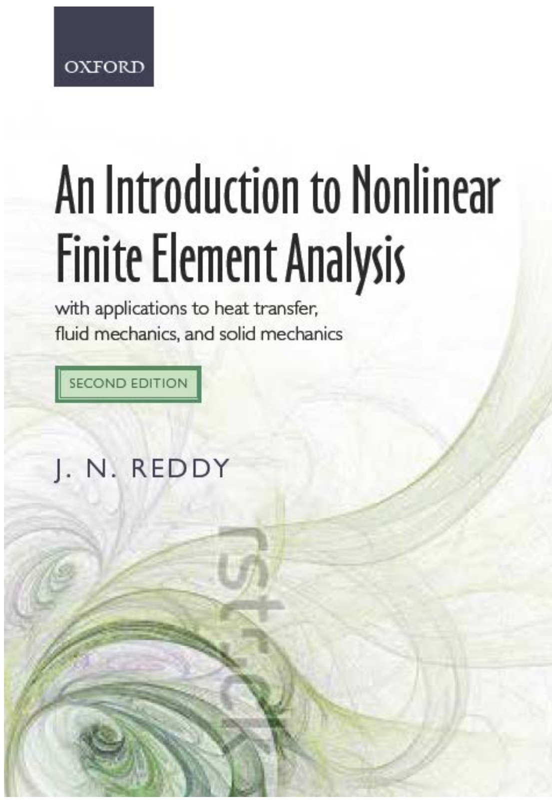
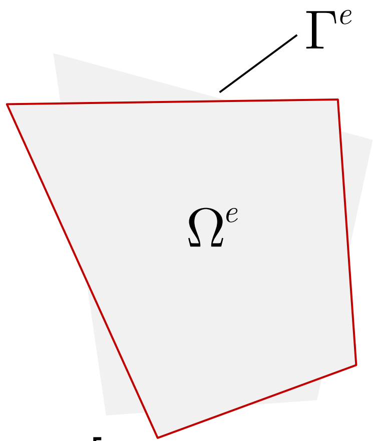
OTHER TPOICS OF INTEREST (Convection BC, Axisymmetric problems, 3D FEM)



CONTENTS

- 2-D Problems with convection BC
- Types of Axisymmetric Problems
- Axisymmetric Problems (2-D)
- 3-D Heat Transfer
- 3-D Elasticity
- Typical 3-D Finite Elements

CONVECTION HEAT TRANSFER



$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0$$

$$\left(a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y\right)+\beta(u-u_\infty)=q_n$$

$$0=\int_{\Omega^e}\left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right)+\frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right)-wf\right]dxdy$$

$$-\oint_{\Gamma^e} w_i\left[\left(a_{11}\frac{\partial u_h}{\partial x}\right)n_x+\left(a_{22}\frac{\partial u_h}{\partial y}\right)n_y\right]ds$$

$$=\int_{\Omega^e}\left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right)+\frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right)-wf\right]dxdy-\oint_{\Gamma^e} w_i\left[q_n-\beta(u_h-u_\infty)\right]ds$$

$$=\int_{\Omega^e}\left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right)+\frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right)-wf\right]dxdy+\beta\oint_{\Gamma^e} w_i u_h\,ds-\oint_{\Gamma^e} w_i(q_n+\beta u_\infty)\,ds$$

CONVECTION HEAT TRANSFER

$$0=\sum_{j=1}^{n}K_{ij}^{e}u_{j}^{e}-f_{i}^{e}-Q_{i}^{e}=\sum_{j=1}^{n}K_{ij}^{e}u_{j}^{e}-F_{i}^{e}\quad \text{or}\quad \mathbf{K}^{e}\mathbf{u}^{e}=\mathbf{F}^{e}$$

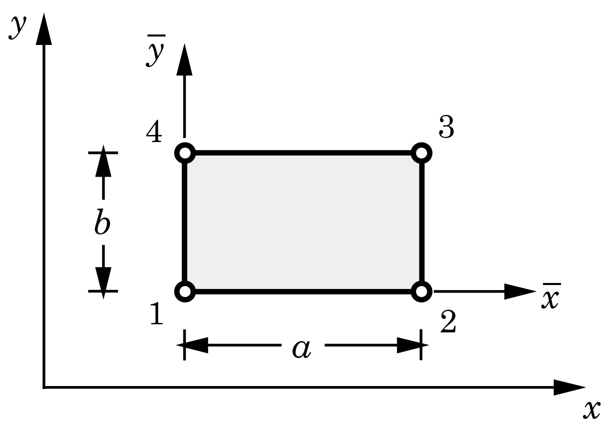
$$K_{ij}^{e}=\int_{\Omega^e}\left(a_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+a_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy+\underbrace{\beta\oint_{\Gamma^e}\psi_i\psi_j\,ds}_{H_{ij}^{e}}$$

