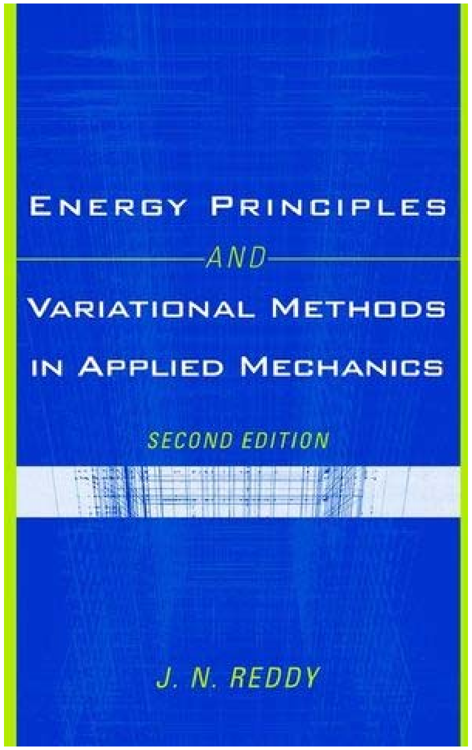
MEEN 618: ENERGY AND VARIATIONAL METHODS

VIRTUAL WORK AND ENERGY PRINCIPLES

Read: Chapter 5



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THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

The Principle

$$\delta W = \delta W_I + \delta W_E = 0$$

$$0 = \int_\Omega \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon}\, d\Omega - \left[\int_\Omega \mathbf{f} \cdot \delta \mathbf{u}\, d\Omega + \int_{\Gamma_\sigma} \hat{\mathbf{t}} \cdot \delta \mathbf{u}\, d\Gamma \right]$$

Application to 3D Linear Elasticity

$$0 = \frac{1}{2} \int_\Omega \left[\sigma_{ij} \left(\delta u_{i,j} + \delta u_{j,i} \right) \right] d\Omega - \left[\int_\Omega f_i \delta u_i\, d\Omega + \int_{\Gamma_\sigma} \hat{t}_i \delta u_i\, d\Gamma \right]$$

$$= -\int_\Omega \left(\sigma_{ji,j} + f_i \right) \delta u_i\, d\Omega - \int_{\Gamma_\sigma} \hat{t}_i u_i\, ds + \int_\Gamma \sigma_{ji} n_j \delta u_i\, d\Gamma$$

$$= -\int_\Omega \left(\sigma_{ji,j} + f_i \right) \delta u_i\, d\Omega + \int_{\Gamma_\sigma} \left(\sigma_{ij} n_j - \hat{t}_i \right) \delta u_i\, d\Gamma + \int_{\Gamma_u} \sigma_{ij} n_j \underbrace{\delta u_i}_{0}\, d\Gamma$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \ \text{ in } \Omega; \ \ \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} - \hat{\mathbf{t}} = \mathbf{0} \ \text{ on } \Gamma_\sigma$$

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

Application to Timoshenko Beams

Displacement field and the von Karman nonlinear strains

$$u_1(x,z) = u(x) + z\phi_x(x),\ u_2 = 0,\ u_3 = w(x)$$

$$\varepsilon_{11}(x,z) = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx},\ 2\varepsilon_{xz}(x) = \phi_x + \frac{dw}{dx}$$

Principle of virtual displacements

$$0 = \int_0^L \int_A (\sigma_{11}\,\delta\varepsilon_{11} + \sigma_{13}\,\delta\varepsilon_{13})dAdx - \left[\int_0^L f\cdot\delta u\,dx + \int_0^L q\cdot\delta w\,dx\right]$$

$$= \int_0^L \int_A \left[\sigma_{11}\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} + z\frac{d\delta\phi_x}{dx}\right) + \sigma_{13}\left(\delta\phi_x + \frac{d\delta w}{dx}\right)\right]dAdx - \int_0^L (f\delta u + q\delta w)\,dx$$

$$= \int_0^L \left[N\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) + M\frac{d\delta\phi_x}{dx} + Q\left(\delta\phi_x + \frac{d\delta w}{dx}\right) - (f\delta u + q\delta w)\right]dx$$

where

$$N = \int_A \sigma_{11}dA,\ M = \int_A z\sigma_{11}dA,\ Q = \int_A \sigma_{13}dA$$

