NONLOCAL THEORY OF ERINGEN

According to Eringen (1972, 1983, 2002), the stress field at a point **x** in an elastic continuum not only depends on the strain field at the point (hyperelastic case) but also on strains at all other points of the body. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion. Thus, the nonlocal stress tensor $\boldsymbol{\sigma}$ at point **x** is expressed as

$$\boldsymbol{\sigma} = \int_V K(|\mathbf{x}' - \mathbf{x}|, \tau)\mathbf{t}(\mathbf{x}')d\mathbf{x}'$$

where **t**(**x**) is the classical, macroscopic stress tensor at point **x** and the $K(|\mathbf{x}' - \mathbf{x}|, \tau)$ kernel function that represents the nonlocal modulus, $|\mathbf{x}' - \mathbf{x}|$ being the distance (in Euclidean norm) and τ is a material constant that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength, respectively).

NONLOCAL THEORY OF ERINGEN

$$\left(1-\boldsymbol{\tau}^2\ell^2\nabla^2\right)\boldsymbol{\sigma}=\mathbf{t},\quad \boldsymbol{\tau}=\frac{e_0 a}{\ell}$$

where e_0 is a material constant, and a and and ℓ are the internal and external characteri-stic lengths,respectively.

In particular, we have the following nonlocal stress-strain relations for **beams**:

$$\boldsymbol{\sigma}_{xx}-\boldsymbol{\mu}\frac{\partial^2\boldsymbol{\sigma}_{xx}}{\partial x^2}=E\boldsymbol{\varepsilon}_{xx}$$

$$\boldsymbol{\sigma}_{xz}-\boldsymbol{\mu}\frac{\partial^2\boldsymbol{\sigma}_{xz}}{\partial x^2}=2G\boldsymbol{\varepsilon}_{xz}\quad(\boldsymbol{\mu}=e_0^2a^2)$$

NONLOCAL THEORIES OF BEAMS

For example, the axial stress resultant (**in all theories**) is

$$N_{xx} - \mu \frac{\partial^2 N_{xx}}{\partial x^2} = EA\varepsilon^0_{xx}$$

Bending moment in the nonlocal **Euler-Bernoulli beam theory** is

$$M^E_{xx} - \mu \frac{\partial^2 M^E_{xx}}{\partial x^2} = EI\kappa^E$$

Bending moment and shear force in the nonlocal **Timoshenko beam theory** becomes

$$M^T - \mu \frac{\partial^2 M^T}{\partial x^2} = EI\kappa^T, \quad Q^T - \mu \frac{\partial^2 Q^T}{\partial x^2} = GAK_s\gamma^T$$

NONLOCAL THEORIES OF BEAMS
(continued)

The stress resultants in the nonlocal **Reddy beam theory** are

$$M^R - \mu \frac{\partial^2 M^R}{\partial x^2} = EI\kappa^R + EJ\rho^R, \; P^R - \mu \frac{\partial^2 P^R}{\partial x^2} = EJ\kappa^R + EK\rho^R,$$

$$Q^R - \mu \frac{\partial^2 Q^R}{\partial x^2} = GA\gamma^R + GI\beta^R, \; R^R - \mu \frac{\partial^2 R^R}{\partial x^2} = GI\gamma^R + GJ\beta^R,$$

$$(A, I, J, K) = \int_A (1, z^2, z^4, z^6)\, dA$$

The axial force can be expressed as

$$N = EA\frac{\partial u}{\partial x} + \mu\left(m_0 \frac{\partial^3 u}{\partial x \partial t^2} - \frac{\partial f}{\partial x}\right)$$

and the **axial equation of motion** becomes

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) + f - \mu\frac{\partial^2 f}{\partial x^2} = m_0\left(\frac{\partial^2 u}{\partial t^2} - \mu\frac{\partial^4 u}{\partial x^2 \partial t^2}\right)$$