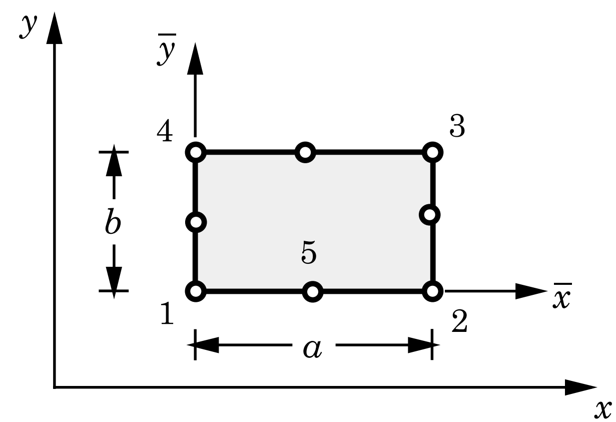
$$F_{i}^{e}=\int_{\Omega^e}f\psi_i\,dxdy+\oint_{\Gamma^e}\psi_i(q_n+\beta u_\infty)\,ds\equiv f_i^e+Q_i^e+P_i^e$$

- **Convective heat transfer contributions (H_{ij}^{e} and P_{i}^{e}) are only from element boundary**
- **The contributions need to be calculated only for elements with convective boundary**

CONVECTION HEAT TRANSFER

(continued)





$$H_{ij}^{e} = \beta \oint_{\Gamma^e} \psi_i \psi_j \, ds = \beta \left(\int_{1-2} \psi_i \psi_j \, ds + \int_{2-3} \psi_i \psi_j \, ds + \int_{3-4} \psi_i \psi_j \, ds + \int_{4-1} \psi_i \psi_j \, ds \right)$$

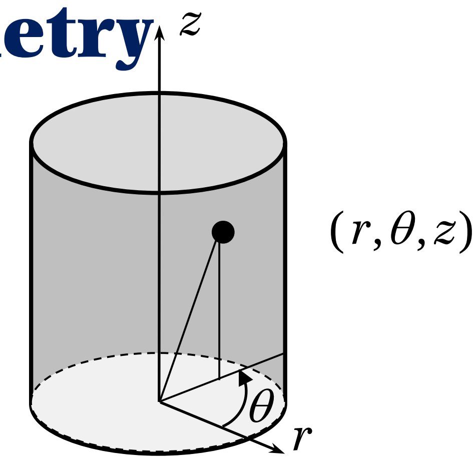
Conditions for Solution Symmetry

- The solution of a problem may be symmetric about a line or plane, allowing one to model only a part of the domain and thereby reducing the computational effort.
- The solution is symmetric about a line or plane, if and only if
  (a) geometry is symmetric,
  (b) material properties are symmetric,
  (c) loads are symmetrically applied, and
  (d) boundary conditions are symmetric
  about the line or plane.
- Use of the solution symmetry allows us to identify a subdomain whose analysis yields the solution in the entire domain.
- The subdomain necessarily will have boundaries that coincide with the lines or planes of symmetry, and one must identify the boundary conditions along these lines or on the planes of symmetry

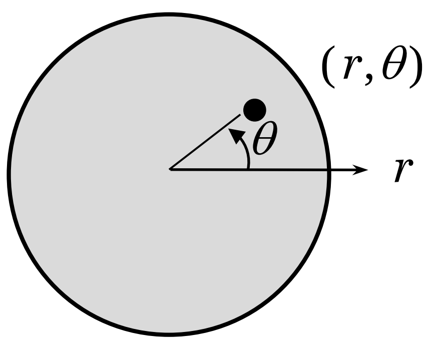
Reduction of Problem Size from 3-D using solution symmetry

