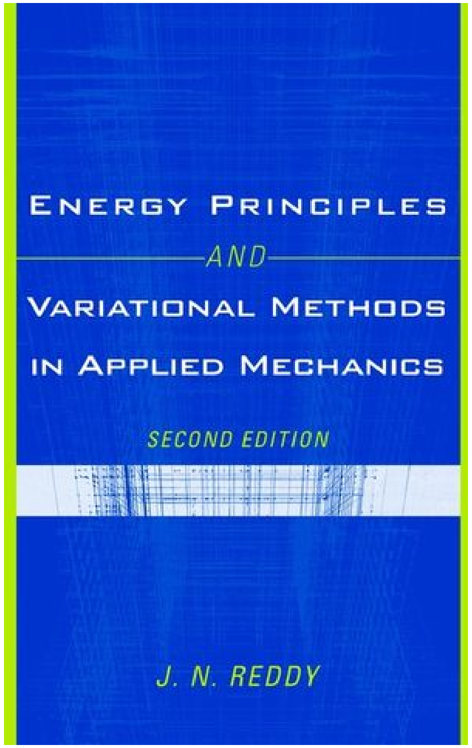
MEEN 618:
ENERGY AND VARIATIONAL METHODS

Clapeyron's, Betti's, and Maxwell's Theorems

Read: Chapter 5



CONTENTS

- Principle of superposition
- Clapeyron's Theorem
- Elasticity problems, uniqueness
- Betti's Reciprocity Theorem
- Maxwell's Reciprocity Theorem

THE PRINCIPLE OF SUPERPOSITION

The principle of superposition is said to hold for a problem if the responses (i.e., displacements) under two sets of boundary conditions and loads are equal to the sum of the responses obtained by applying each set of boundary conditions and loads separately.

$$\text{Set 1}: \quad \mathbf{u} = \hat{\mathbf{u}}^{(1)} \text{ on } \Gamma_u; \quad \mathbf{t} = \hat{\mathbf{t}}^{(1)} \text{ on } \Gamma_{\boldsymbol{\sigma}}; \quad \mathbf{f} = \mathbf{f}^{(1)} \text{ in } \Omega$$

$$\text{Set 2}: \quad \mathbf{u} = \hat{\mathbf{u}}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \hat{\mathbf{t}}^{(2)} \text{ on } \Gamma_{\boldsymbol{\sigma}}; \quad \mathbf{f} = \mathbf{f}^{(2)} \text{ in } \Omega$$

Then if

$$\mathbf{u} = \hat{\mathbf{u}}^{(1)} + \hat{\mathbf{u}}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \hat{\mathbf{t}}^{(1)} + \hat{\mathbf{t}}^{(2)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} \text{ in } \Omega$$

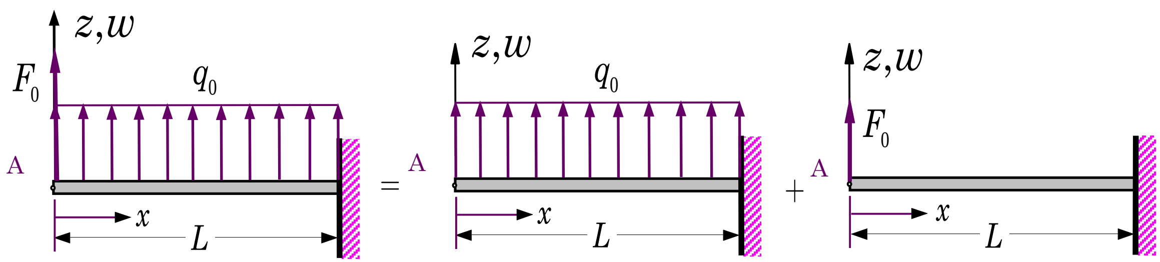
and the solution is

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{(1)}(\mathbf{x}) + \mathbf{u}^{(2)}(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \Omega$$

We say that the principle of superposition holds.