The Nonlocal Euler-Bernoulli Beam Theory

The bending moment in the EBT is

$$M^E = -EI\frac{\partial^2 w^E}{\partial x^2} + \boldsymbol{\mu}\left[\frac{\partial}{\partial x}\left(\bar{N}^E\frac{\partial w^E}{\partial x}\right) - q + m_0\frac{\partial^2 w^E}{\partial t^2} - m_2\frac{\partial^4 w^E}{\partial x^2\partial t^2}\right]$$

and the equation of motion becomes

$$\frac{\partial^2}{\partial x^2}\left(-EI\frac{\partial^2 w^E}{\partial x^2}\right) + \boldsymbol{\mu}\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial x}\left(\bar{N}^E\frac{\partial w^E}{\partial x}\right) - q + m_0\frac{\partial^2 w^E}{\partial t^2} - m_2\frac{\partial^4 w^E}{\partial x^2\partial t^2}\right]$$
$$+ q - \frac{\partial}{\partial x}\left(\bar{N}^E\frac{\partial w^E}{\partial x}\right) = m_0\frac{\partial^2 w^E}{\partial t^2} - m_2\frac{\partial^4 w^E}{\partial x^2\partial t^2}$$

The Nonlocal Timoshenko Beam Theory

The stress resultants in the TBT are

$$M^T = EI\frac{\partial \phi^T}{\partial x} + \mu\left[-q + \frac{\partial}{\partial x}\left(\bar{N}^T\frac{\partial w^T}{\partial x}\right) + m_0\frac{\partial^2 w^T}{\partial t^2} + m_2\frac{\partial^3 \phi^T}{\partial x \partial t^2}\right]$$

$$Q^T = GAK_s\left(\phi^T + \frac{\partial w^T}{\partial x}\right) + \mu\frac{\partial}{\partial x}\left[-q + \frac{\partial}{\partial x}\left(\bar{N}^T\frac{\partial w^T}{\partial x}\right) + m_0\frac{\partial^2 w^T}{\partial t^2}\right]$$

and the equations of motion becomes

$$\frac{\partial}{\partial x}\left[GAK_s\left(\phi^T + \frac{\partial w^T}{\partial x}\right)\right] + q - \frac{\partial}{\partial x}\left(\bar{N}^T\frac{\partial w^T}{\partial x}\right)$$

$$-\mu\frac{\partial^2}{\partial x^2}\left[q - \frac{\partial}{\partial x}\left(\bar{N}^T\frac{\partial w^T}{\partial x}\right)\right] = m_0\left(\frac{\partial^2 w^T}{\partial t^2} - \mu\frac{\partial^4 w^T}{\partial x^2 \partial t^2}\right)$$

$$\frac{\partial}{\partial x}\left(EI\frac{\partial \phi^T}{\partial x}\right) - GAK_s\left(\phi^T + \frac{\partial w^T}{\partial x}\right) = m_2\frac{\partial^2 \phi^T}{\partial t^2} - \mu m_2\frac{\partial^4 \phi^T}{\partial x^2 \partial t^2}$$

The Nonlocal Reddy Third-Order Beam Theory

The stress resultants in the RBT are

$$\hat{M}^R = E\hat{I}\frac{\partial\phi^R}{\partial x} - c_1 E\hat{J}\left(\frac{\partial\phi^R}{\partial x} + \frac{\partial^2 w^R}{\partial x^2}\right) + \mu\left[-c_1\frac{\partial^2 P^R}{\partial x^2} - q + m_0\frac{\partial^2 w^R}{\partial t^2}\right.$$

$$\left. + \frac{\partial}{\partial x}\left(\bar{N}^R\frac{\partial w^R}{\partial x}\right) + m_2\frac{\partial^3\phi^R}{\partial x\partial t^2} - c_1 m_4\left(\frac{\partial^3\phi^R}{\partial x\partial t^2} + \frac{\partial^4 w^R}{\partial x^2\partial t^2}\right)\right]$$

$$\hat{Q}^R = G\tilde{A}\left(\phi^R + \frac{\partial w^R}{\partial x}\right) + \mu\frac{\partial}{\partial x}\left[-c_1\frac{\partial^2 P^R}{\partial x^2} + \frac{\partial}{\partial x}\left(\bar{N}^R\frac{\partial w^R}{\partial x}\right) - q\right]$$