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

Simplifying, we obtain

$$0=\int_0^L\left[N\frac{d\delta u}{dx}+\left(N\frac{dw}{dx}+Q\right)\frac{d\delta w}{dx}+M\frac{d\delta\phi_x}{dx}+Q\,\delta\phi_x-\left(f\delta u+q\delta w\right)\right]dx$$

$$=\int_0^L\left\{-\left(\frac{dN}{dx}+f\right)\delta u-\left[\frac{d}{dx}\left(N\frac{dw}{dx}+Q\right)+q\right]\delta w+\left(-\frac{dM}{dx}+Q\right)\delta\phi_x\right\}dx$$

$$+\left[N\delta u+\left(N\frac{dw}{dx}+Q\right)\delta w+M\delta\phi_x\right]_0^L$$

Using the fundamental lemma, we obtain

$$-\frac{dN}{dx}-f=0,\ -\frac{d}{dx}\left(N\frac{dw}{dx}+Q\right)-q=0,\ -\frac{dM}{dx}+Q=0$$

Specify

$$N\ \text{ or }\ u;\quad N\frac{dw}{dx}+Q\ \text{ or }\ w;\quad M\ \text{ or }\ \phi_x$$

THE UNIT DUMMY DISPLACEMENT METHOD

The principle of virtual displacements can also be used, in addition to deriving equations of equilibrium, to directly determine reaction forces and displacements in structural problems. If $\mathbf{F}_0$ is the force at point O in a structure, we can prescribe a virtual displacement $\delta\mathbf{u}_0 = \hat{\mathbf{e}}_F$ at the point and assume that the virtual displacements at all other points are zero. Then

$$\mathbf{F}_0 \cdot \delta\mathbf{u}_0 = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}^0 \, d\Omega \;\Rightarrow F_0 = \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}^0 \, d\Omega$$

This is known as the *unit* dummy displacement method.

Trusses

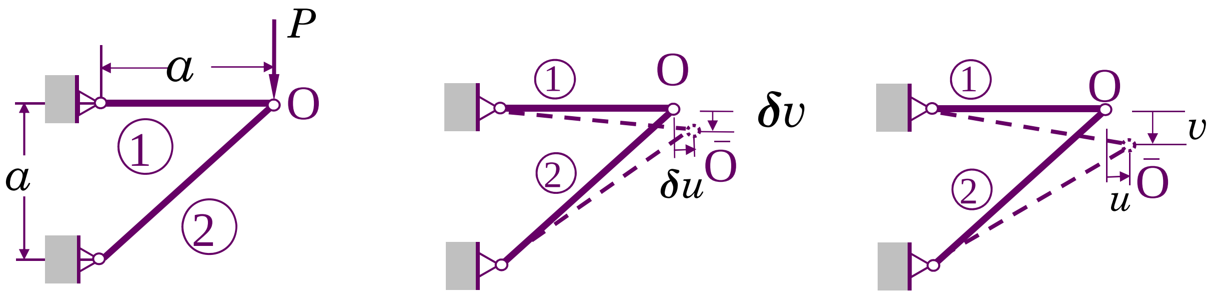
$$P_0 = \sum_{i=1}^{N} A_i L_i \sigma^{(i)} \delta\varepsilon^{(i)}, \quad \sigma^{(i)} = E_i \varepsilon^{(i)} \text{ (no sum on } i)$$

where $\delta\varepsilon^{(i)}$ is the strain in ith member due to unit displacement at point where P_0 is applied.