THE PRINCIPLE OF SUPERPOSITION

Application to Beams



Combined loads Load system 1 Load system 2

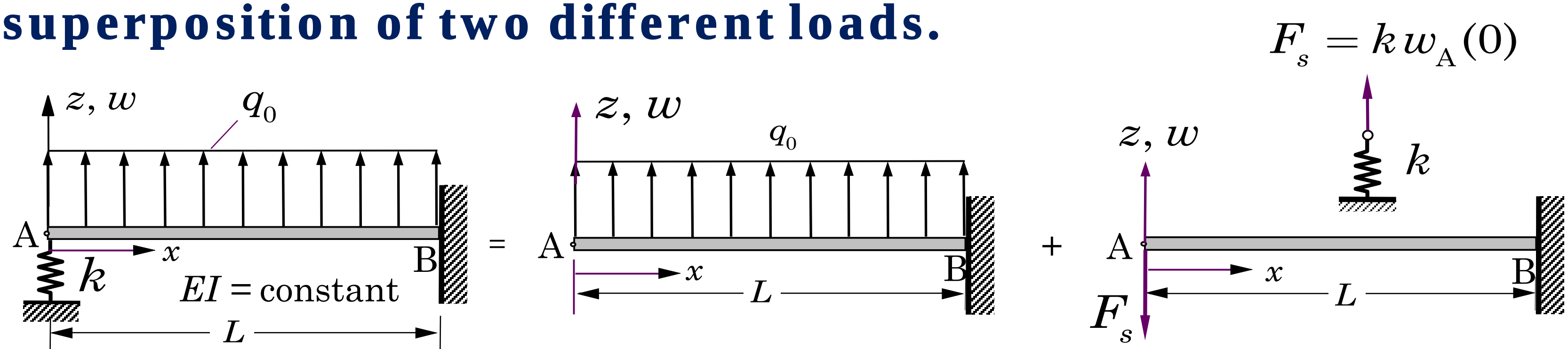
The displacement field of a beam subjected to two different loads

$$w^q(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^4\right], \; w^F(x) = \frac{F_0 L^3}{6EI}\left[2 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]$$

$$w(x) = w^q(x) + w^F(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^4\right] + \frac{F_0 L^3}{6EI}\left[2 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]$$

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

Consider the beam shown in the figure. It can be viewed as a superposition of two different loads.



Given Problem

Problem 1

Problem 2

$$w(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^4\right] + \frac{(-F_s)L^3}{6EI}\left[2 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]$$

$$w(0) = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI} \quad \Rightarrow \quad w(0) = \frac{q_0 L^4}{8EI}\left(1 + \frac{kL^3}{3EI}\right)^{-1} \qquad [F_s = kw(0)]$$

$$w(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^4\right] - \frac{kL^3}{6EI}\left[2 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^3\right] w(0)$$

$$= \frac{q_0 L^4}{24EI}\left[3 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^4\right] - \frac{q_0 L^4}{16EI}\left[2 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]\left(1 + \frac{3EI}{kL^3}\right)^{-1}$$

CLAYPERON'S THEOREM

Theorem: The strain energy stored in a linear elastic body is equal to one-half of the work done by external forces on the body:

$$U = -\tfrac{1}{2} V_E$$

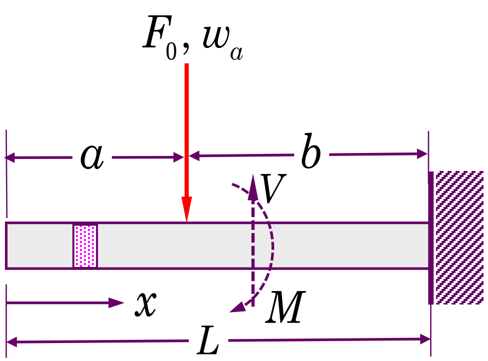
Proof: We begin with the strain energy stored in a linear elastic body