$$+ \mu\frac{\partial}{\partial x}\left[m_0\frac{\partial^2 w^R}{\partial t^2} + c_1 m_4\frac{\partial^3\phi^R}{\partial x\partial t^2} - c_1^2 m_6\left(\frac{\partial^3\phi^R}{\partial x\partial t^2} + \frac{\partial^4 w^R}{\partial x^2\partial t^2}\right)\right]$$

$$c_1\frac{\partial^2}{\partial x^2}\left(P^R - \mu\frac{\partial^2 P^R}{\partial x^2}\right) = c_1\left[EJ\frac{\partial^3\phi^R}{\partial x^3} - c_1 EK\left(\frac{\partial^3\phi^R}{\partial x^3} + \frac{\partial^4 w^R}{\partial x^4}\right)\right]$$

The Nonlocal Reddy Third-Order Beam Theory (continued)

The first equation of motion is

$$G\tilde{A}\left(\frac{\partial \phi^R}{\partial x}+\frac{\partial^2 w^R}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(\bar{N}^R\frac{\partial w^R}{\partial x}\right)+q+\mu\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial x}\left(N^R\frac{\partial w^R}{\partial x}\right)-q\right]$$

$$+c_1\left[EJ\frac{\partial^3 \phi^R}{\partial x^3}-c_1 EK\left(\frac{\partial^3 \phi^R}{\partial x^3}+\frac{\partial^4 w^R}{\partial x^4}\right)\right]$$

$$=m_0\frac{\partial^2 w^R}{\partial t^2}+c_1 m_4\frac{\partial^3 \phi^R}{\partial x\partial t^2}-c_1^2 m_6\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial \phi^R}{\partial t}+\frac{\partial^2 w^R}{\partial x\partial t}\right)$$

$$-\mu\left[m_0\frac{\partial^4 w^R}{\partial x^2\partial t^2}+c_1 m_4\frac{\partial^5 \phi^R}{\partial x^3\partial t^2}-c_1^2 m_6\left(\frac{\partial^5 \phi^R}{\partial x^3\partial t^2}+\frac{\partial^6 w^R}{\partial x^4\partial t^2}\right)\right] \quad (1)$$

The Nonlocal Reddy Third-Order Beam Theory (continued)

The second equation of motion is

$$E\hat{I}\frac{\partial^2\phi^R}{\partial x^2} - c_1 E\hat{J}\left(\frac{\partial^2\phi^R}{\partial x^2} + \frac{\partial^3 w^R}{\partial x^3}\right) - G\tilde{A}\left(\phi^R + \frac{\partial w^R}{\partial x}\right)$$

$$= \hat{m}_2\frac{\partial^2\phi^R}{\partial t^2} - c_1\hat{m}_4\left(\frac{\partial^2\phi^R}{\partial t^2} + \frac{\partial^3 w^R}{\partial x\partial t^2}\right)$$

$$-\mu\left[\hat{m}_2\frac{\partial^4\phi^R}{\partial x^2\partial t^2} - c_1\hat{m}_4\left(\frac{\partial^4\phi^R}{\partial x^2\partial t^2} + \frac{\partial^5 w^R}{\partial x^3\partial t^2}\right)\right] \quad (2)$$

ANALYTICAL SOLUTIONS

Solution form

$$w(x,t) = \sum_{n=1}^{\infty} W_n \sin\frac{n\pi x}{L}\, e^{i\omega_n t}, \quad \phi(x,t) = \sum_{n=1}^{\infty} \Phi_n \cos\frac{n\pi x}{L}\, e^{i\omega_n t}$$

Bending Solution -EBT

$$w^E(x) = \sum_{n=1}^{\infty} \frac{\lambda_n Q_n L^4}{n^4 \pi^4 EI} \sin\frac{n\pi x}{L}$$

$$Q_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}\, dx$$

ANALYTICAL SOLUTIONS (CONTINUED)

Bending Solution -TBT

$$w^T(x) = \sum_{n=1}^{\infty} \boldsymbol{\lambda}_n \Lambda_n \frac{Q_n L^4}{n^4 \boldsymbol{\pi}^4 EI} \sin \frac{n\boldsymbol{\pi} x}{L}$$

$$\phi^T(x) = -\sum_{n=1}^{\infty} \boldsymbol{\lambda}_n \frac{Q_n L^3}{n^3 \boldsymbol{\pi}^3 EI} \cos \frac{n\boldsymbol{\pi} x}{L}$$

$$\Lambda_n = \left(1 + n^2 \boldsymbol{\pi}^2 \Omega\right), \quad \Omega = \frac{EI}{GAK_s L^2}$$

$$\boldsymbol{\lambda}_n = 1 + \boldsymbol{\mu}\left(\frac{n\boldsymbol{\pi}}{L}\right)^2$$

