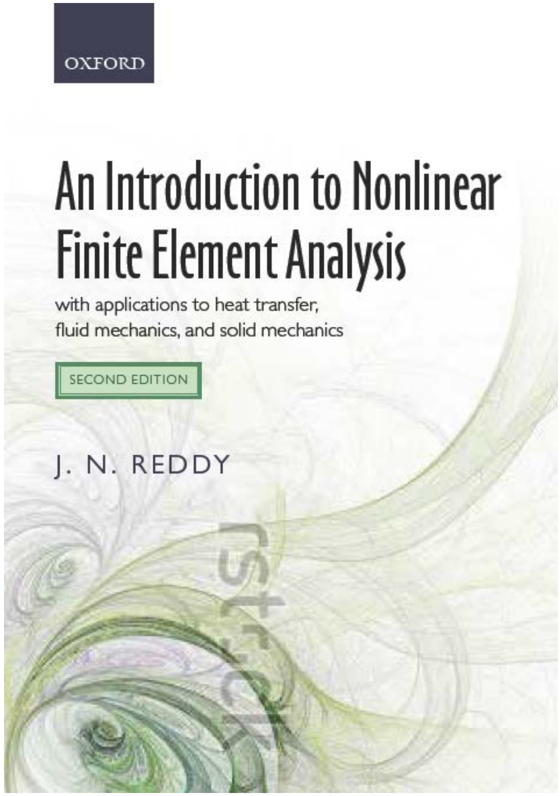
Finite Element Models for Steady Flows of Viscous Incompressible Fluids

Read: Chapter 10



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Governing Equations of Flows of Viscous incompressible Fluids

Equations of motion

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} \;\Rightarrow\; \begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho \frac{Dv_x}{Dt} \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho \frac{Dv_y}{Dt} \end{aligned}$$

Material time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \;\Rightarrow\; \frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$$

Conservation of mass

$$\nabla \cdot \mathbf{v} = 0 \;\Rightarrow\; \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Constitutive relations

$$\boldsymbol{\sigma} = -P\,\mathbf{I} + \boldsymbol{\tau}, \qquad \boldsymbol{\tau} = 2\mu \mathbf{D}$$

Governing Equations of Flows of Viscous incompressible Fluids

Kinematics relations

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right] \Rightarrow \; D_{xx} = \frac{\partial v_x}{\partial x},\; D_{yy} = \frac{\partial v_y}{\partial y},\; 2D_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

Stress-velocity-pressure relations

$$\sigma_{xx} = 2\mu\frac{\partial v_x}{\partial x} - P,\; \sigma_{yy} = 2\mu\frac{\partial v_y}{\partial y} - P,\; \sigma_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

Boundary conditions

$$(v_x, t_x) \text{ and } (v_y, t_y)$$

$$t_x = \sigma_{xx}n_x + \sigma_{xy}n_y,\quad t_y = \sigma_{xy}n_x + \sigma_{yy}n_y$$

Governing Equations in Terms of Velocities and Pressure (2D)

Differential equations

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - f_x = 0$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) - \frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} - f_y = 0$$

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = 0$$

Boundary conditions

$$(v_x, t_x) \text{ and } (v_y, t_y)$$

$$\sigma_{xx} = 2\mu\frac{\partial v_x}{\partial x} - P, \quad \sigma_{yy} = 2\mu\frac{\partial v_y}{\partial y} - P, \quad \sigma_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

WEAK FORMS OF THE EQUATIONS

$$0 = \int_{\Omega^e} w_1 \left[\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - f_x \right] dxdy \qquad w_1 \sim v_x$$

$$= \int_{\Omega^e} \left[\rho w_1 \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) + \frac{\partial w_1}{\partial x} \sigma_{xx} + \frac{\partial w_1}{\partial y} \sigma_{xy} - w_1 f_x \right] dxdy - \oint_{\Gamma^e} w_1 t_x \, ds,$$

$$0 = \int_{\Omega^e} w_2 \left[\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) - \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial x} - f_y \right] dxdy \qquad w_2 \sim v_y$$

$$= \int_{\Omega^e} \left[\rho w_2 \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) + \frac{\partial w_2}{\partial y} \sigma_{yy} + \frac{\partial w_2}{\partial x} \sigma_{xy} - w_2 f_y \right] dxdy - \oint_{\Gamma^e} w_2 t_y \, ds,$$

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y, \quad t_y = \sigma_{xy} n_x + \sigma_{yy} n_y$$

$$0 = \int_{\Omega^e} w_3 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dxdy \qquad w_3 \sim -P$$

Mixed Finite Element Model for the steady-state case

$$v_x = \sum_{j=1}^{m} v_{xj}\psi_j, \quad v_y = \sum_{j=1}^{m} v_{yj}\psi_j, \quad P = \sum_{j=1}^{n} P_j\phi_j,$$

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_x \\ \dot{\mathbf{v}}_y \\ \dot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} + \mathbf{G}(\mathbf{v}) & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} + \mathbf{G}(\mathbf{v}) & \mathbf{K}^{23} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{\Omega^e} \mu \left(2\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy, \quad G_{ij}(v_x, v_y) = \int_{\Omega^e} \rho\psi_i \left(v_x \frac{\partial \psi_j}{\partial x} + v_y \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial x} dxdy, \quad K_{ij}^{13} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial x}\phi_j \, dxdy, \quad K_{ij}^{23} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y}\phi_j \, dxdy,$$