$$\begin{aligned}
U &= \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega = \frac{1}{2}\int_{\Omega} \sigma_{ij}\varepsilon_{ij}\, d\Omega = U^{*} \\
&= \frac{1}{4}\int_{\Omega} \sigma_{ij}\left(u_{i,j} + u_{j,i}\right) d\Omega = \frac{1}{2}\int_{\Omega} \sigma_{ij} u_{i,j}\, d\Omega \\
&= -\frac{1}{2}\int_{\Omega} \sigma_{ij,j} u_i\, d\Omega + \frac{1}{2}\oint_{\Gamma} n_j \sigma_{ij} u_i\, d\Gamma \\
&= \frac{1}{2}\int_{\Omega} f_i u_i\, d\Omega + \frac{1}{2}\oint_{\Gamma} t_i u_i\, d\Gamma = -\tfrac{1}{2} V_E
\end{aligned}$$

USES OF CLAYPERON'S THEOREM

Problem: **Find w_a using Clayperon's Theorem.**

Solution: **By Clayperon's Theorem,**



$$\tfrac{1}{2}F_0 w_a = U^* \equiv \tfrac{1}{2}\int_0^L \left(\frac{M^2}{EI} + \frac{V^2}{GAK_s}\right)dx$$

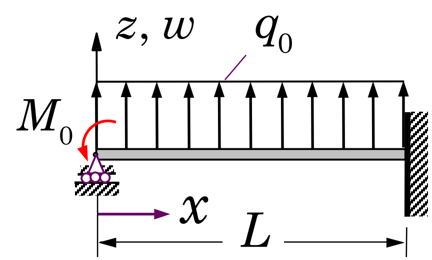
$$M(x) = \begin{cases} 0, & 0 \le x \le a \\ F_0(x-a), & a \le x \le L \end{cases}, \quad V(x) = \begin{cases} 0 \\ F_0 \end{cases}$$

$$\tfrac{1}{2}F_0 w_a = \tfrac{1}{2}\int_a^L \left(\frac{F_0^2}{EI}(x-a)^2 + \frac{F_0^2}{GAK_s}\right)dx$$

$$F_0 w_a = \left(\frac{F_0^2}{3EI}(x-a)^3 + \frac{F_0^2}{GAK_s}x\right)\Bigg|_a^L \Rightarrow w_a = \frac{F_0 b^3}{3EI} + \frac{F_0 b}{GAK_s}$$

USES OF CLAYPERON'S THEOREM

Problem: **Find θ_0 using Clayperon's Theorem.**



EI = constant

Solution: **By Clayperon's Theorem,**

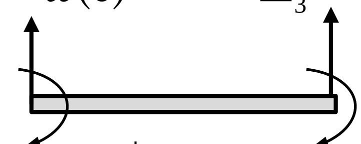
$$\tfrac{1}{2}\int_0^L q_0 w(x)\, dx - \tfrac{1}{2} M_0\theta_0 = \frac{EI}{2}\int_0^L \left(\frac{d^2 w}{dx^2}\right)^2 dx$$

$$w(x) = \sum_{j=1}^{4} \Delta_j \varphi_j(x) = \Delta_2 \varphi_2(x) = -x\left(1 - \frac{x}{L}\right)^2 \theta_0$$

$$\int_0^L q_0 w(x)\, dx - M_0\theta_0 = EI\int_0^L \left(\frac{d^2 w}{dx^2}\right)^2 dx$$

$$-\frac{q_0 L^2}{12}\theta_0 - M_0\theta_0 = \frac{4EI}{L}\theta_0^2 \Rightarrow \theta_0 = -\left(\frac{q_0 L^3}{48EI} + \frac{M_0 L}{4EL}\right)$$

