MODIFIED COUPLE STESS THEORIES OF FUNCTIONALLY GRADED PLATES

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CONTENTS

- Formulate a general thermo-mechanical model of through-the-thickness functionally graded plates using the first-order shear deformation plate theory, accounting for
  - (a) the von Karman nonlinearity
  - (b) temperature dependent properties
  - (c) microstructure-dependent constitutive model
- Present numerical results for certain cases

Modified Couple Stress Theory of FGM Plates

$$\delta U = \int_V \left(\delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \delta\boldsymbol{\chi} : \mathbf{m}\right) dv = \int_V \left(\sigma_{ij}\,\delta\varepsilon_{ij} + m_{ij}\,\delta\chi_{ij}\right) dv$$

$$0 = \delta W = \int_{\Omega} \left(\delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \delta\boldsymbol{\chi} : \mathbf{m}\right) d\mathbf{v} - \int_{\Omega} \left(\delta w\, q + \delta\boldsymbol{\omega}\cdot\mathbf{c}\right) d\mathbf{x} - WDE$$

$$\boldsymbol{\chi} = \frac{1}{2}\left[\nabla\boldsymbol{\omega} + \left(\nabla\boldsymbol{\omega}\right)^{\mathrm{T}}\right], \quad \boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{u} \quad \text{or} \quad \chi_{ij} = \frac{1}{2}\left(\frac{\partial\omega_i}{\partial x_j} + \frac{\partial\omega_j}{\partial x_i}\right), \; i,j = 1,2,3$$

curvature tensor

$\mathbf{m}$ = Deviatoric part of the couple stress tensor

$= G\, l^2\, \boldsymbol{\chi}$

length scale parameter

$\mathbf{c}$ = Body couple resultant

A General Third-Order Plate Theory

Displacement Field

$$\begin{aligned}
u_1(x,y,z,t) &= u(x,y,t) + z\theta_x(x,y,t) + z^2\phi_x(x,y,t) + z^3\psi_x(x,y,t) \\
u_2(x,y,z,t) &= v(x,y,t) + z\theta_y(x,y,t) + z^2\phi_y(x,y,t) + z^3\psi_y(x,y,t) \\
u_3(x,y,z,t) &= w(x,y,t) + z\theta_z(x,y,t) + z^2\phi_z(x,y,t)
\end{aligned}$$

Strain Field

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{zz}^{(0)} \\ 2\varepsilon_{xy}^{(0)} \\ 2\varepsilon_{xz}^{(0)} \\ 2\varepsilon_{yz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ 2\varepsilon_{yz}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{zz}^{(3)} \\ 2\varepsilon_{xy}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ 2\varepsilon_{yz}^{(3)} \end{Bmatrix}$$

A General Third-Order Plate Theory

Strain-Displacement Relations

$$
\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{zz}^{(0)} \\ 2\varepsilon_{xy}^{(0)} \\ 2\varepsilon_{xz}^{(0)} \\ 2\varepsilon_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \theta_z \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix}, \quad
\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ 2\phi_z \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \\ 2\phi_x + \dfrac{\partial \theta_z}{\partial x} \\ 2\phi_y + \dfrac{\partial \theta_z}{\partial y} \end{Bmatrix}, \quad
\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ 2\varepsilon_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ 0 \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \\ 3\psi_x + \dfrac{\partial \phi_z}{\partial x} \\ 3\psi_y + \dfrac{\partial \phi_z}{\partial y} \end{Bmatrix}, \quad
\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{zz}^{(3)} \\ 2\varepsilon_{xy}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ 2\varepsilon_{yz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ 0 \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix}
$$

A General Third-Order Plate Theory

Curvature-Displacement Relations

$$\omega_x = \omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \ \omega_y = \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \ \omega_z = \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$\chi_{xx} = \chi_{11} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_3}\right), \ \chi_{yy} = \chi_{22} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)$$

$$\chi_{zz} = \chi_{33} = \frac{1}{2}\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_3} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3}\right), \ \chi_{yz} = 2\chi_{23} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_3^2} - \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2^2}\right)$$

$$\chi_{xy} = 2\chi_{12} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_1}{\partial x_1 \partial x_3} - \frac{\partial^2 u_3}{\partial x_1^2}\right)$$

$$\chi_{xz} = 2\chi_{13} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_2 \partial x_3} - \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)$$