ANALYTICAL SOLUTIONS (CONTINUED)

Bending Solution -RBT

$$w^R(x) = \sum_{n=1}^{\infty} \boldsymbol{\lambda}_n \left[B_n + E\hat{I} \left(\frac{n\pi}{L} \right)^2 \right] \left(\frac{1}{A_n + c_1 B_n \frac{J}{\hat{I}}} \right) \frac{Q_n L^4}{n^4 \pi^4 E\hat{I}} \sin \frac{n\pi x}{L}$$

$$\phi^R(x) = -\sum_{n=1}^{\infty} \boldsymbol{\lambda}_n \left(\frac{B_n}{A_n + c_1 B_n \frac{J}{\hat{I}}} \right) \frac{Q_n L^3}{n^3 \pi^3 E\hat{I}} \cos \frac{n\pi x}{L}, \quad \boldsymbol{\lambda}_n = 1 + \mu \left(\frac{n\pi}{L} \right)^2$$

ANALYTICAL SOLUTIONS (CONTINUED)

Buckling Solution - EBT

$$\lambda_n \left\{ \bar{N}^E \left(\frac{n\pi}{L}\right)^2 + \omega_n^2 \left[m_0 + m_2 \left(\frac{n\pi}{L}\right)^2 \right] \right\} = EI \left(\frac{n\pi}{L}\right)^4$$

$$\bar{N}^E = \frac{1}{\lambda_1} \frac{\pi^2 EI}{L^2}, \quad \lambda_n = 1 + \mu \left(\frac{n\pi}{L}\right)^2$$

Buckling Solution - TBT

$$\bar{N}^T = \frac{1}{\lambda_1 \Lambda_1} \frac{\pi^2 EI}{L^2}, \quad \Lambda_n = \left(1 + n^2 \pi^2 \Omega\right), \quad \Omega = \frac{EI}{GAK_s L^2}$$

ANALYTICAL SOLUTIONS (CONTINUED)

Buckling Solution - RBT

$$\bar{N}^R = \frac{1}{\lambda_1}\frac{\pi^2 E\hat{I}}{L^2}\left[\frac{A_1 + c_1 B_1 \dfrac{J}{\hat{I}}}{B_1 + E\hat{I}\left(\dfrac{\pi}{L}\right)^2}\right]$$

Vibration Solution - EBT

$$\omega_n^2 = \frac{1}{\lambda_n M_n}\left(\frac{n\pi}{L}\right)^4 EI, \quad M_n = m_0 + m_2\left(\frac{n\pi}{L}\right)^2$$

ANALYTICAL SOLUTIONS (CONTINUED)

Vibration Solution - TBT

$$\frac{m_0 m_2}{GAK_s}\lambda_n^2\omega_n^4 - \left[m_0\Lambda_n + m_2\left(\frac{n\pi}{L}\right)^2\right]\lambda_n\omega_n^2 + EI\left(\frac{n\pi}{L}\right)^4 = 0$$