THE UNIT DUMMY DISPLACEMENT METHOD

Example (1a)

$$\sigma = E\varepsilon$$



(a) (b) (c)

Determine the horizontal and vertical deflections at point O using the *unit* dummy displacement method. **Assume linear elastic behavior.**

$$0\cdot\delta u + P\cdot\delta v = L_1 A^{(1)}\sigma^{(1)}\delta\varepsilon^{(1)} + L_2 A^{(2)}\sigma^{(2)}\delta\varepsilon^{(2)}$$

$$\varepsilon^{(1)} = \frac{1}{a}\left(\sqrt{(a+u)^2+v^2}-a\right)\approx\frac{u}{a},\quad \varepsilon^{(2)} = \frac{1}{\sqrt{2}a}\left(\sqrt{(a+u)^2+(a-v)^2}-\sqrt{2}a\right)\approx\frac{u-v}{2a}$$

$$\sigma^{(1)} = E\frac{u}{a},\quad \sigma^{(2)} = E\frac{u-v}{2a},\quad \delta\varepsilon^{(1)} = \frac{\delta u}{a},\quad \delta\varepsilon^{(2)} = \frac{\delta u-\delta v}{2a}$$

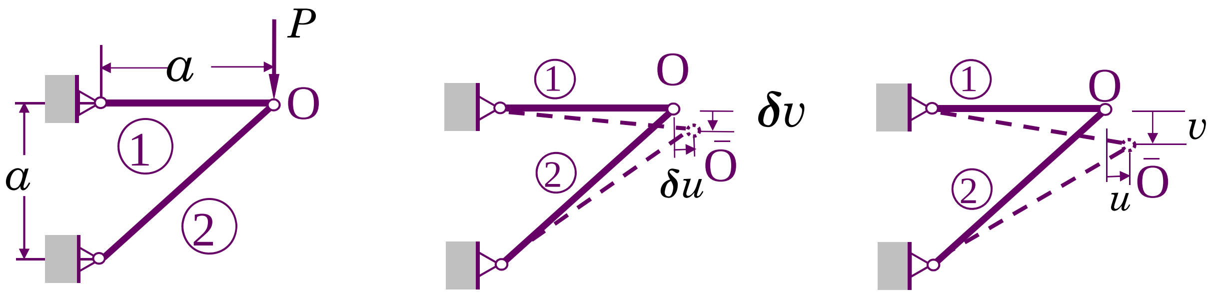
$$0\cdot\delta u + P\cdot\delta v = aA\sigma^{(1)}\frac{\delta u}{a} + \sqrt{2}aA\sigma^{(2)}\frac{\delta u-\delta v}{2a} = A\left(\sigma^{(1)}+\frac{1}{\sqrt{2}}\sigma^{(2)}\right)\delta u + A\left(-\frac{1}{\sqrt{2}}\sigma^{(2)}\right)\delta v$$

$$\boxed{u = \frac{Pa}{AE},\quad v = \frac{Pa}{AE}\left(1+2\sqrt{2}\right)}$$

THE UNIT DUMMY DISPLACEMENT METHOD

Example (1b)

$$\sigma = \begin{cases} K\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -K\sqrt{-\varepsilon}, & \varepsilon \leq 0 \end{cases}$$



(a) (b) (c)

Determine the horizontal and vertical deflections at point O using the *unit* dummy displacement method. **Assume nonlinear elastic behavior.**

$$0 \cdot \delta u + P \cdot \delta v = L_1 A^{(1)} \sigma^{(1)} \delta\varepsilon^{(1)} + L_2 A^{(2)} \sigma^{(2)} \delta\varepsilon^{(2)}$$

$$\sigma^{(1)} = K\sqrt{\frac{u}{a}}, \;\; \sigma^{(2)} = -K\sqrt{\frac{(v-u)}{2a}}, \;\; \varepsilon^{(1)} \approx \frac{u}{a}, \;\; \varepsilon^{(2)} \approx \frac{u-v}{2a}, \; \delta\varepsilon^{(1)} = \frac{\delta u}{a}, \;\; \delta\varepsilon^{(2)} = \frac{\delta u - \delta v}{2a}$$