3-D Models

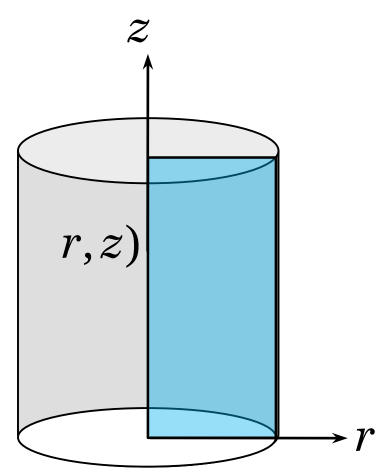
General loading, boundary conditions, and material properties, all of which may change along the length and around the circumference (i.e., with z and θ)



2-D Models



The loading, boundary conditions, and material properties do not change along the length

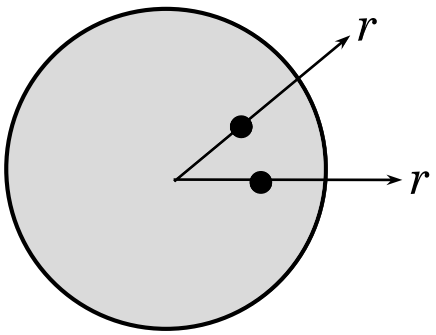


The loading, boundary conditions, and material properties do not change around the circumference

Reduction of Problem Size from 3-D

1-D Models

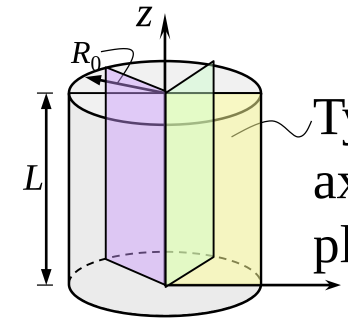
The loading, boundary conditions, and material properties do not change around the circumference as well as the length



AXISYMMETRIC PROBLEMS (2-D)

Governing Equation

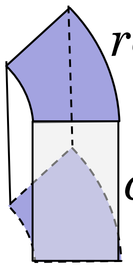
$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{11}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{22}\frac{\partial u}{\partial z}\right)=f(r,z)$$



Weak Form

$$0=\int_{\Omega^e} w_i\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{11}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{22}\frac{\partial u}{\partial z}\right)-f(r,z)\right]rdrdz$$

$$=\int_{\Omega^e}\left(a_{11}\frac{\partial w_i}{\partial r}\frac{\partial u}{\partial r}+a_{22}\frac{\partial w_i}{\partial z}\frac{\partial u}{\partial z}-w_i f\right)rdrdz-\oint_{\Gamma^e} q_n w_i\, rds$$



$$dv = rdr\,d\theta\,dz$$

$$q_n \equiv \mathbf{q}\cdot\hat{\mathbf{n}} = a_{11}\frac{\partial u}{\partial r}n_r + a_{22}\frac{\partial u}{\partial z}n_z$$

AXISYMMETRIC PROBLEMS (cont.)

Finite Element Model

$$u_h = \sum_{j=1}^{n} u_j \psi_j(r,z)$$

$$0 = \sum_{j=1}^{n} u_j \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial r} \frac{\partial \psi_j}{\partial r} + a_{22} \frac{\partial \psi_i}{\partial z} \frac{\partial \psi_j}{\partial z} \right) r dr dz - \int_{\Omega^e} f \psi_i \, r dr dz - \oint_{\Gamma^e} q_n w_i \, r ds$$

$$= \sum_{j=1}^{n} K_{ij} u_j - f_i - Q_i$$

$$K_{ij} = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial r} \frac{\partial \psi_j}{\partial r} + a_{22} \frac{\partial \psi_i}{\partial z} \frac{\partial \psi_j}{\partial z} \right) r dr dz$$

$$f_i = \int_{\Omega^e} f \psi_i \, r dr dz, \quad Q_i = \oint_{\Gamma^e} q_n w_i \, r ds$$

SINGLE-VARIABLE PROBLEMS IN 3-D

Governing Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial z}\left(a_{33}\frac{\partial u}{\partial z}\right)=f(x,y,z)$$