Modified Couple Stress Theory of FGM Plates

Curvature-displacement relations for the FSDT

$$\omega_1 = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \phi_y\right), \quad \omega_1 = -\frac{1}{2}\left(\frac{\partial w}{\partial x} - \phi_x\right)$$

$$\omega_3 = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + z\,\frac{1}{2}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right)$$

$$\chi_{xx} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \phi_y}{\partial x}\right), \quad \chi_{yy} = -\frac{1}{2}\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \phi_x}{\partial y}\right)$$

$$\chi_{xy} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} + \frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y}\right)$$

$$\chi_{xz} = \frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\right) + z\,\frac{1}{4}\left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y}\right)$$

$$\chi_{yz} = \frac{1}{4}\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right) + z\,\frac{1}{4}\left(\frac{\partial^2 \phi_y}{\partial x \partial y} - \frac{\partial^2 \phi_x}{\partial y^2}\right)$$

Plate Constitutive Equations for the General Third-Order Theory

$$\begin{Bmatrix} M_{xx}^{(i)} \\ M_{yy}^{(i)} \\ M_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{11}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{12}^{(k)} & A_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k-i)} \\ \varepsilon_{yy}^{(k-i)} \\ \varepsilon_{zz}^{(k-i)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xy}^{(i)} \\ M_{xz}^{(i)} \\ M_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^{(k-i)} \\ \gamma_{xz}^{(k-i)} \\ \gamma_{zz}^{(k-i)} \end{Bmatrix}$$

Plate Constitutive Equations for the General Third-Order Theory

$$\begin{Bmatrix} \mathfrak{M}_{xx}^{(i)} \\ \mathfrak{M}_{yy}^{(i)} \\ \mathfrak{M}_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{zz} \end{Bmatrix} (z)^{i}\, dz = \sum_{k=i}^{2+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xx}^{(k-i)} \\ \chi_{yy}^{(k-i)} \\ \chi_{zz}^{(k-i)} \end{Bmatrix}$$

$$\begin{Bmatrix} \mathfrak{M}_{xy}^{(i)} \\ \mathfrak{M}_{xz}^{(i)} \\ \mathfrak{M}_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xy} \\ m_{xz} \\ m_{yz} \end{Bmatrix} (z)^{i}\, dz = \sum_{k=i}^{3+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xy}^{(k-i)} \\ \chi_{xz}^{(k-i)} \\ \chi_{yz}^{(k-i)} \end{Bmatrix}$$

Plate Stiffnesses

For the General Third-Order Theory

$$A_{11}^{(k)} = \frac{1-\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz$$

$$A_{12}^{(k)} = \frac{\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz$$

$$B_{11}^{(k)} = \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz$$

$$B_{11}^{(k)} = \frac{l^2}{(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz$$

Equations of Motion for the General Third-Order Theory

$$
\begin{aligned}
&A_{11}^{(0)}\frac{\partial^2 u}{\partial x^2}+A_{12}^{(0)}\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial \theta_z}{\partial x}\right)+A_{11}^{(1)}\frac{\partial^2\theta_x}{\partial x^2}+A_{12}^{(1)}\left(\frac{\partial^2\theta_y}{\partial x\partial y}+2\frac{\partial\phi_z}{\partial x}\right)\\
&+A_{11}^{(2)}\frac{\partial^2\phi_x}{\partial x^2}+A_{12}^{(2)}\frac{\partial^2\phi_y}{\partial x\partial y}+A_{11}^{(3)}\frac{\partial^2\psi_x}{\partial x^2}+A_{12}^{(3)}\frac{\partial^2\psi_y}{\partial x\partial y}+B_{11}^{(0)}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)\\
&+B_{11}^{(1)}\left(\frac{\partial^2\theta_x}{\partial y^2}+\frac{\partial^2\theta_y}{\partial x\partial y}\right)+B_{11}^{(2)}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+B_{11}^{(3)}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)\\
&+\frac{1}{4}\left\{\tilde{B}_{11}^{(0)}\left[\frac{\partial^4 v}{\partial x^3\partial y}-\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 v}{\partial x\partial y^3}-\frac{\partial^4 u}{\partial y^4}-2\left(\frac{\partial^2\phi_y}{\partial x\partial y}-\frac{\partial\phi_x}{\partial y^2}\right)\right]\right.\\
&+\tilde{B}_{11}^{(1)}\left[\frac{\partial^4\theta_y}{\partial x^3\partial y}-\frac{\partial^4\theta_x}{\partial x^2\partial y^2}+\frac{\partial^4\theta_y}{\partial x\partial y^3}-\frac{\partial^4\theta_x}{\partial y^4}-6\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\\
&\left.+\tilde{B}_{11}^{(2)}\left(\frac{\partial^4\phi_y}{\partial x^3\partial y}-\frac{\partial^4\phi_x}{\partial x^2\partial y^2}+\frac{\partial^4\phi_y}{\partial x\partial y^3}-\frac{\partial^4\phi_x}{\partial y^4}\right)+\tilde{B}_{11}^{(3)}\left(\frac{\partial^4\psi_y}{\partial x^3\partial y}-\frac{\partial^4\psi_x}{\partial x^2\partial y^2}+\frac{\partial^4\psi_y}{\partial x\partial y^3}-\frac{\partial^4\psi_x}{\partial y^4}\right)\right\}\\
&+F_x^{(0)}+\frac{1}{2}\frac{\partial c_z^{(0)}}{\partial y}=m_0\frac{\partial^2 u}{\partial t^2}+m_1\frac{\partial^2\theta_x}{\partial t^2}+m_2\frac{\partial^2\phi_x}{\partial t^2}+m_3\frac{\partial^2\psi_x}{\partial t^2}
\end{aligned}
$$