$\Delta_1 = w(0)$ $\quad$ $\Delta_3 = w(L)$



$\Delta_2 = -\left.\frac{dw}{dx}\right|_{x=0}$ $\quad$ $\Delta_4 = -\left.\frac{dw}{dx}\right|_{x=L}$

$\Delta_2 = \theta(0) = \theta_0$

BETTI'S RECIPROCITY THEOREM

Consider a linear elastic body undergoing small strains. Let W_1 be the work produced by $\mathbf{F}_1$. Then, we apply force $\mathbf{F}_2$, which produces work W_1. When force $\mathbf{F}_2$ is applied, force $\mathbf{F}_1$ does additional work because its point of application is displaced due to the deformation caused by force $\mathbf{F}_2$. Let us denote this work by W_{12}, which is the work done by force $\mathbf{F}_1$ in moving through the displacement produced by force $\mathbf{F}_2$. Thus, the total work done is

$$W = W_1 + W_2 + W_{12}$$

When the order of application of the forces is reversed, we obtain

$$W = W_1 + W_2 + W_{21}$$

BETTI'S & MAXWELL'S RECIPROCITY THEOREMS

The work done in both cases should be the same because at the end the elastic body is loaded by the same pair of external forces. Thus we have

$$W_{12} = W_{21}$$

This is known as **Betti's reciprocity theorem**.

Let $\mathbf{u}_{12}$ be the displacement of point 1 produced by unit force $\mathbf{F}^2$ in the direction of force $\mathbf{F}^1$ and $\mathbf{u}_{21}$ be the displacement of point 2 produced by unit force $\mathbf{F}^1$ in the direction of force $\mathbf{F}^2$. Then by Betti's reciprocity theorem we have

$$\mathbf{F}^1 \cdot \mathbf{u}_{12} = \mathbf{F}^2 \cdot \mathbf{u}_{21} \;\Rightarrow u_{12} = u_{21}$$

This is the **Maxwell's reciprocity theorem**.

Theorem 4.6.2: If a linear elastic body is subjected to two different sets of forces, the work done by the first system of forces in moving through the displacements produced by the second system of forces is equal to the work done by the second system of forces in moving through the displacements produced by the first system of forces:

$$\int_{\Omega} \mathbf{f}^{(1)} \cdot \mathbf{u}^{(2)}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)}\, d\Gamma = \int_{\Omega} \mathbf{f}^{(2)} \cdot \mathbf{u}^{(1)}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)}\, d\Gamma, \quad (4.6.25)$$

where $\mathbf{u}^{(1)}$ is the displacement produced by body forces $\mathbf{f}^{(1)}$ and surface forces $\mathbf{t}^{(1)}$, and $\mathbf{u}^{(2)}$ is the displacement produced by body forces $\mathbf{f}^{(2)}$ and surface forces $\mathbf{t}^{(2)}$. The left-hand side of Eq. (4.6.25), for example, denotes the work done by forces $\mathbf{f}^{(1)}$ and $\mathbf{t}^{(1)}$ in moving through the displacement $\mathbf{u}^{(2)}$ produced by forces $\mathbf{f}^{(2)}$ and $\mathbf{t}^{(2)}$.

Proof: The proof of Betti's reciprocity theorem is straightforward. Let W_{12} denote the work done by forces $(\mathbf{f}^{(1)}, \mathbf{t}^{(1)})$ acting through the displacement $\mathbf{u}^{(2)}$ produced by the forces $(\mathbf{f}^{(2)}, \mathbf{t}^{(2)})$. Then