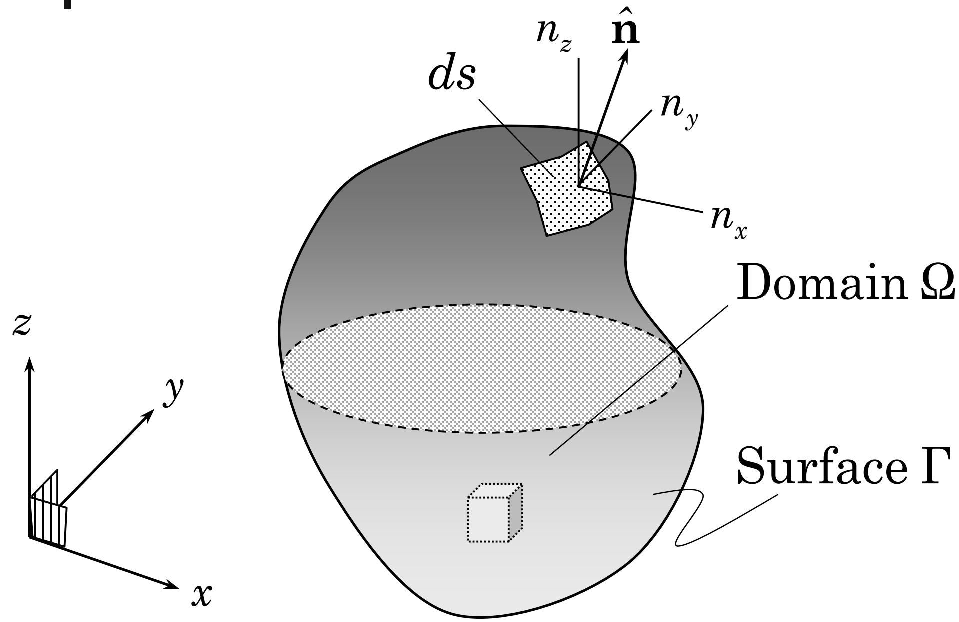
Boundary Conditions

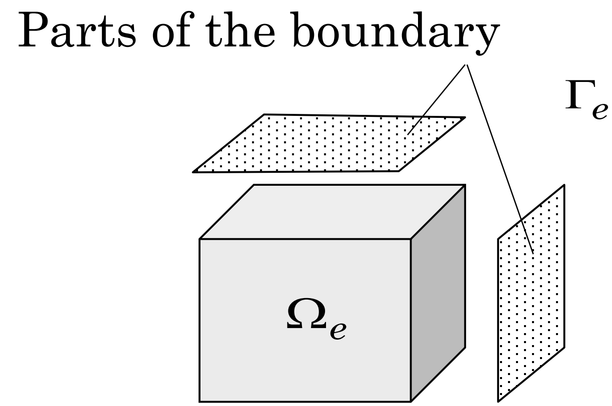
$$\text{Specify:}\quad u \quad \text{or} \quad q_{cnd}+q_{cnv}=q_n$$

$$q_{cnd}\equiv \mathrm{q}\cdot\hat{\mathrm{n}}=a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y+a_{33}\frac{\partial u}{\partial z}n_z$$

$$q_{cnv}=\beta(u-u_\infty)$$

3-D HEAT TRANSFER (continued)





A six-face 3-D finite element

Weak Form

$$0 = \int_{\Omega^e} \left(a_{11} \frac{\partial w_i}{\partial x}\frac{\partial u_h}{\partial x} + a_{22} \frac{\partial w_i}{\partial y}\frac{\partial u_h}{\partial y} + a_{33} \frac{\partial w_i}{\partial z}\frac{\partial u_h}{\partial z} \right) dxdydz + \oint_{\Gamma^e} \beta w_i u_h \, ds$$

$$- \int_{\Omega^e} f\, w_i \, dxdydz - \oint_{\Gamma^e} \left(q_n + \beta u_\infty \right) w_i \, ds$$

3-D HEAT TRANSFER (continued)

Finite element approximation

$$u_h = \sum_{j=1}^{n} u_j \psi_j(x,y,z)$$

Finite element model

$$0 = \sum_{j=1}^{n} K_{ij} u_j - f_i - Q_i$$

$$K_{ij} = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} + a_{33} \frac{\partial \psi_i}{\partial z}\frac{\partial \psi_j}{\partial z} \right) dxdydz + \oint_{\Gamma^e} \beta \psi_i \psi_j ds$$

$$f_i = \int_{\Omega^e} f\, \psi_i \, dxdydz,\ \ Q_i = \oint_{\Gamma^e} \left(q_n + \beta u_\infty \right) \psi_i \, ds$$

3-D ELASTICITY

Equations of Motion

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = \rho \frac{\partial u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = \rho \frac{\partial u_y}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = \rho \frac{\partial u_z}{\partial t^2}$$

Strain-Displacement Relations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

3-D ELASTICITY (continued)

Constitutive Relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

The material axes are assumed coincide with the global axes and the material is orthotropic with respect to the global axes.