Equations of Motion of the Modified Couple Stress Theory (FSDT)

$$Y_{ij} = \int_{-h/2}^{h/2} m_{ij}\, dz, \quad H_{ij} = \int_{-h/2}^{h/2} m_{ij}\, z\, dz$$

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 Y_{xz}}{\partial x \partial y} + \frac{\partial^2 Y_{yz}}{\partial y^2} + \frac{\partial c_z}{\partial y}\right) = I_0 \frac{\partial^2 u}{\partial t^2}$$

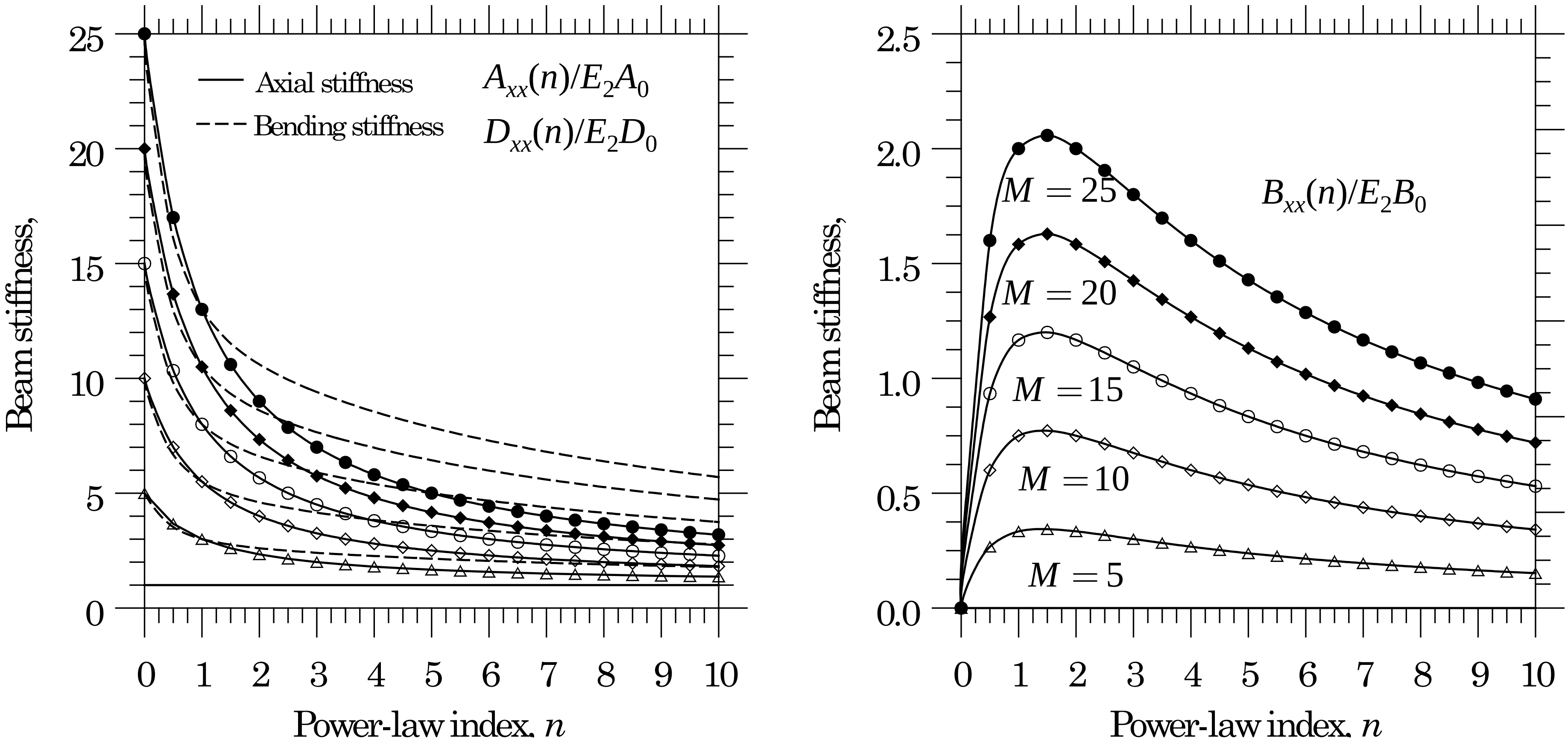
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} - \frac{1}{2}\left(\frac{\partial^2 Y_{xz}}{\partial x^2} + \frac{\partial^2 Y_{yz}}{\partial x \partial y} + \frac{\partial c_z}{\partial x}\right) = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{1}{2}\left(\frac{\partial^2 Y_{xx}}{\partial x \partial y} + \frac{\partial^2 Y_{xy}}{\partial y^2} + \frac{\partial c_x}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial^2 Y_{yy}}{\partial x \partial y} + \frac{\partial^2 Y_{xy}}{\partial x^2} + \frac{\partial c_y}{\partial x}\right) + N + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + \frac{1}{2}\left(\frac{\partial^2 H_{xz}}{\partial x \partial y} + \frac{\partial^2 H_{yz}}{\partial y^2} + \frac{\partial Y_{yy}}{\partial y} + \frac{\partial Y_{xy}}{\partial x} - \frac{\partial Y_{zz}}{\partial y} + c_y\right) = I_2 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y - \frac{1}{2}\left(\frac{\partial^2 H_{xz}}{\partial x^2} + \frac{\partial^2 H_{yz}}{\partial x \partial y} + \frac{\partial Y_{xy}}{\partial y} + \frac{\partial Y_{xx}}{\partial x} - \frac{\partial Y_{zz}}{\partial x} + c_y\right) = I_2 \frac{\partial^2 \phi_y}{\partial t^2}$$

VARIATION OF THE PLATE STIFFNESSES of the FGM Beams



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Numerical Results for the FGM Plates