$$\begin{aligned}
W_{12} &= \int_{\Omega} \mathbf{f}^{(1)} \cdot \mathbf{u}^{(2)}\, d\Omega + \oint_{\Gamma} \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)}\, d\Gamma \\
&= \int_{\Omega} f_i^{(1)} u_i^{(2)}\, d\Omega + \oint_{\Gamma} t_i^{(1)} u_i^{(2)}\, d\Gamma \\
&= \int_{\Omega} f_i^{(1)} u_i^{(2)}\, d\Omega + \oint_{\Gamma} n_j \sigma_{ji}^{(1)} u_i^{(2)}\, d\Gamma \\
&= \int_{\Omega} f_i^{(1)} u_i^{(2)}\, d\Omega + \int_{\Omega} \left(\sigma_{ji}^{(1)} u_i^{(2)}\right)_{,j} d\Omega \\
&= \int_{\Omega} \left(\sigma_{ij,j}^{(1)} + f_i^{(1)}\right) u_i^{(2)}\, d\Omega + \int_{\Omega} \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, d\Omega \\
&= \int_{\Omega} \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, d\Omega = \int_{\Omega} \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}\, d\Omega.
\end{aligned}$$

Using Hooke's law $\sigma_{ij}^{(1)} = C_{ijk\ell}\, \varepsilon_{k\ell}^{(1)}$, we obtain

$$W_{12} = \int_{\Omega} C_{ijk\ell}\, \varepsilon_{k\ell}^{(1)}\, \varepsilon_{ij}^{(2)}\, d\Omega. \quad (4.6.26)$$

Since $C_{ijk\ell} = C_{k\ell ij}$, it follows that

$$W_{12} = \int_\Omega C_{ijk\ell}\,\varepsilon_{k\ell}^{(1)}\,\varepsilon_{ij}^{(2)}\,d\Omega = \int_\Omega C_{k\ell ij}\,\varepsilon_{ij}^{(2)}\,\varepsilon_{k\ell}^{(1)}\,d\Omega = \int_\Omega \sigma_{k\ell}^{(2)}\,\varepsilon_{k\ell}^{(1)}\,d\Omega = W_{21}.$$

One can trace back to show that W_{21} is equal to the right-hand side of Eq. (4.6.25). This completes the proof.

During the proof we have also established the equality

$$\begin{aligned} \int_\Omega \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\,d\Omega &= \int_\Omega \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}\,d\Omega, \\ \int_\Omega \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)}\,d\Omega &= \int_\Omega \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)}\,d\Omega. \end{aligned} \tag{4.6.27}$$

Example 4.6.4

Consider a cantilever beam of length L subjected to two sets of loads: a uniformly distributed load of intensity q_0 throughout the span, as shown in Fig. 4.6.6(a), and a concentrated load F at the free end, as shown in Fig. 4.6.6(b). Verify Betti's reciprocity theorem, that is, the work done by the point load F in moving through the displacement $w^q(0)$ produced by q_0 is equal to the work done by the distributed force q_0 in moving through the displacement $w^F(x)$ produced by the point load F.

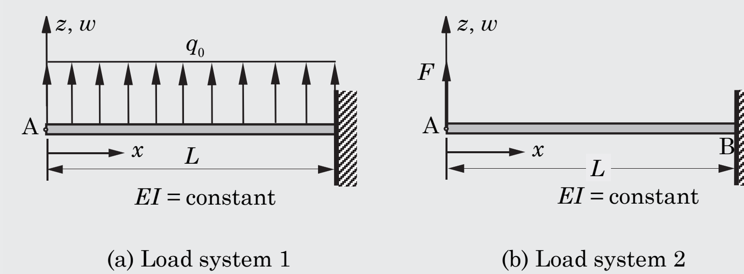


Fig. 4.6.6: (a) A cantilever beam under uniformly distributed load. (b) A cantilever beam with a point load at its free end.