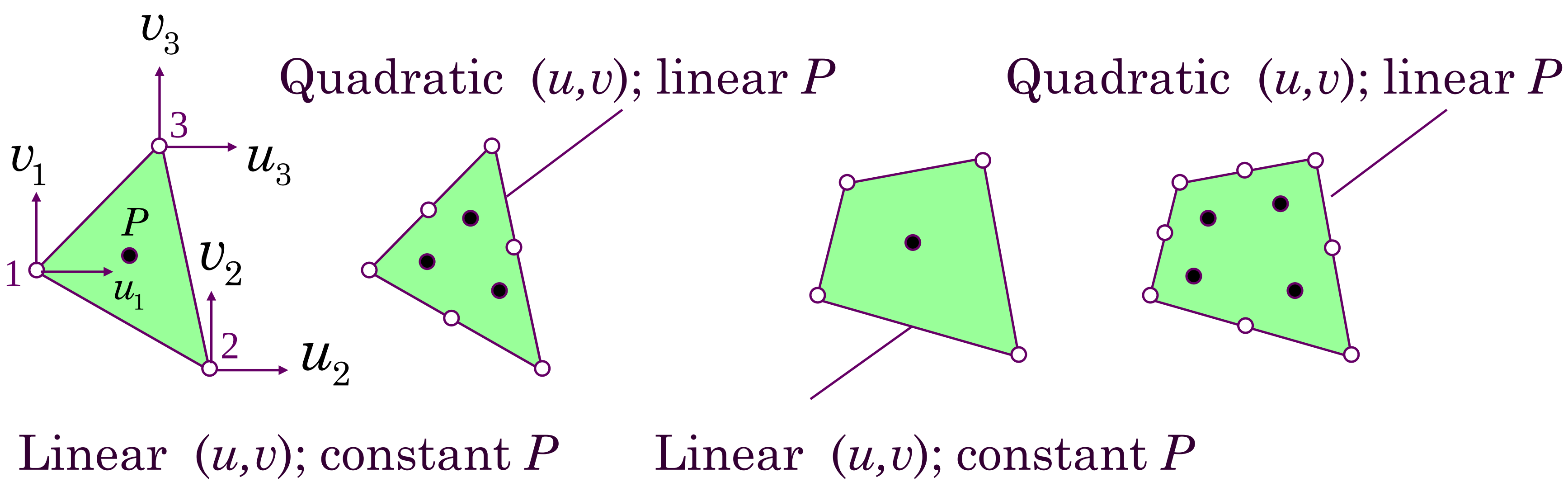
$$K_{ij}^{22} = \int_{\Omega^e} \mu \left(\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + 2\frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy, \; K_{ij}^{21} = K_{ji}^{12}, \; K_{ij}^{31} = K_{ji}^{13}, \; K_{ij}^{32} = K_{ji}^{23}, \; K_{ij}^{33} = 0$$

$$F_i^1 = \int_{\Omega^e} f_x\psi_i \, dxdy + \oint_{\Gamma^e} t_x\psi_i \, ds, \quad F_i^2 = \int_{\Omega^e} f_y\psi_i \, dxdy + \oint_{\Gamma^e} t_y\psi_i \, ds$$

Mixed Finite Element Model

(continued)

$$v_x = u, \quad v_y = v$$



Penalty Function Method-*algebraic*

Problem: Find the minimum of the function $F(x, y)$ subject to the constraint $G(x, y) = 0$

$$dF \equiv \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

Lagrange multiplier method

$$F_L(x, y, \lambda) \equiv F(x, y) + \lambda G(x, y)$$

$$\begin{aligned} dF_L &\equiv \frac{\partial F_L}{\partial x} dx + \frac{\partial F_L}{\partial y} dy + \frac{\partial F_L}{\partial \lambda} d\lambda = 0 \\ &= \left(\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} \right) dx + \left(\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} \right) dy + G(x, y) d\lambda \end{aligned}$$

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0, \quad G(x, y) = 0$$

Penalty Function Method-*algebraic* (continued)

Penalty function method

$$F_P(x,y) = F(x,y) + \frac{\gamma}{2}[G(x,y) - 0]^2$$

$$dF_P \equiv \frac{\partial F_P}{\partial x}dx + \frac{\partial F_P}{\partial y}dy = 0$$

$$= \left(\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y}\right)dy$$

$$\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y} = 0$$

$$dF_L \equiv \frac{\partial F_L}{\partial x}dx + \frac{\partial F_L}{\partial y}dy + \frac{\partial F_L}{\partial \lambda}d\lambda = 0$$

$$= \left(\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y}\right)dy + G(x,y)d\lambda$$

$$\boxed{\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x} = 0,} \quad \boxed{\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y} = 0,} \quad G(x,y) = 0$$

$$dF_P \equiv \frac{\partial F_P}{\partial x}dx + \frac{\partial F_P}{\partial y}dy = 0$$

$$= \left(\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y}\right)dy$$

$$\boxed{\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x} = 0,} \quad \boxed{\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y} = 0}$$

Approximation of the Lagrange multiplied can be computed in the penalty method from

$$\boxed{\lambda_\gamma = \gamma\, G(x_\gamma, y_\gamma)}$$

Penalty Function Method-*algebraic*

(An Example)

$$F(x,y) = 2x^2 + y^2 - 8x + y + 1, \quad G(x,y) \equiv 2x - y = 0$$

Lagrange Multiplier Method

$$4x - 8 + 2\lambda = 0, \quad 2y + 1 - \lambda = 0, \quad 2x - y = 0$$

$$x = 0.5, \qquad y = 1.0, \qquad \lambda = 3.0$$

Penalty Function Method

$$4x - 8 + 2\gamma(2x - y) = 0, \qquad 2y + 1 - \gamma(2x - y) = 0$$

$$x_\gamma = \frac{8 + 3\gamma}{4 + 6\gamma}, \qquad y_\gamma = \frac{3\gamma - 1}{2 + 3\gamma}$$

Clearly, as $\gamma \to \infty$, we have

$$\lim_{\gamma \to \infty} x_\gamma = 0.5 = x, \qquad \lim_{\gamma \to \infty} y_\gamma = 1.0 = y$$

Penalty Function Method-*algebraic*

(Example - continued)

Table: A comparison of the penalty solution with the exact for various values of the penalty parameter γ.