$a = 20h,\ b = 20h,$

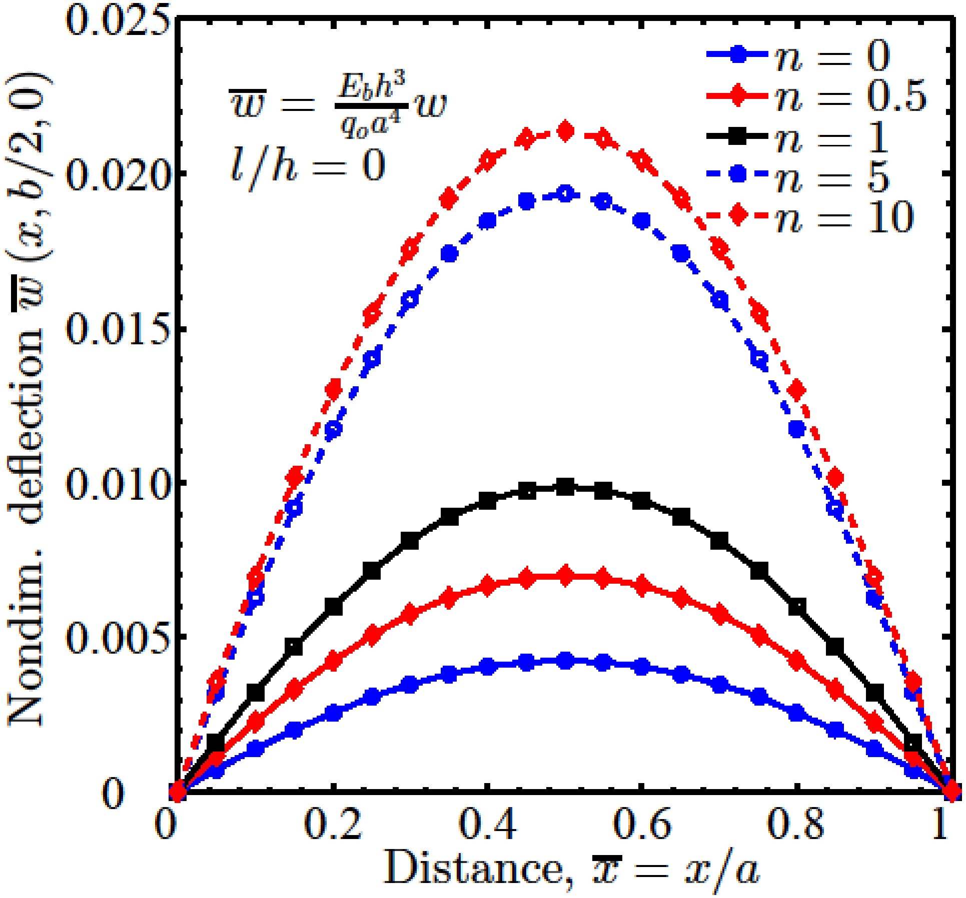
$h = 17.6\times10^{-6}\,\mathrm{m}$

$E_t = 14.4\,\mathrm{GPa},$

$E_b = 1.44\,\mathrm{GPa}$

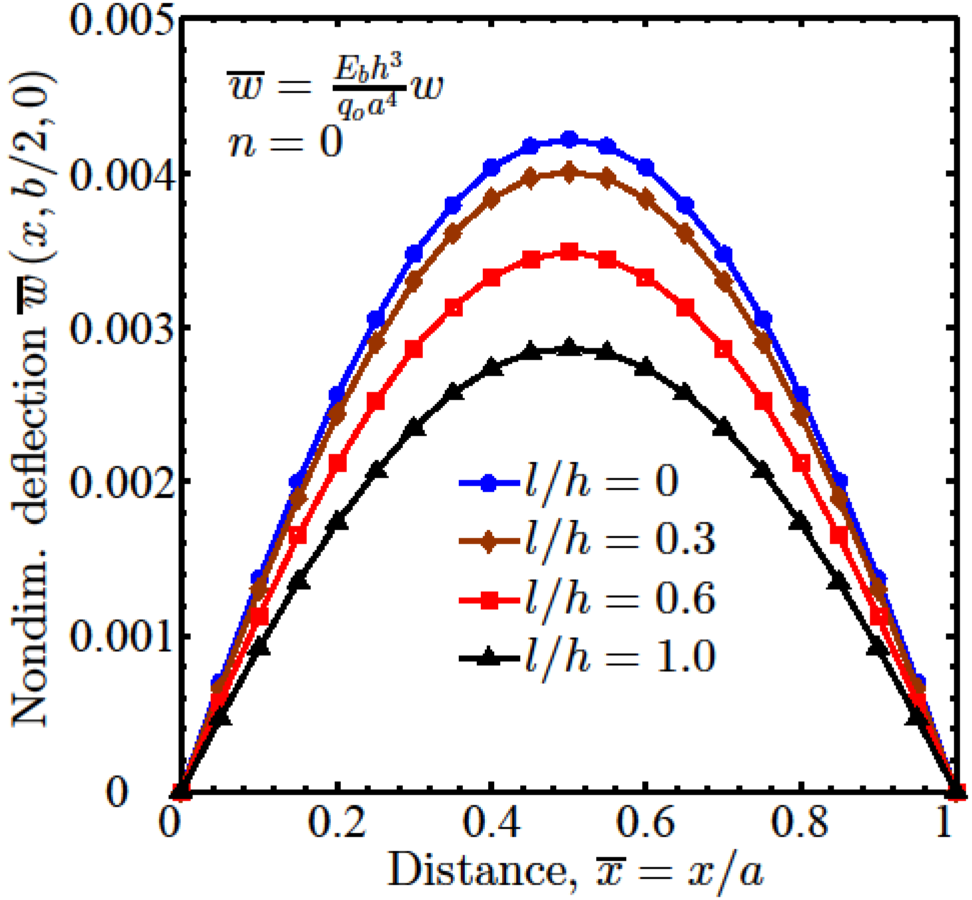
$\rho_t = 12.2\times10^{3}\,kg/m,$

$\rho_b = 1.22\times10^{3}\,kg/m$



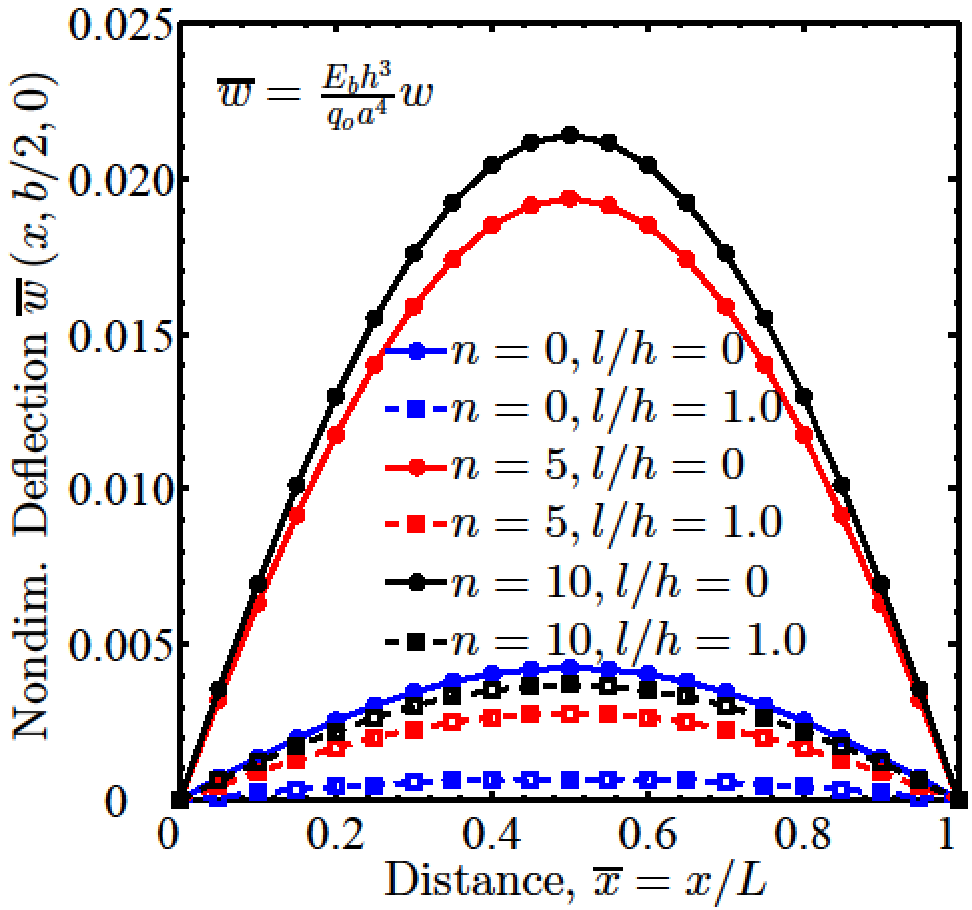
Non-dimensional deflection $\bar{w}(x, b/2, 0)$ versus *x/a* distance along a FGM simply supported plate with various values of the power-law index, *n*

Numerical Results for the FGM Plates



Non-dimensional deflection $\bar{w}(x, b/2, 0)$ versus distance *x/a* along a homogeneous simply supported plate with various values of the length scale parameter, *l*

Numerical Results for the FGM Plates



Non-dimensional deflection $\bar{w}(x, b/2, 0)$ versus distance *x/a* along a simply supported FGM plate with various power-law index *n* and length scale parameter *l* values.

SUMMARY, CONCLUSIONS AND FUTURE WORK

- A general thermomechanical model of functionally graded plates is developed using the general third-order shear deformation plate theory, accounting for (a) the von Karman nonlinearity, (b) functionally graded material, and (c) microstructure-dependent constitutive model.
- The thermomechnical coupling in the nonlinear case makes the FGM plates to have a response that is NOT in between the ceramic and metal plates. The micro-structure dependent constitutive model brings a length scale effect that makes the beam behave stiffer.
- Nonlinear analysis of FGM plates with temperature-dependent material properties and length scale effects is awaiting attention.