Solution: From Eqs. (4.6.5) and (4.6.6), the expression for deflection of the cantilever beam with uniformly distributed load q_0 alone is

$$w_0^q(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^4\right]. \tag{1}$$

and the expression for deflection of the cantilever beam with the point load F at the free end alone is

$$w_0^F(x) = \frac{FL^3}{6EI}\left[2 - 3\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^3\right], \tag{2}$$

The work done by the load F in moving through the displacement due to the application of the uniformly distributed load q_0 is

$$W_{Fq} = Fw_0^q(0) = \frac{Fq_0L^4}{8EI}.$$

The work done by the uniformly distributed q_0 in moving through the displacement field $w^F(x)$ due to the application of point load F is

$$W_{qF} = \int_0^L \frac{FL^3}{6EI}\left[2 - 3\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^3\right] q_0\, dx = \frac{Fq_0L^4}{8EI},$$

which is in agreement with W_{Fq}.

Example 4.6.5

Use Betti's reciprocity theorem to determine the deflection at the free end of a cantilever beam with distributed load of intensity q_0 in the span between $x = a$ and $x = L$, as shown in Fig. 4.6.7. The deflection $w^F(x)$ due to a point load F at the free end (acting upward) is

$$w^F(x) = \frac{FL^3}{6EI}\left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]. \tag{1}$$

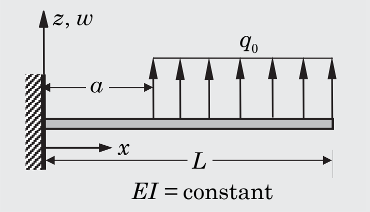


Fig. 4.6.7: A cantilever beam with uniformly distributed load on a portion of the beam.

Solution: The work done by the point load F in moving through the displacement due to the application of the uniformly distributed load q_0 is

$$W_{Fq} = Fw_0^q(L). \tag{2}$$

The work done by the uniformly distributed load q_0 in moving through the displacement field $w^F(x)$ due to the application of point load F is

$$W_{qF} = \int_a^L \frac{FL^3}{6EI}\left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] q_0\, dx = \frac{Fq_0}{24EI}\left(3L^4 - 4La^3 + a^4\right). \tag{3}$$

By Betti's reciprocity theorem, we have $W_{qF} = W_{Fq}$. Hence, the deflection at the free end of the beam due to the distributed load is

$$w_0^q(L) = \frac{q_0}{24EI}\left(3L^4 - 4La^3 + a^4\right). \tag{4}$$

Example 4.6.6

Consider a cantilever beam of length L and constant EI and subjected to a point load F_0 at the free end [see Fig. 4.6.9(a)]. Use Maxwell's theorem to determine the deflection at $x = a$ from the free end. Use the following data: $E = 24 \times 10^6$ psi, $I = 120$ in^4, $F_0 = 1,000$ lb, $a = 36$ in, and $b = 108$ in.

Solution: By Maxwell's theorem, the displacement w_{BA} at point B ($x = a$) produced by unit load at point A ($x = 0$) is equal to the deflection w_{AB} at point A produced by unit load at point B. We are required to find $w(0) = w_{\mathrm{BA}} F_0$. Thus, we must determine w_{AB} (which,

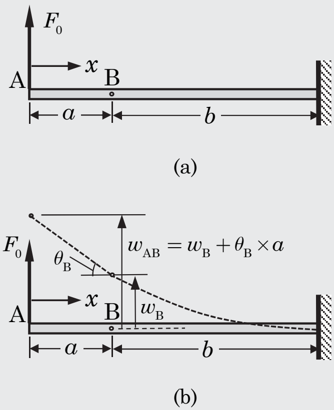


Fig. 4.6.9: A cantilever beam with a point load at the free end.

presumably, is easier to compute by some way than to compute $w(0)$ directly). Let $w_{\rm B}$ and $\theta_{\rm B}$ denote the deflection and slope, respectively, at point B owing to a load $F = 1$ applied at point B. Then the deflection at point A due to load $F = 1$ is [see Fig. 4.6.9(b)]

$$w_{\rm AB} = w_{\rm B} + \theta_{\rm B} a \tag{1}$$

and the required solution is

$$w(0) = w_{\rm BA} F_0 = w_{\rm AB} F_0 = F_0 \left(w_{\rm B} + a\,\theta_{\rm B} \right). \tag{2}$$