γ	1.0	10.0	25.0	50.0	100.0	1000.0
x_γ	1.1	0.5938	0.5390	0.5197	0.5099	0.5010
y_γ	0.4	0.9063	0.9610	0.9803	0.9901	0.9990
$G(x_\gamma, y_\gamma)$	1.8	0.2813	0.1169	0.0592	0.0298	0.0030
λ_γ	1.8	2.8125	2.9221	2.9605	2.9801	2.9980

$$\lambda_\gamma = \gamma\, G(x_\gamma, y_\gamma)$$

PENALTY FINITE ELEMENT FORMULATION for the Steady-State Case

Consider the weak forms

$$0 = \int_{\Omega^e} \left[\rho w_1 \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) + \frac{\partial w_1}{\partial x} \sigma_{xx} + \frac{\partial w_1}{\partial y} \sigma_{xy} - w_1 f_x \right] dxdy - \oint_{\Gamma^e} w_1 t_x \, ds,$$

$$0 = \int_{\Omega^e} \left[\rho w_2 \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) + \frac{\partial w_2}{\partial y} \sigma_{yy} + \frac{\partial w_2}{\partial x} \sigma_{xy} - w_2 f_y \right] dxdy - \oint_{\Gamma^e} w_2 t_y \, ds,$$

$$0 = \int_{\Omega^e} w_3 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dxdy, \qquad t_x = \sigma_{xx} n_x + \sigma_{xy} n_y, \quad t_y = \sigma_{xy} n_x + \sigma_{yy} n_y$$

Penalty Finite Element Formulation (continued)

Now suppose that the velocity field satisfies the constraint

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \;\; \Rightarrow \;\; \frac{\partial \delta v_x}{\partial x} + \frac{\partial \delta v_y}{\partial y} \equiv \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$$

Then adding the three weak statements, we obtain

$$0 = \int_{\Omega^e} \left[2\mu \frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x} + 2\mu \frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y} + \mu \left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x} \right)\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right.$$
$$\left. - P\underbrace{\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right)}_{\mathbf{0}} - w_3 \underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right)}_{\mathbf{0}} - w_1 f_x - w_2 f_y \right] dxdy$$
$$+ \int_{\Omega^e} \left[\rho w_1 \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) + \rho w_2 \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) \right] dxdy - \oint_{\Gamma^e} \left(w_1 t_x + w_2 t_y \right) ds$$

Penalty Finite Element Formulation
- continued

Thus, the weak form of the problem, subjected to the constraint is

$$0 = \int_{\Omega^e} \left[2\mu \frac{\partial w_1}{\partial x}\frac{\partial u}{\partial x} + 2\mu \frac{\partial w_2}{\partial y}\frac{\partial v}{\partial y} + \mu \left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x} \right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - w_1 f_x - w_2 f_y \right] dxdy - \oint_{\Gamma^e} \left(w_1 t_x + w_2 t_y \right) ds$$

$$+ \int_{\Omega^e} \left[\rho w_1 \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) + \rho w_2 \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) \right] dxdy$$

Thus, the variational problem is $0 = \delta I(v_x, v_y)$

subject to the constraint $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$

Penalty Finite Element Formulation
- continued

Then, the modified weak form with the constraint is

$$0=\delta I_p(v_x,v_y)=\delta I(v_x,v_y)+\delta\int_{\Omega_e}\frac{\gamma}{2}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)^2 dxdy$$

$$0=\int_{\Omega^e}\left[2\mu\frac{\partial w_1}{\partial x}\frac{\partial u}{\partial x}+2\mu\frac{\partial w_2}{\partial y}\frac{\partial v}{\partial y}+\mu\left(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right.$$

$$\left.-w_1 f_x-w_2 f_y\right]dxdy-\oint_{\Gamma^e}\left(w_1 t_x+w_2 t_y\right)ds$$

$$+\int_{\Omega^e}\left[\rho w_1\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)+\rho w_2\left(v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right)\right]dxdy$$

$$+\int_{\Omega^e}\gamma\left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}\right)\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)dxdy$$

← Penalty expression

Penalty Finite Element Formulation
- continued

The weak form of the problem can be separated into the following two statements:

$$0=\int_{\Omega^e}\left[2\mu\frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x}+\mu\frac{\partial w_1}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\gamma\frac{\partial w_1}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)-w_1 f_x\right]dxdy$$

$$-\oint_{\Gamma^e} w_1 t_x\, ds+\int_{\Omega^e}\rho w_1\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)dxdy$$

$$0=\int_{\Omega^e}\left[2\mu\frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y}+\mu\frac{\partial w_2}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\gamma\frac{\partial w_2}{\partial y}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)-w_2 f_y\right]dxdy$$

$$-\oint_{\Gamma^e} w_2 t_y\, ds+\int_{\Omega^e}\rho w_2\left(v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right)dxdy$$

These statements form the basis of the penalty FE Model.

Penalty Finite Element Formulation
continued

Alternatively, the pressure (negative of the Lagrange multiplier) in the governing equations can be replaced by

$$P_\gamma = -\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

We obtain

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial}{\partial x}\left[\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] - f_x = 0$$

$$\rho\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) - \frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial}{\partial y}\left[\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] - f_y = 0$$

The weak forms of these equations are precisely the same as those on the previous slide.

Penalty Finite Element Model

$$v_x = \sum_{j=1}^{m} v_{xj}\psi_j(x,y), \quad v_y = \sum_{j=1}^{m} v_{yj}\psi_j(x,y)$$

Substitution into the weak forms (adding inertia terms) yields the equations

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_x \\ \dot{\mathbf{v}}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} + \mathbf{G}(\mathbf{v}) & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} + \mathbf{G}(\mathbf{v}) \end{bmatrix} \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix}$$

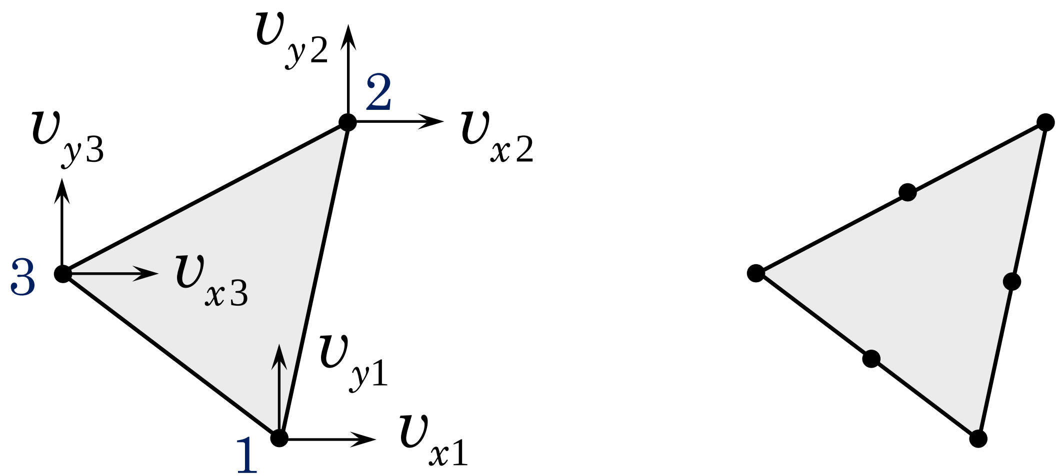
$$K_{ij}^{11} = \int_{\Omega^e} \mu \left(2\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial x} dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial y} dxdy, \quad K_{ij}^{21} = K_{ji}^{12}$$