Boundary Conditions

$$\left.\begin{aligned} t_x &\equiv \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z = \hat{t}_x \\ t_y &\equiv \sigma_{xy}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z = \hat{t}_y \\ t_z &\equiv \sigma_{xz}n_x + \sigma_{yz}n_y + \sigma_{zz}n_z = \hat{t}_z \end{aligned}\right\} \quad \text{on } \Gamma_\sigma \quad \underline{\text{or}} \quad \mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u$$

3-D ELASTICITY (continued)

MATRIX FORM OF THE GOVERNING EQUATIONS

Notation

$$\mathbf{D}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial z & 0 & \partial/\partial y \\ 0 & \partial/\partial y & 0 & 0 & \partial/\partial z & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial x & \partial/\partial y & 0 \end{bmatrix}$$

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad \varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

Governing equations

$$\mathbf{D}^{\mathrm{T}}\sigma + \mathbf{f} = \rho\ddot{\mathbf{u}} \qquad \varepsilon = \mathbf{D}\mathbf{u}, \qquad \sigma = \mathbf{C}\varepsilon$$

3-D ELASTICITY (continued)

Principle of virtual displacements (in matrix form)

$$0 = \int_{\Omega_e} \left[(\mathbf{D}\delta\mathbf{u})^{\mathrm{T}} \mathbf{C}\,(\mathbf{D}\mathbf{u}) + \rho \mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}} \right] d\mathbf{x} - \int_{\Omega_e} (\delta\mathbf{u})^{\mathrm{T}} \mathbf{f}\; d\mathbf{x} - \oint_{\Gamma_e} (\delta\mathbf{u})^{\mathrm{T}} \mathbf{t}\; ds$$

Finite element approximation (in matrix form)

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \mathbf{\Psi}\mathbf{\Delta}, \quad \mathbf{w} = \delta\mathbf{u} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \mathbf{\Psi}\delta\mathbf{\Delta}$$

$$\mathbf{\Psi} = \begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & 0 & 0 & \ldots & \psi_n & 0 & 0 \\ 0 & \psi_1 & 0 & 0 & \psi_2 & 0 & \ldots & \psi_n & 0 & \\ 0 & 0 & \psi_1 & 0 & 0 & \psi_2 & 0 & \ldots & 0 & \psi_n \\ & & & & & & & & & \end{bmatrix}$$

$$\mathbf{\Delta} = \{ u_x^1 \quad u_y^1 \quad u_z^1 \quad u_x^2 \quad u_y^2 \quad u_z^2 \quad \ldots \quad u_x^n \quad u_y^n \quad u_z^n \}^{\mathrm{T}}$$

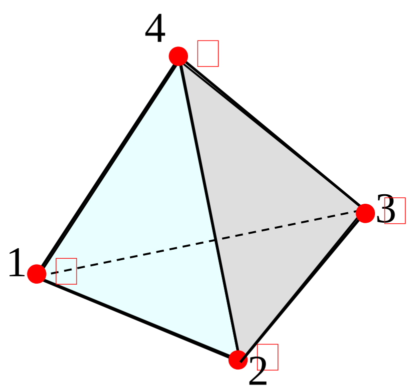
Finite Element Model

$$\mathbf{M}^e \ddot{\mathbf{\Delta}}^e + \mathbf{K}^e \mathbf{\Delta}^e = \mathbf{F}^e + \mathbf{Q}^e$$

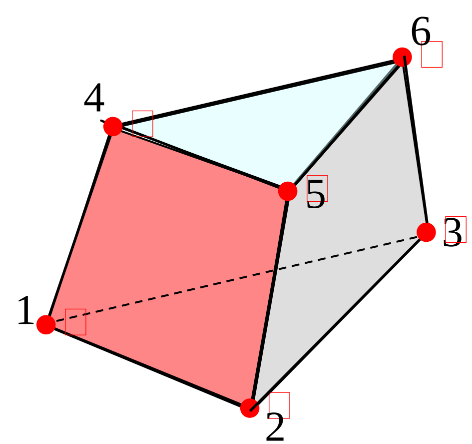
where

$$\mathbf{K}^e = \int_{\Omega_e} \mathbf{B}^{\mathrm{T}} \mathbf{C} \mathbf{B} \, d\mathbf{x}, \quad \mathbf{M}^e = \int_{\Omega_e} \rho \quad \mathbf{\Psi}^{\mathrm{T}} \mathbf{\Psi}^e \, d\mathbf{x}$$