The values of $w_{\rm B}$ and $\theta_{\rm B}$ can be computed using Eq. (4.6.6) as

$$w_{\rm B} = \frac{b^3}{6EI}\left[2 - 3\,\frac{\bar{x}}{b} + \left(\frac{\bar{x}}{b}\right)^3\right]_{\bar{x}=0} = \frac{b^3}{3EI}, \tag{3}$$

$$\theta_{\rm B} = -\frac{dw}{dx}\bigg|_{\bar{x}=0} = \frac{b^2}{2EI}\left[1 - \left(\frac{\bar{x}}{b}\right)^2\right]_{\bar{x}=0} = \frac{b^2}{2EI}\,. \tag{4}$$

Therefore, we have

$$\begin{aligned} w(0) &= F_0 \left(w_{\rm B} + a\,\theta_{\rm B} \right) = F_0 \left(\frac{b^3}{3EI} + \frac{b^2 a}{2EI} \right) = \frac{F_0 b^2}{6EI}(3a + 2b) \\ &= \frac{1,000 \times (108)^2}{6 \times 24 \times 10^6 \times 120}(3 \times 36 + 2 \times 108) = 0.2187 \text{ in.} \end{aligned} \tag{5}$$

Principle of virtual displacements and the associated Euler equations [see Eqs. (4.2.2), (4.2.5), and (4.2.6)]

$$\delta W = \delta W_I + \delta W_E = 0. \tag{4.7.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \ \text{ in } \Omega; \quad \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} - \hat{\mathbf{t}} = \mathbf{0} \ \text{ on } \Gamma_\sigma. \tag{4.7.2}$$

Unit dummy-displacement method [see Eq. (4.2.7)]

$$\mathbf{F}_0 \cdot \delta \mathbf{u}_0 = \int_\Omega \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon}^0 \, d\Omega. \tag{4.7.3}$$

The principle of minimum total potential energy and the associated Euler equations [see Eqs. (4.3.5) and (4.3.18)]

$$\delta \Pi \equiv \delta(U + V_E) = 0. \tag{4.7.4}$$

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \text{and} \quad \mathbf{t} - \hat{\mathbf{t}} = \mathbf{0} \text{ on } \Gamma_\sigma. \tag{4.7.5}$$

Castigliano's Theorem I [see Eq. (4.3.22)]

$$\frac{\partial U}{\partial \mathbf{u}_i} = \mathbf{F}_i. \tag{4.7.6}$$

Principle of virtual forces and the associated Euler equations [see Eqs. (4.4.5) and (4.4.10)]

$$\delta W^* = \delta W_I^* + \delta W_E^* = 0. \tag{4.7.7}$$

$$\boldsymbol{\varepsilon} - \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right] = \mathbf{0} \;\text{ in } \Omega; \quad \mathbf{u} - \hat{\mathbf{u}} = \mathbf{0} \;\text{ on } \Gamma_u. \tag{4.7.8}$$

Unit dummy-load method [see Eq. (4.4.11)]

$$\delta\mathbf{F}_0 \cdot \mathbf{u}_0 = \int_\Omega \boldsymbol{\varepsilon} : \delta\boldsymbol{\sigma}^0 \, d\Omega. \tag{4.7.9}$$

The principle of minimum total complementary energy and the associated Euler equations [see Eqs. (4.5.2) and (4.5.5)]

$$\delta\Pi^* \equiv \delta(U^* + V_E^*) = 0. \tag{4.7.10}$$

$$\mathbf{C}^* : \boldsymbol{\sigma} - \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right] = \mathbf{0} \;\text{ in } \Omega; \quad \mathbf{u} - \hat{\mathbf{u}} = \mathbf{0} \;\text{ on } \Gamma_u. \tag{4.7.11}$$