$$K_{ij}^{22} = \int_{\Omega^e} \mu \left(\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + 2\frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} dxdy$$

$$F_i^1 = \int_{\Omega^e} f_x \psi_i \, dxdy + \oint_{\Gamma^e} t_x \psi_i \, ds, \; F_i^2 = \int_{\Omega^e} f_y \psi_i \, dxdy + \oint_{\Gamma^e} t_y \psi_i \, ds$$

Elements Used for Penalty FE Model



- **Nodes with v_x and v_y**

Computational Aspects of the Penalty FEM

General form of the Penalty FEM:

$$\left(\mu[K^1] + \rho[K^2] + \gamma[K^3]\right)\{\Delta\} = \{F\}$$

Element 'locking':

$$\lim \gamma \to 0 \ \ \left(\mu[K^1] + \rho[K^2] + \gamma[K^3]\right)\{\Delta\} = \{F\} \ \to \ \gamma[K^3]\{\Delta\} = \{F\}$$

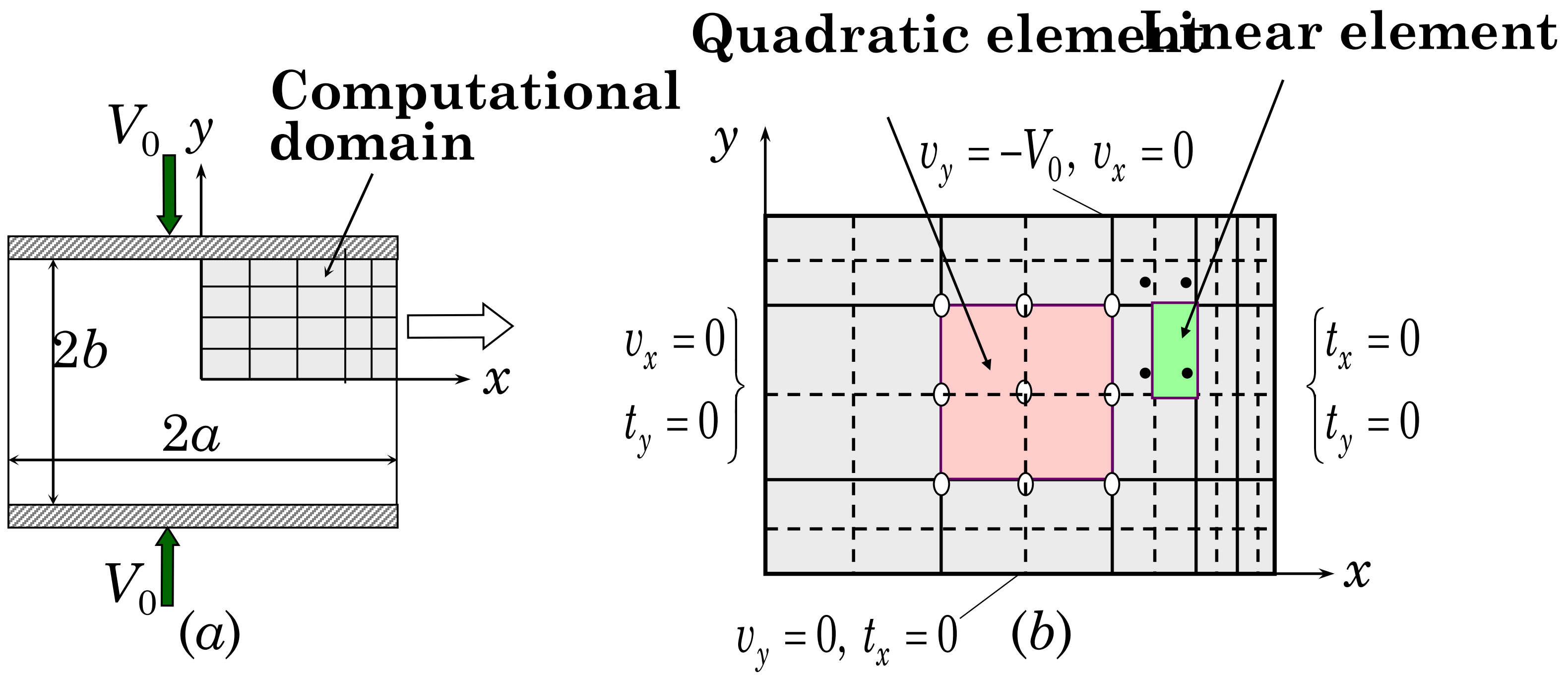
Choice of the penalty parameter:

$$\gamma = 10^4\mu \text{ to } \ \gamma = 10^{12}\mu$$

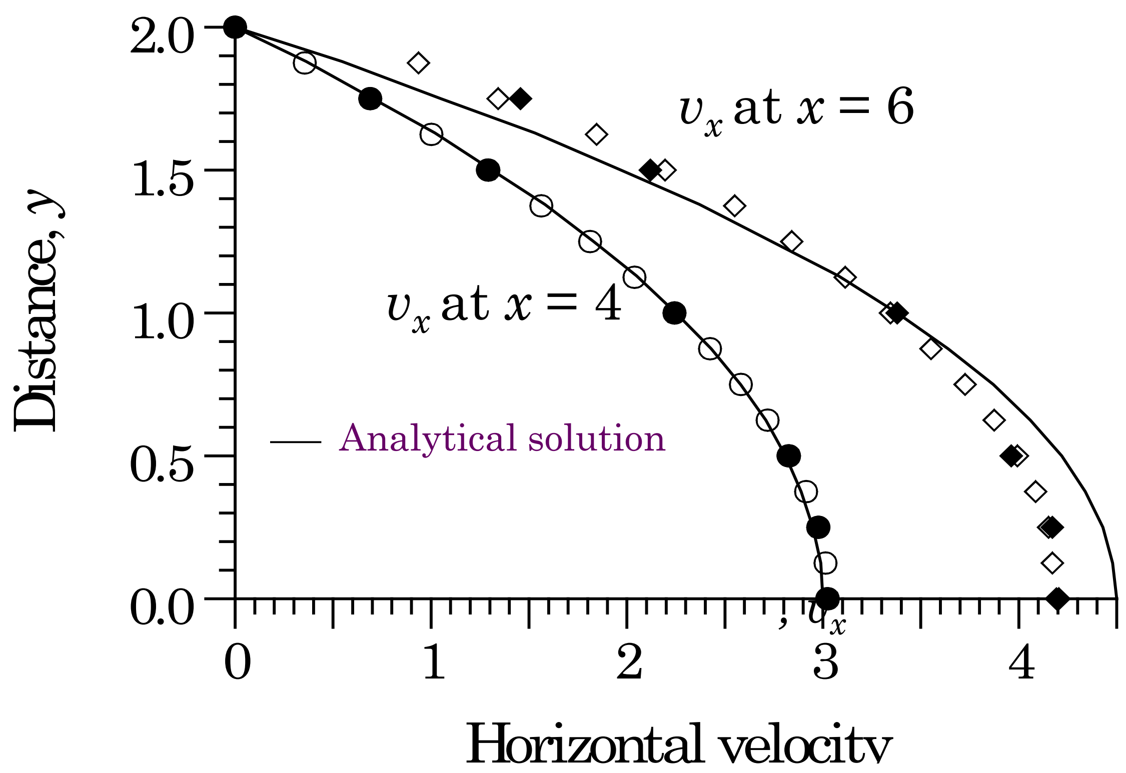
Reduced integration of the penalty terms

Numerical Examples

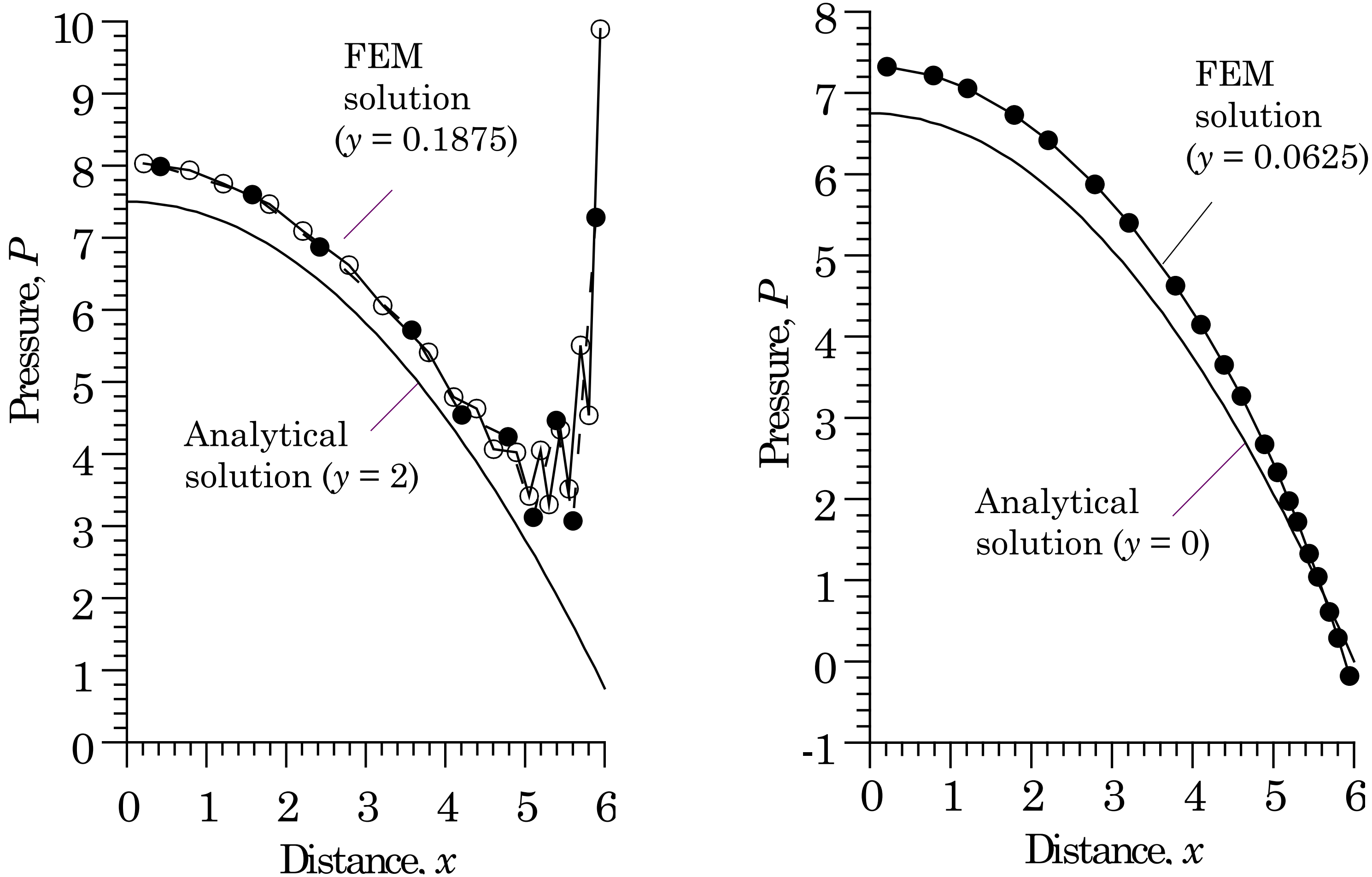
Viscous fluid squeezed between parallel plates