$$0 \cdot \delta u + P \cdot \delta v = aA\sigma^{(1)} \frac{\delta u}{a} + \sqrt{2} aA\sigma^{(2)} \frac{\delta u - \delta v}{2a} = A\left(\sigma^{(1)} + \tfrac{1}{\sqrt{2}}\sigma^{(2)}\right)\delta u + A\left(-\tfrac{1}{\sqrt{2}}\sigma^{(2)}\right)\delta v$$

$$u = \frac{P^2 a}{A^2 K^2}, \;\; v = \frac{5P^2 a}{A^2 K^2}$$

THE UNIT DUMMY DISPLACEMENT METHOD

Beams

First we must write the axial displacement $u(x)$ and transverse deflection $w(x)$ in terms of suitable quantities, called the **generalized coordinates**:

$$u(x) \approx \sum_{i=1}^{m} u_i \psi_i(x), \quad w(x) \approx \sum_{i=1}^{n} \Delta_i \varphi_i(x)$$

These expansions are typically constructed using the exact solutions to the respective governing equations. Then we apply the unit dummy-displacement method to determine the required generalized displacements in terms of the applied loads.

$$\sum_{i=1}^{m} (F_i + f_i)\delta u_i + \sum_{i=1}^{n} (Q_i + q_i)\delta \Delta_i$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m}\left(\int_0^L EA \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} dx\right) u_j \delta u_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_0^L EI \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2} dx\right) \Delta_j \delta \Delta_i$$

where the horizontal and transverse distributed loads are converted into point loads using

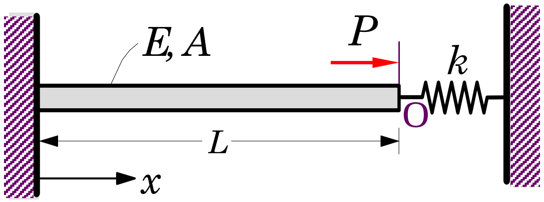
$$f_i = \int_0^L f(x)\psi_i(x)\, dx, \quad q_i = \int_0^L q(x)\varphi_i(x)\, dx$$

THE UNIT DUMMY DISPLACEMENT METHOD

Thus we have $[\mathbf{K}_u]\{\mathbf{u}\}=\{\mathbf{F}+\mathbf{f}\}, \quad [\mathbf{K}_w]\{\Delta\}=\{\mathbf{Q}+\mathbf{q}\}$

The method, for continuous systems, is close the finite element method.

Example 2



Problem: Determine the displacement u of point O of the spring supported bar. Assume linearly elastic behavior.

Solution: The displacement u is expanded as (solution to $d^2u/dx^2=0$)

$$u(x)=u(0)\left(1-\frac{x}{L}\right)+u(L)\frac{x}{L}\equiv u_1\psi_1(x)+u_2\psi_2(x);\quad \varepsilon=\frac{du}{dx}=\frac{u_2-u_1}{L}$$

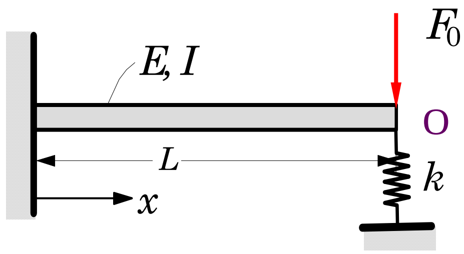
Then $[\mathbf{K}_u]\{\mathbf{u}\}=\{\mathbf{F}+\mathbf{f}\}$ becomes ($f=0$)

$$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}\Rightarrow\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ P-ku_2\end{Bmatrix}$$

Then the displacement at $x=L$ is $u_2=u(L)=\dfrac{PL}{kL+AE}$

THE UNIT DUMMY DISPLACEMENT METHOD

Example 3



Problem: Determine the transverse displacement w of point O of the spring supported beam. Assume linearly elastic behavior and use the Euler-Bernoulli beam theory.