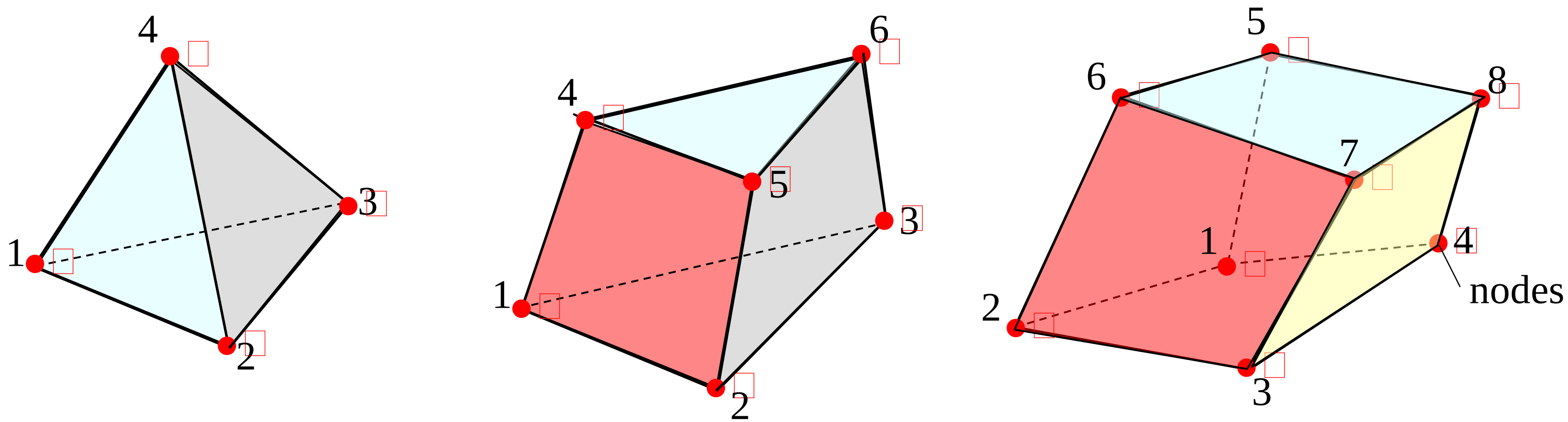
$$\mathbf{F}^e = \int_{\Omega_e} \mathbf{\Psi}^{\mathrm{T}} \mathbf{f} \, d\mathbf{x}, \qquad \mathbf{Q}^e = \oint_{\Gamma_e} \mathbf{\Psi}^{\mathrm{T}} \mathbf{t} \, ds$$



At each node (u,v,w)

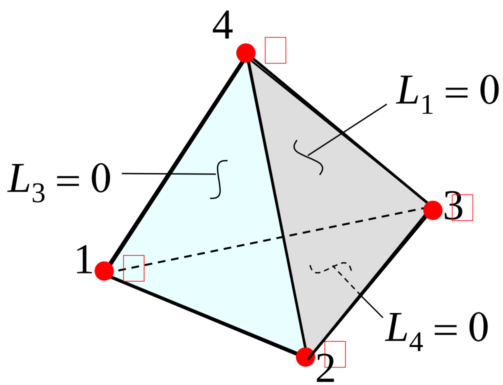


TYPICAL 3-D FINITE ELEENTS



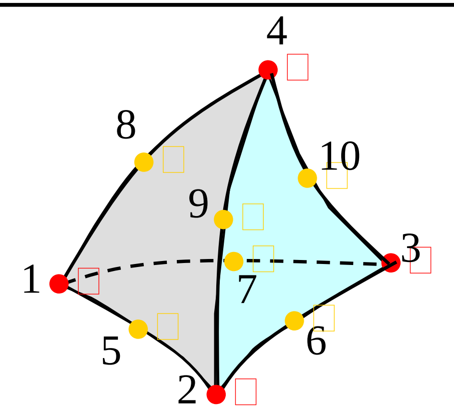
TYPICAL 3-D FINITE ELEMENTS

Linear tetrahedral element



$$\{\boldsymbol{\Psi}^e\} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix}$$