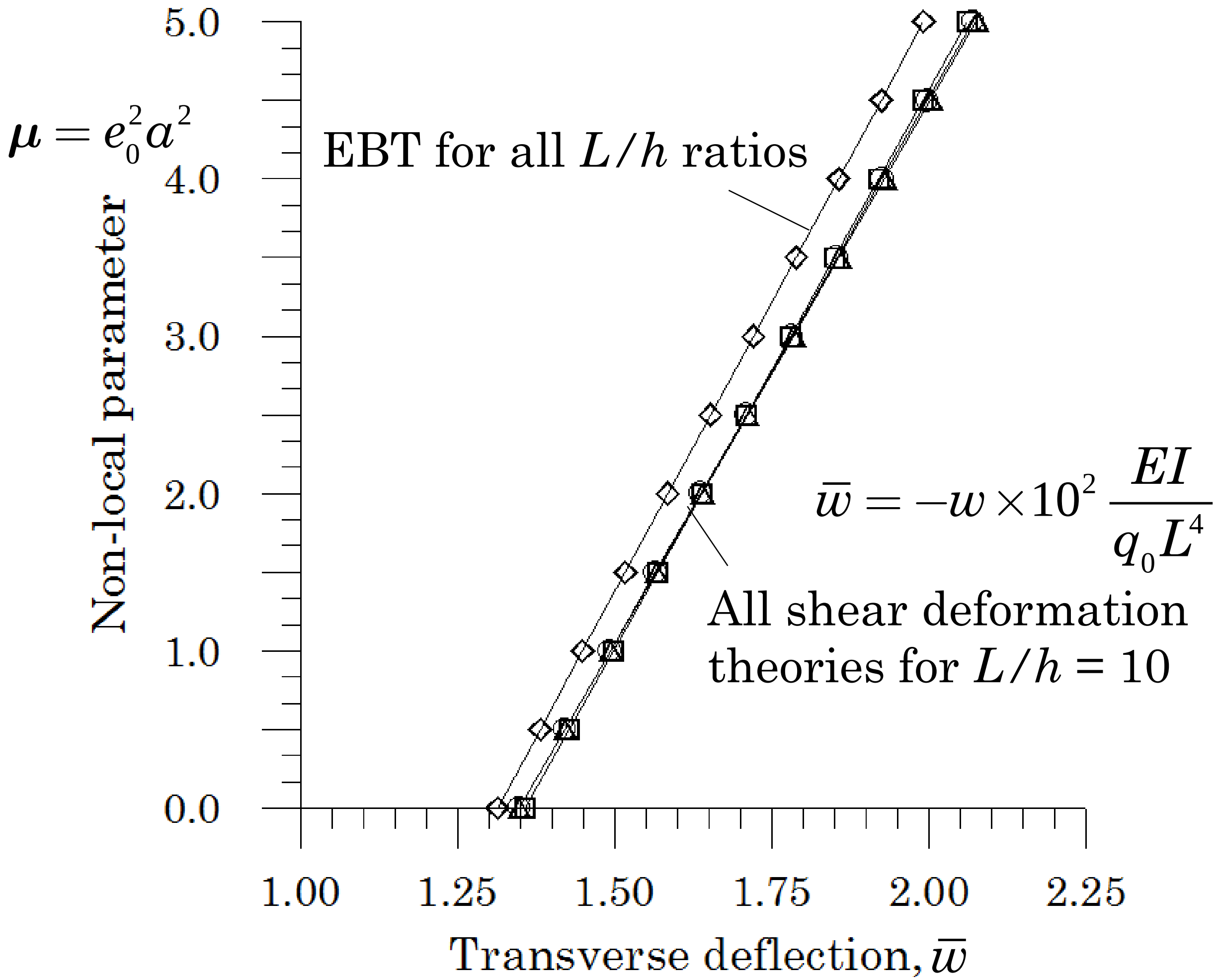
$$\omega_n^2 = \frac{1}{\lambda_n\Sigma_n}\left(\frac{n\pi}{L}\right)^4 EI, \quad \Sigma_n = m_0\Lambda_n + m_2\left(\frac{n\pi}{L}\right)^2$$