Castigliano's Theorem II [see Eq. (4.5.13)]

$$\frac{\partial U^*}{\partial \mathbf{F}_i} = \mathbf{u}_i. \tag{4.7.12}$$

Clapeyron's Theorem [see Eq. (4.6.10)]

$$\tfrac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega = \tfrac{1}{2}\left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\Omega + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, d\Gamma\right]. \tag{4.7.13}$$

Betti's Reciprocity Theorem [see Eqs. (4.6.25) and (4.6.27)]

$$\int_\Omega \mathbf{f}^{(1)}\cdot\mathbf{u}^{(2)}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{t}^{(1)}\cdot\mathbf{u}^{(2)}\, d\Gamma = \int_\Omega \mathbf{f}^{(2)}\cdot\mathbf{u}^{(1)}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{t}^{(2)}\cdot\mathbf{u}^{(1)}\, d\Gamma. \tag{4.7.14}$$

$$\int_\Omega \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)}\, d\Omega = \int_\Omega \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)}\, d\Omega. \tag{4.7.15}$$

Maxwell's Reciprocity Theorem [see Eqs. (4.6.28)]

$$\mathbf{F}^1\cdot\mathbf{u}_{12} = \mathbf{F}^2\cdot\mathbf{u}_{21} \quad \text{or} \quad u_{12} = u_{21}. \tag{4.7.16}$$

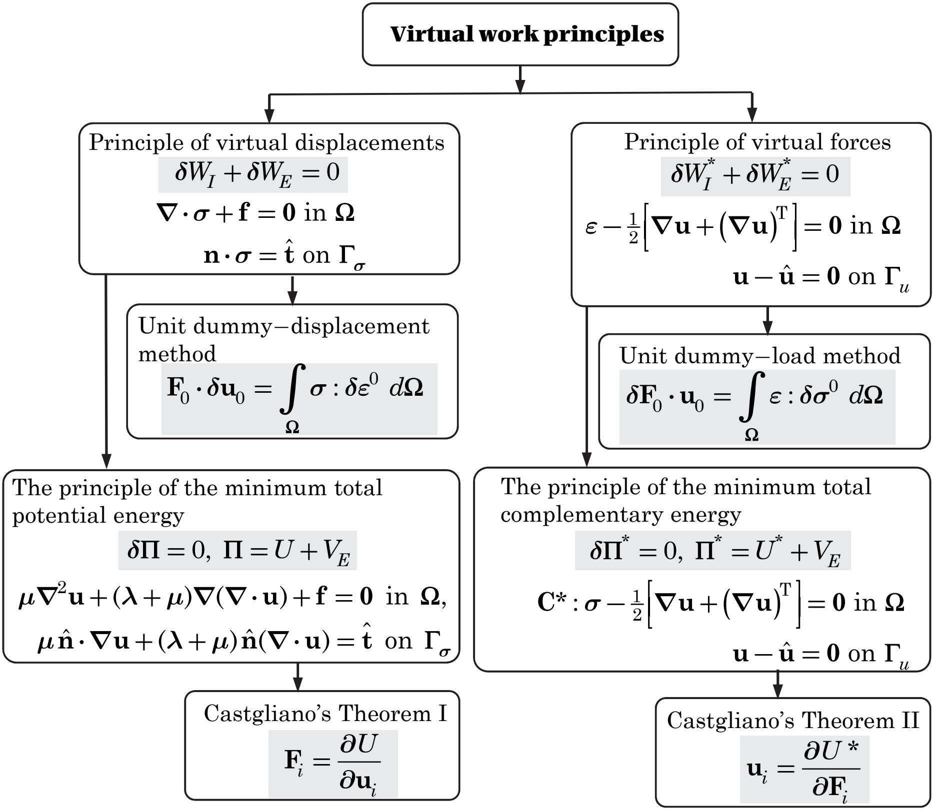


Fig. 4.7.1: A flow chart of various energy principles and methods.