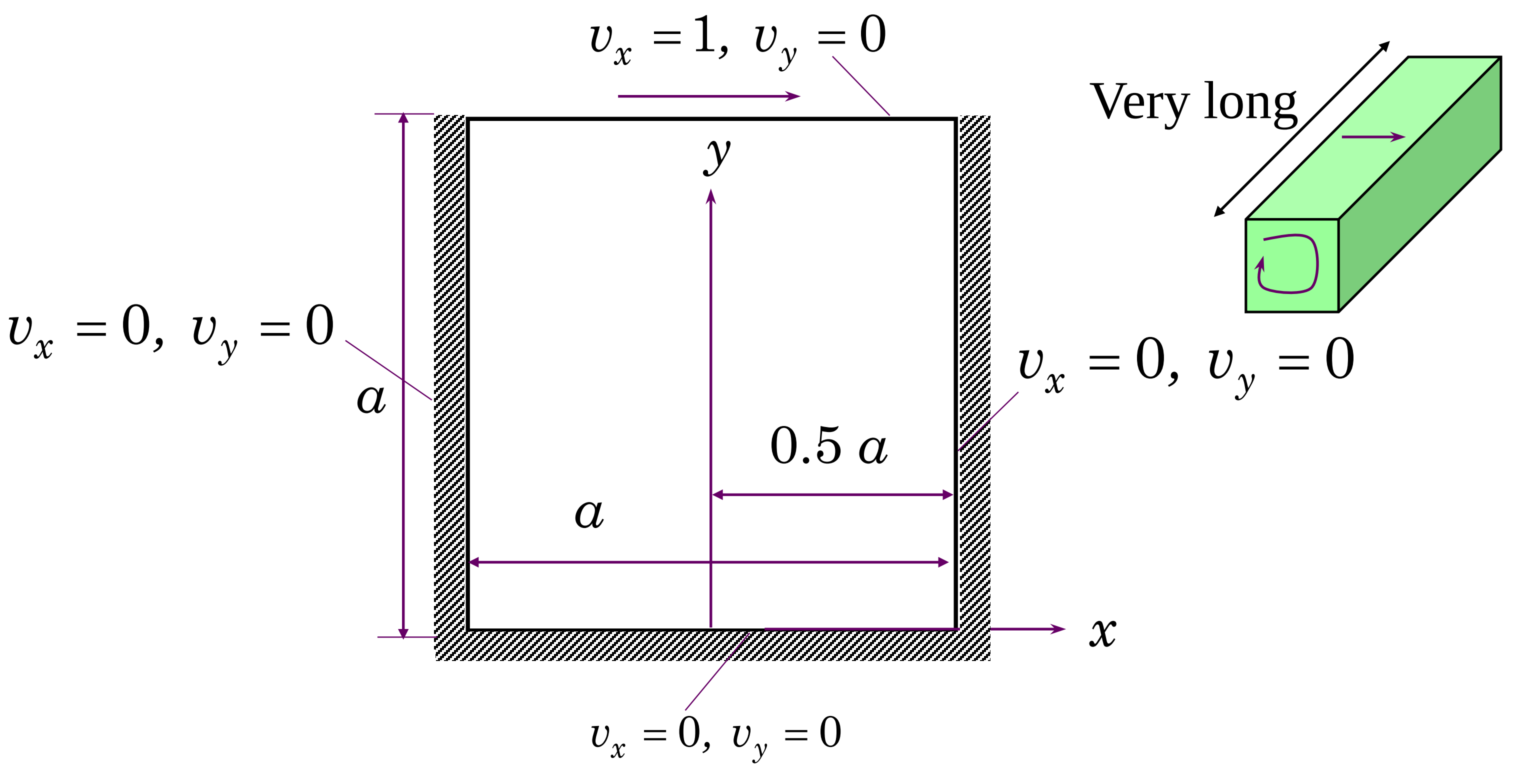
Viscous fluid squeezed between parallel plates: Velocity field



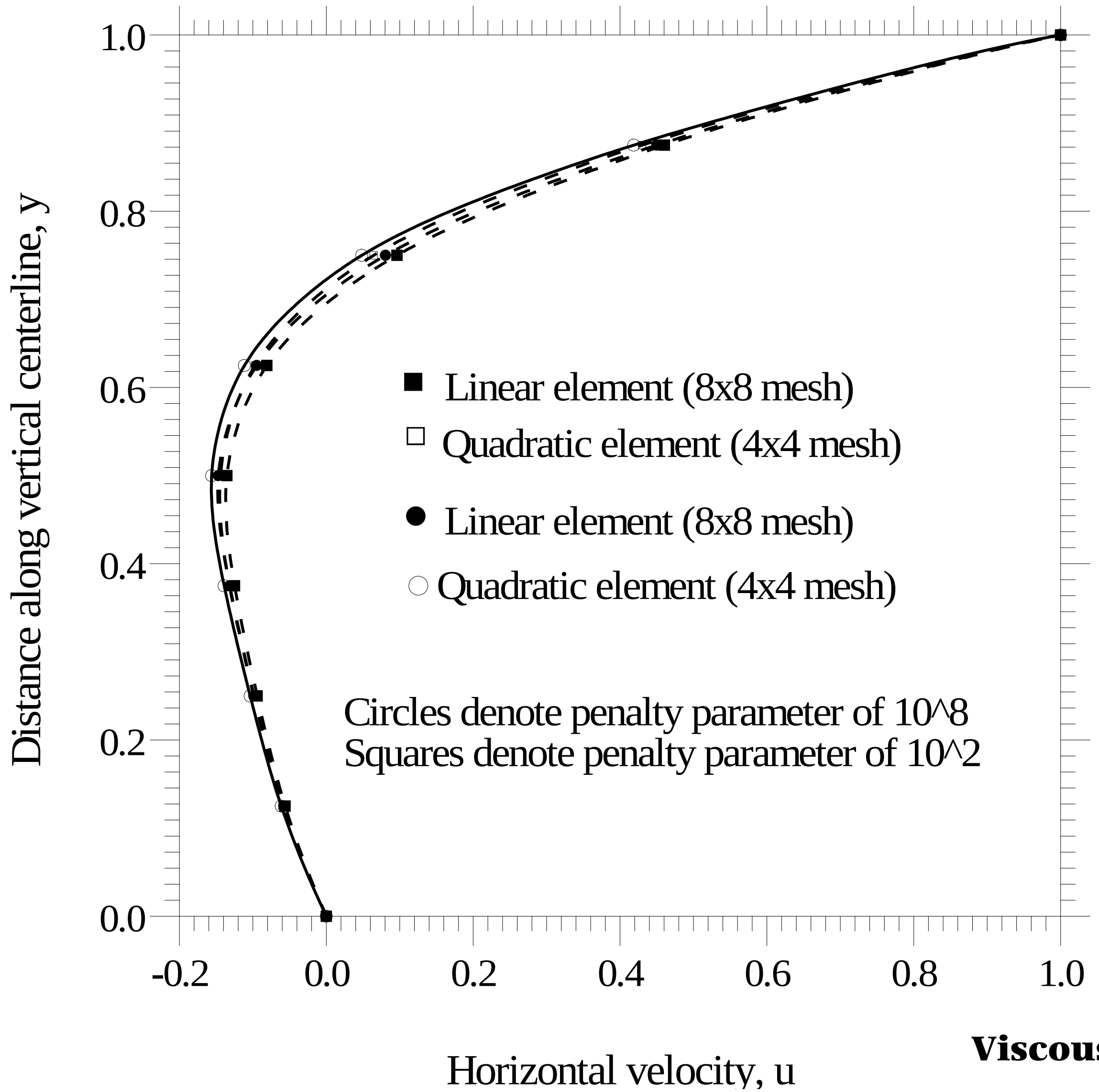
Viscous fluid squeezed between parallel plates: Pressure field