Solution: The displacement w is expanded as (solution to $d^4w/dx^4 = 0$)

$$w(x) = \varphi_1(x)\Delta_1 + \varphi_2(x)\Delta_2 + \varphi_3(x)\Delta_3 + \varphi_4(x)\Delta_4 = \sum_{i=1}^{4} \varphi_i(x)\Delta_i,$$

where (Δ_1, Δ_3) denote the values of w at x=0 and x=L, respectively, (Δ_2, Δ_4) are the values of the rotation $(-dw/dx)$ at x=0 and x=L, respectively, and

$$\varphi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \quad \varphi_2(x) = -x\left[1 - 2\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right],$$

$$\varphi_3(x) = \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right), \qquad \varphi_4(x) = x\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

THE UNIT DUMMY DISPLACEMENT METHOD

Then $[\mathbf{K}_w]\{\Delta\}=\{\mathbf{Q}\}$ becomes ($q=0$)

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\end{Bmatrix}=\begin{Bmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\end{Bmatrix}$$

Using the boundary conditions

$$\Delta_1=\Delta_2=0,\text{ and }\ Q_3=-k\Delta_3+F_0,\ \ Q_4=0$$

$$\begin{bmatrix} \dfrac{12EI}{L^3}+k & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}\begin{Bmatrix}\Delta_3\\ \Delta_4\end{Bmatrix}=\begin{Bmatrix}F_0\\ 0\end{Bmatrix}\ \Rightarrow\ \Delta_3=\frac{2F_0L^3}{6EI+kL^3},\ \Delta_4=-\frac{3F_0L^2}{6EI+kL^3}$$

When k=0, we have

$$\Delta_3=\frac{F_0L^3}{3EI},\quad \Delta_4=-\frac{F_0L^2}{2EI}$$

CASTIGLIANO'S THEOREM I

The principle of the minimum total potential energy

$$\delta\Pi = \delta U + \delta V_E = 0 \Rightarrow \delta U = -\delta V_E$$

Castigliano's Theorem I

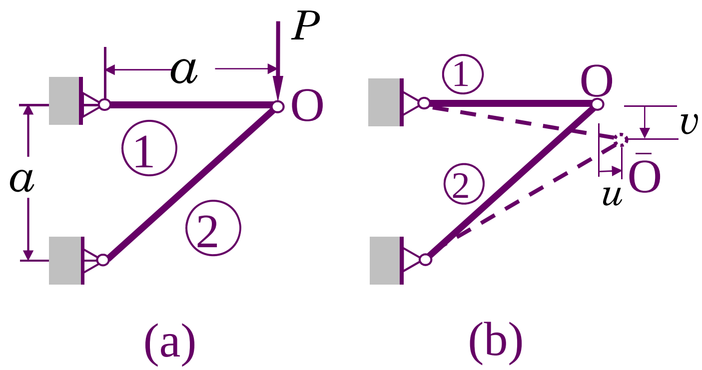
$$\delta U = \frac{\partial U}{\partial \mathbf{u}_i} \cdot \delta \mathbf{u}_i \quad \text{and} \quad \delta V_E = \frac{\partial V_E}{\partial \mathbf{u}_i} \cdot \delta \mathbf{u}_i = -\mathbf{F}_i \cdot \delta \mathbf{u}_i$$

$$\text{Hence, we have } \left(\frac{\partial U}{\partial \mathbf{u}_i} - \mathbf{F}_i \right) \cdot \delta \mathbf{u}_i = 0 \quad \Rightarrow \boxed{\frac{\partial U}{\partial \mathbf{u}_i} = \mathbf{F}_i}$$

It is understood that the strain energy is a function of displacement parameters in order to apply Castigliano's Theorem I. The unit dummy displacement method and Castigliano's Theorem I are equivalent. Hence, the examples presented for trusses, bars, and beams are also valid here.

CASTIGLIANO'S THEOREM I

Example 4(a)



Problem: Determine the displacements u and v of point O. Assume linearly elastic behavior.