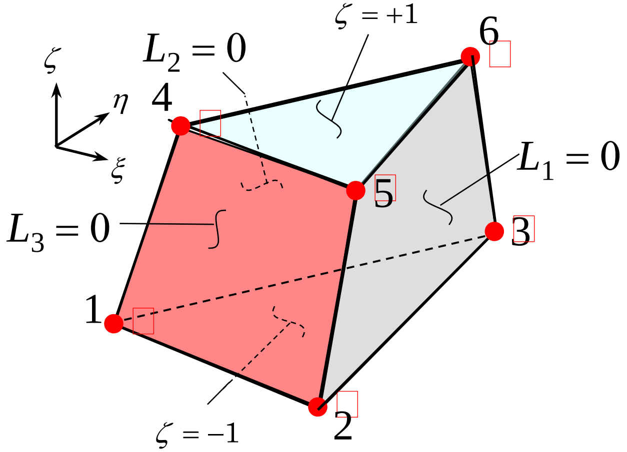
Quadratic tetrahedral element



$$\{\boldsymbol{\Psi}^e\} = \begin{Bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ L_4(2L_4 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \\ 4L_1L_4 \\ 4L_2L_4 \\ 4L_3L_4 \end{Bmatrix}$$

TYPICAL 3-D FINITE ELEMENTS

Linear prism element

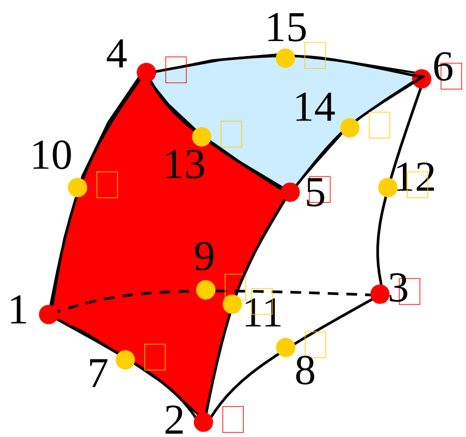


$$u_h(\xi,\eta,\zeta) = c_0 + c_1\xi + c_2\eta + \left(c_3 + c_4\xi + c_5\eta\right)\zeta$$

$$\{\Psi^e\} = \frac{1}{2}\begin{Bmatrix} L_1(1-\zeta) \\ L_2(1-\zeta) \\ L_3(1-\zeta) \\ L_1(1+\zeta) \\ L_2(1+\zeta) \\ L_3(1+\zeta) \end{Bmatrix}$$

TYPICAL 3-D FINITE ELEMENTS (cont…)

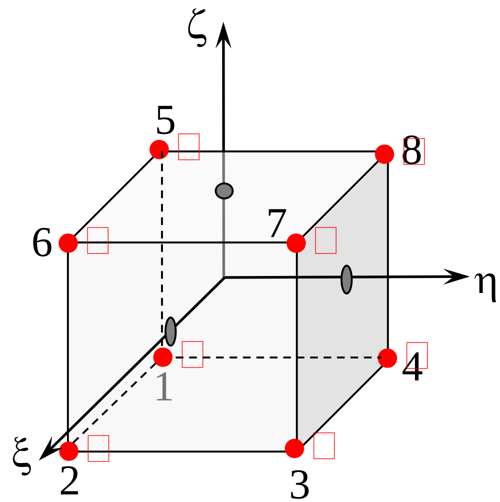
Quadratic prism element



$$
\{\Psi^e\} = \frac{1}{2}\left\{\begin{array}{c}
L_1[(2L_1-1)(1-\zeta)-(1-\zeta^2)] \\
L_2[(2L_2-1)(1-\zeta)-(1-\zeta^2)] \\
L_3[(2L_3-1)(1-\zeta)-(1-\zeta^2)] \\
L_1[(2L_1-1)(1+\zeta)-(1-\zeta^2)] \\
L_2[(2L_2-1)(1+\zeta)-(1-\zeta^2)] \\
L_3[(2L_3-1)(1+\zeta)-(1-\zeta^2)] \\
4L_1L_2(1-\zeta) \\
4L_2L_3(1-\zeta) \\
4L_3L_1(1-\zeta) \\
2L_1(1-\zeta^2) \\
2L_2(1-\zeta^2) \\
2L_3(1-\zeta^2) \\
4L_1L_2(1+\zeta) \\
4L_2L_3(1+\zeta) \\
4L_3L_1(1+\zeta)
\end{array}\right\}
$$

TYPICAL 3-D FINITE ELEMENTS (cont...)