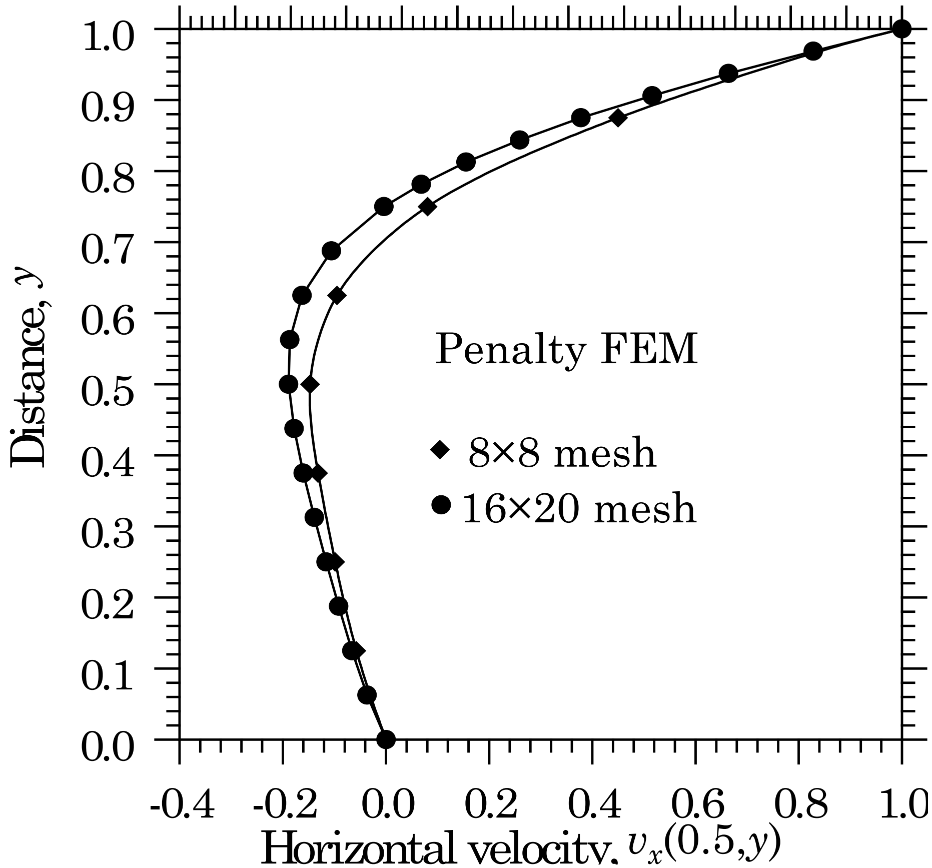
LID-DRIVEN CAVITY FLOW

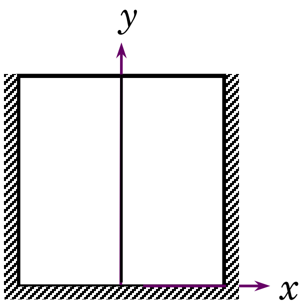


LID-DRIVEN CAVITY FLOW

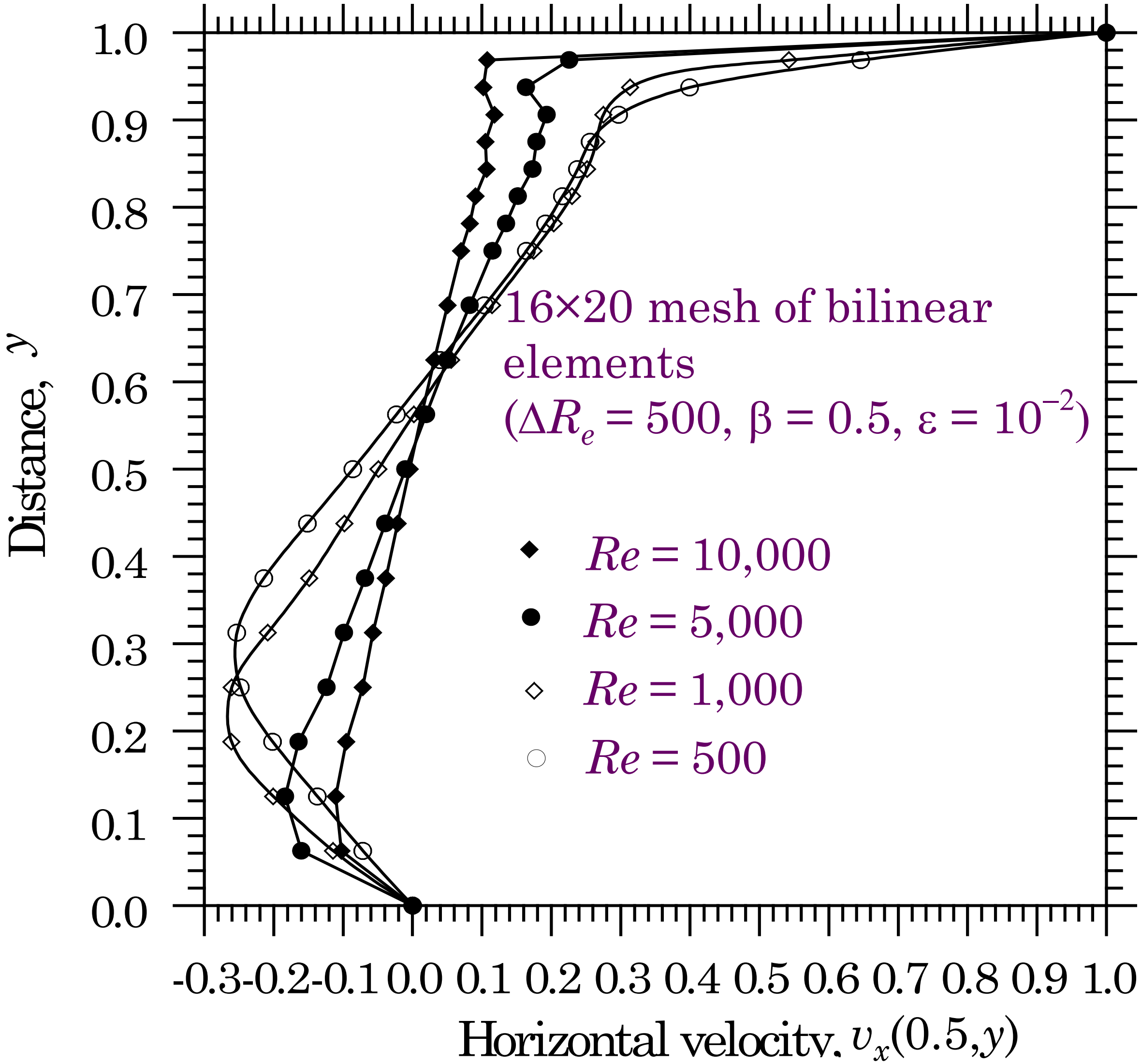


Wall-driven cavity flow – velocity profiles

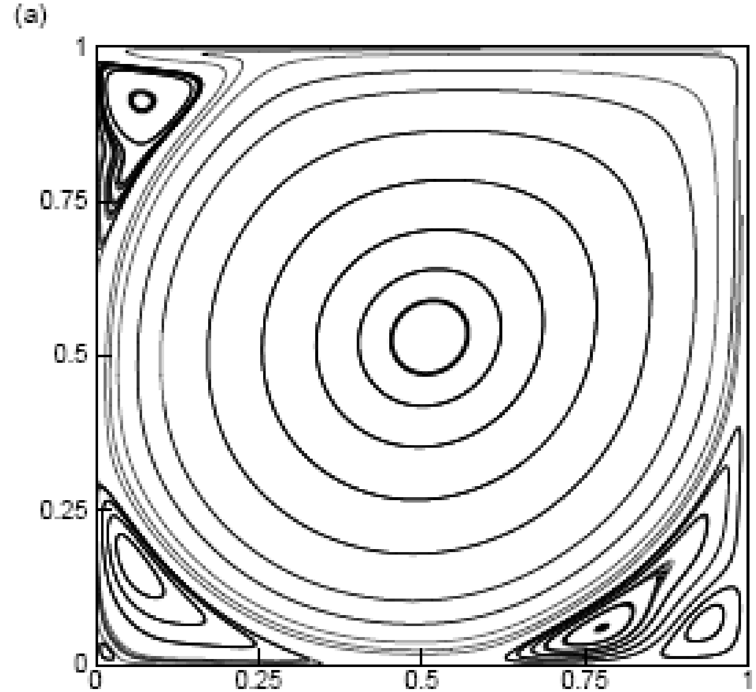




Wall-Driven Cavity results (continued)

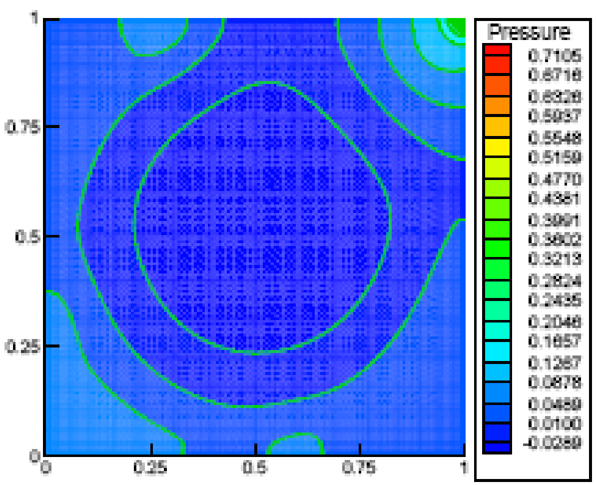


$$\mathrm{Re} = 10^4$$

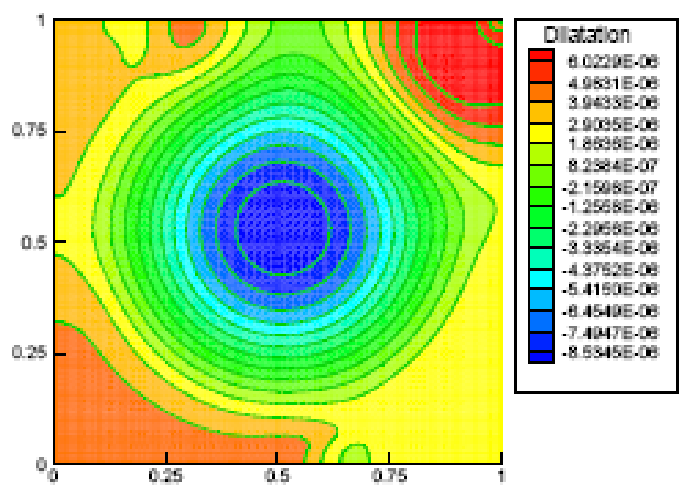


Streamlines

Pressure contours



Dilatation contours



SUMMARY

The following topics were covered in this lecture:

- **Governing equations of flows of incompressible fluids**
- **Mixed (velocity-pressure) finite element model**
- **Penalty function method -** ***algebraic problem***
- **Penalty finite element model of viscous incompressible fluids**
- **Numerical results**