Solution: The strains are

$$\varepsilon^{(1)} = \frac{u}{a}, \quad \varepsilon^{(2)} = \frac{u-v}{2a}$$

The strain energy is given by

$$U(u,v) = \frac{1}{2}\sum_{i=1}^{2} A_i L_i E_i \left(\varepsilon^{(i)}\right)^2 = \frac{EA}{2}\left[a\left(\frac{u}{a}\right)^2 + \sqrt{2}a\left(\frac{u-v}{2a}\right)^2\right]$$

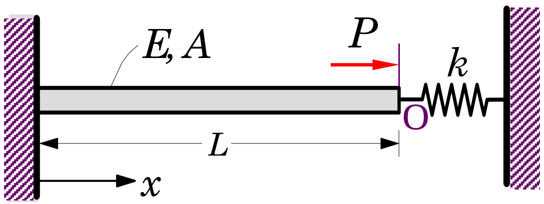
$$0 = \frac{\partial U}{\partial u} = EA\left(\frac{u}{a} + \frac{\sqrt{2}}{2}\frac{u-v}{2a}\right), \quad P = \frac{\partial U}{\partial v} = EA\left(0 - \frac{\sqrt{2}}{2}\frac{u-v}{2a}\right)$$

$$u = \frac{Pa}{EA}, \qquad v = (1+2\sqrt{2})\frac{Pa}{EA}.$$

Prob (4b): Solve the problem for nonlinear elastic case [see Example 1(b)].

CASTIGLIANO'S THEOREM I

Example 5



Problem: Determine the displacement u of point O of the spring supported bar. Assume linearly elastic behavior.

Solution: The strain in the bar can be expressed as (an approximation)

$$u_1 = u_0,\; u_2 = u_L,\; \varepsilon = \frac{u_L - u_0}{L} = \frac{u_L}{L}$$

The strain energy is given by

$$U(u_L) = \frac{EAL}{2}\left(\frac{u_L}{L}\right)^2 + \frac{k}{2}(u_L)^2$$

Applying the Castigliano's Theorem I, we obtain

$$\frac{\partial U}{\partial u_L} = P \;\Rightarrow\; \frac{AE}{L}u_L + ku_L = P \;\text{ or }\; \boxed{u_L = \frac{PL}{kL + AE}}$$

PRINCIPLE OF VIRTUAL FORCES

The principle of complementary virtual work (or virtual forces) states that *the strains and displacements in a deformable body are compatible and consistent with the constraints if and only if the total complementary virtual work is zero:*

$$\delta W_I^* + \delta W_E^* = 0$$

where

$$\delta W_I^* = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \delta\boldsymbol{\sigma}\, d\Omega = \int_\Omega \varepsilon_{ij}\delta\sigma_{ij}\, d\Omega,$$

$$\delta W_E^* = \delta V_E^* = -\int_\Omega \mathbf{u}\cdot\delta\mathbf{f}\, d\Omega - \int_{\Gamma_u} \hat{\mathbf{u}}\cdot\delta\mathbf{t}\, ds$$

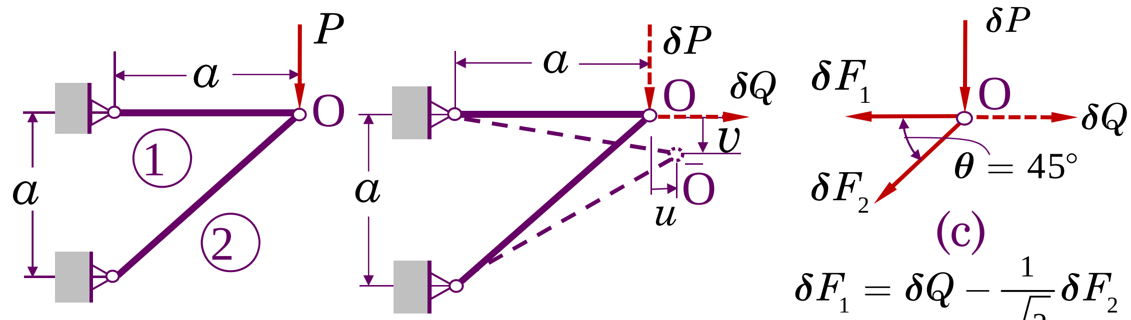
Unit Dummy Load Method Assuming virtual force δF_0 at a point and zero virtual forces elsewhere, we can write