Linear brick element

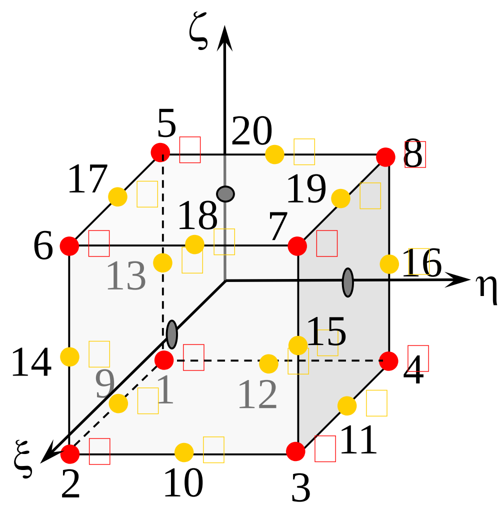


$$u_h(\xi,\eta,\zeta) = c_0 + c_1\xi + c_2\eta + c_3\zeta + c_4\xi\eta + c_5\xi\zeta + c_6\eta\zeta + c_7\xi\eta\zeta$$

$$\{\boldsymbol{\Psi}^e\} = \frac{1}{8}\left\{\begin{matrix}(1-\xi)(1-\eta)(1-\zeta)\\(1+\xi)(1-\eta)(1-\zeta)\\(1+\xi)(1+\eta)(1-\zeta)\\(1-\xi)(1+\eta)(1-\zeta)\\(1-\xi)(1-\eta)(1+\zeta)\\(1+\xi)(1-\eta)(1+\zeta)\\(1+\xi)(1+\eta)(1+\zeta)\\(1-\xi)(1+\eta)(1+\zeta)\end{matrix}\right\}$$

TYPICAL 3-D FINITE ELEMENTS (cont…)

Quadratic Brick Element



$$
\{\Psi^e\} = \frac{1}{8}\left\{\begin{array}{c}
(1-\xi)(1-\eta)(1-\zeta)(-\xi-\eta-\zeta-2)\\
(1+\xi)(1-\eta)(1-\zeta)(\xi-\eta-\zeta-2)\\
(1+\xi)(1+\eta)(1-\zeta)(\xi+\eta-\zeta-2)\\
(1-\xi)(1+\eta)(1-\zeta)(-\xi+\eta-\zeta-2)\\
(1-\xi)(1-\eta)(1+\zeta)(-\xi-\eta+\zeta-2)\\
(1+\xi)(1-\eta)(1+\zeta)(\xi-\eta+\zeta-2)\\
(1+\xi)(1+\eta)(1+\zeta)(\xi+\eta+\zeta-2)\\
(1-\xi)(1+\eta)(1+\zeta)(-\xi+\eta+\zeta-2)\\
2(1-\xi^2)(1-\eta)(1-\zeta)\\
2(1+\xi)(1-\eta^2)(1-\zeta)\\
2(1-\xi^2)(1+\eta)(1-\zeta)\\
2(1-\xi)(1-\eta^2)(1-\zeta)\\
2(1-\xi)(1-\eta)(1-\zeta^2)\\
2(1+\xi)(1-\eta)(1-\zeta^2)\\
2(1+\xi)(1+\eta)(1-\zeta^2)\\
2(1-\xi)(1+\eta)(1-\zeta^2)\\
2(1-\xi^2)(1-\eta)(1+\zeta)\\
2(1+\xi)(1-\eta^2)(1+\zeta)\\
2(1-\xi^2)(1+\eta)(1+\zeta)\\
2(1-\xi)(1-\eta^2)(1+\zeta)
\end{array}\right\}
$$

SUMMARY

In this lecture, the following topics were covered:

- **2-D problem with convection**
- **Conditions for solution symmetry**
- **Types of axisymmetric problems**
- **FEM of axisymmetric problems (2-D)**
- **3-D heat transfer**
- **3-D elasticity**
- **Typical 3-D finite elements**