Vibration Solution - RBT

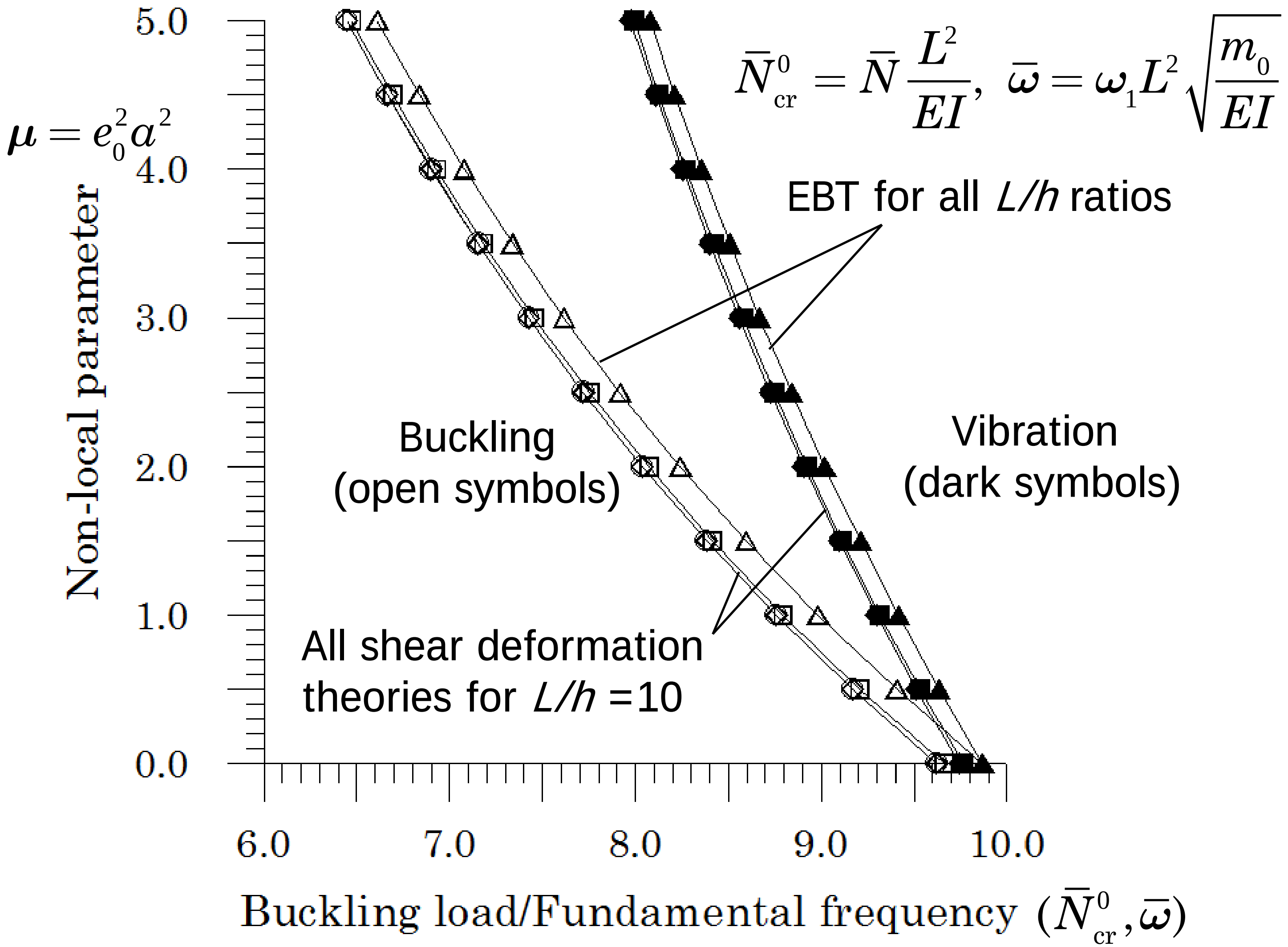
$$\omega_n^2 = \frac{E\hat{I}}{m_0\lambda_n}\left(\frac{n\pi}{L}\right)^4\left[\frac{c_1 EJB_n + A_n E\hat{I}}{E\hat{I}\left(\frac{n\pi}{L}\right)^2 + B_n}\right]$$

NUMERICAL RESULTS - BENDING

$$E = 30 \times 10^6,\ \nu = 0.3,\ L = 10,\ h = \text{varied}$$



NUMERICAL RESULTS – BUCKLING AND VIBRATION



SUMMARY - REMARKS

- The nonlocal theory of Eringen has softening effect on the stiffness characteristics of beams. Consequently,
  - the deflections are larger than those of the conventional beams
  - buckling loads and frequencies are smaller
- The equations of equilibrium of beams based on the nonlocal theory of Eringen cannot be derived from an energy or a functional; it is possible to construct a functional using the inverse method from known governing equations, but the resulting boundary conditions are not the same as those derived.
- The differential nonlocal model of Eringen need to be examined further for its validity.