$$u_0\delta F_0 = \int_\Omega \varepsilon_{ij}\delta\sigma_{ij}^0\, d\Omega$$

THE UNIT DUMMY FORCE METHOD

Example (6)

$$\sigma = E\varepsilon$$



(a) (b) (c)

$$\delta F_1 = \delta Q - \frac{1}{\sqrt{2}}\delta F_2$$

$$\delta F_2 = -\sqrt{2}\,\delta P$$

Problem statement

Determine the horizontal and vertical deflections at point O using the *unit* force method. **Assume <u>linear</u> elastic behavior.**

Solution Let us apply vertical and horizontal virtual forces shown in Fig. (b). Then the virtual forces in the members can be calculated using equilibrium as shown in Fig. (c). The actual forces in the members are (to calculate the actual strains)

$$F^{(1)} = P,\;\; F^{(2)} = -\sqrt{2}P$$

Then the unit dummy force method can be expressed as

$$u\cdot\delta Q + v\cdot\delta P = A^{(1)}L_1\varepsilon^{(1)}\delta\sigma^{(1)} + A^{(2)}L_2\varepsilon^{(2)}\delta\sigma^{(2)} = Aa\frac{P}{EA}\frac{\delta Q+\delta P}{A} + \sqrt{2}Aa\frac{\sqrt{2}P}{EA}\frac{\sqrt{2}\delta P}{A}$$

THE UNIT DUMMY FORCE METHOD

By collecting the coefficients of δu and δv, we obtain

$$u = \frac{Pa}{EA}, \quad v = \frac{Pa}{EA} + 2\sqrt{2}\frac{Pa}{EA} = \frac{Pa}{AE}\left(1 + 2\sqrt{2}\right)$$

The Principle of Minimum Complementary Energy

$$\delta\Pi^* \equiv \delta\left(U^* + V_E^*\right) = 0 \quad \text{with} \quad \Pi^* \equiv U^* + V_E^*$$

where the complementary strain energy is expressed in terms of stresses/forces. For a 3D elastic body we have

$$\Pi^*(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega C^*_{ijkl}\sigma_{ij}\sigma_{kl}\, d\Omega - \left[\int_\Omega u_i f_i\, d\Omega + \int_{\Gamma_u} \hat{u}_i t_i\, d\Gamma\right]$$

The Euler equations of this functional are

$$C^*_{ijkl}\sigma_{kl} - \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) = 0 \ \text{ in } \ \Omega \qquad u_i - \hat{u}_i = 0 \ \text{ on } \ \Gamma_u$$

CASTIGLIANO'S THEOREM II

We can write

$$\delta U^* = \frac{\partial U^*}{\partial F_i}\delta F_i, \quad \delta V^* = \frac{\partial V^*}{\partial F_i}\delta F_i = -u_i \delta F_i$$

Thus, we have

$$\left(\frac{\partial U^*}{\partial F_i} - u_i\right)\delta F_i = 0 \quad \text{or} \quad \boxed{\frac{\partial U^*}{\partial F_i} = u_i}$$

This is known as the **Castigliano's Theorem II**.

Example (7)

Problem statement

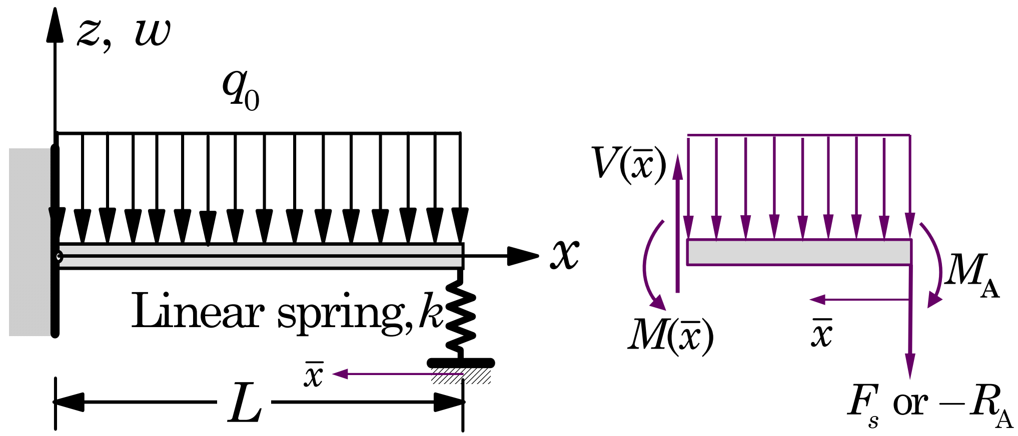
Consider the spring-supported beam shown in the figure. Determine
(a) the compression in the linear elastic spring and
(b) the reaction force and the rotation at $x = L$ when the spring is replaced by a rigid support. Include the energy due to transverse shear force.

CASTIGLIANO'S THEOREM II

Solution

The complementary strain energy is

$$U^* = \frac{1}{2}\int_0^L \left(\frac{M^2}{EI} + \frac{V^2}{GAK_s} \right) d\bar{x}$$



where

$$M(\bar{x}) = \frac{q_0 \bar{x}^2}{2} + F_s \bar{x}, \quad V(\bar{x}) = F_s + q_0 \bar{x}, \quad F_s = kw(0)$$

Hence,

$$U^* = \frac{1}{2EI}\int_0^L \left(\frac{q_0 \bar{x}^2}{2} + F_s \bar{x} \right)^2 d\bar{x} + \frac{1}{2GAK_s}\int_0^L (F_s + q_0 \bar{x})^2 d\bar{x}$$

Then

$$-w(0) = \frac{\partial U^*}{\partial F_s} = \frac{1}{EI}\int_0^L \left(\frac{q_0 \bar{x}^2}{2} + F_s \bar{x} \right)(\bar{x}) d\bar{x} + \frac{1}{GAK_s}\int_0^L (F_s + q_0 \bar{x}) d\bar{x}$$

$$= \frac{q_0 L^4}{8EI} + \frac{F_s L^3}{3EI} + \frac{F_s L}{GAK_s} + \frac{q_0 L^2}{2GAK_s}$$

CASTIGLIANO'S THEOREM II

We have

$$w(0) = -\left(\frac{q_0L^4}{8EI} + \frac{q_0L^2}{2GAK_s}\right)\left(1 + \frac{kL^3}{3EI} + \frac{kL}{GAK_s}\right)^{-1}$$

Note that, when $k = 0$ (cantilevered beam with uniformly distributed load), we have

$$w(0) = -\left(\frac{q_0L^4}{8EI} + \frac{q_0L^2}{2GAK_s}\right) = -\frac{q_0L^4}{8EI}\left(1 + 4\frac{EI}{2GAK_sL^2}\right)$$

When $k \to \infty$ (A beam fixed at the left end roller-supported at the other end and with uniformly distributed load), the reaction at the support is

$$0 = \frac{\partial U^*}{\partial R_A} = \frac{1}{EI}\int_0^L \left(\frac{q_0\bar{x}^2}{2} - R_A\bar{x}\right)(\bar{x})d\bar{x} + \frac{1}{GAK_s}\int_0^L (-R_A + q_0\bar{x})\,d\bar{x}$$

$$R_A = \frac{3q_0L}{8}\left(1 + \frac{4EI}{2GAK_sL^2}\right)\left(1 + \frac{3EI}{GAK_sL^2}\right)^{-1}$$