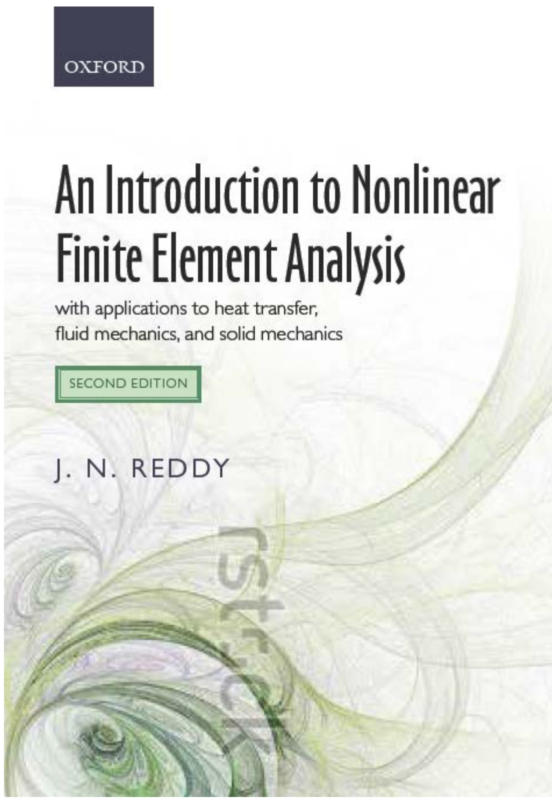
# NONLINEAR FINITE ELEMENT ANALYSIS OF PLATE BENDING

**Read: Chapter 7**

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- Governing Equations of the First-Order Shear Deformation theory (FSDT)
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- Shear and membrane locking
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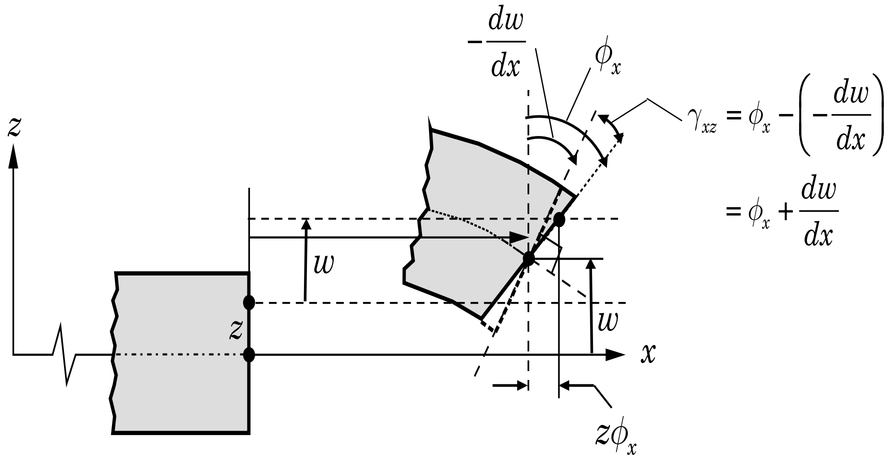
# THE FIRST-ORDER SHEAR DEFORMATION THEORY

## Displacement Field of the FSDT

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$




## Nonlinear strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

## Von Karman Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial u_m}{\partial x_1}\frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

# NONLINEAR STARINS OF THE FSDT

## Actual Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\,\frac{\partial \phi_x}{\partial x} = \varepsilon_{xx}^0 + z\,\varepsilon_{xx}^1$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\,\frac{\partial \phi_y}{\partial y} = \varepsilon_{yy}^0 + z\,\varepsilon_{yy}^1$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) = \gamma_{xy}^0 + z\,\gamma_{xy}^1$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x} = \gamma_{xz}^0, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y} = \gamma_{yz}^0$$

## Virtual Nonlinear strains

$$\delta\varepsilon_{xx} = \frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + z\,\frac{\partial \delta\phi_x}{\partial x} = \delta\varepsilon_{xx}^0 + z\,\delta\varepsilon_{xx}^1$$

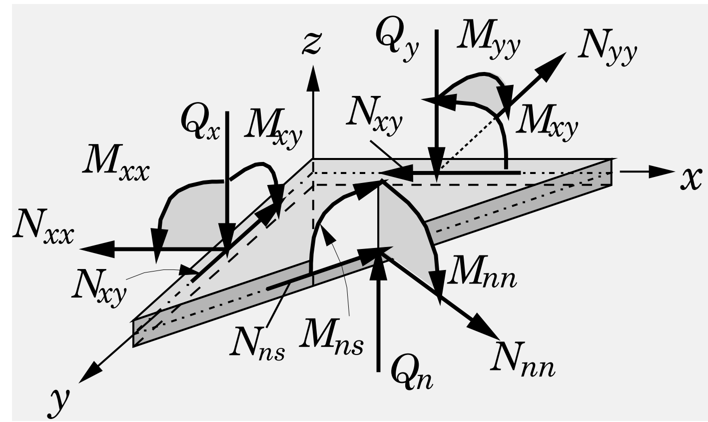
$$\delta\varepsilon_{yy} = \frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + z\,\frac{\partial \delta\phi_y}{\partial y} = \delta\varepsilon_{yy}^0 + z\,\delta\varepsilon_{yy}^1$$

$$\delta\gamma_{xy} = \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + z\left(\frac{\partial \delta\phi_x}{\partial y} + \frac{\partial \delta\phi_y}{\partial x}\right) = \delta\gamma_{xy}^0 + z\,\delta\gamma_{xy}^1$$

$$\delta\gamma_{xz} = \delta\phi_x + \frac{\partial \delta w}{\partial x} = \delta\gamma_{xz}^0, \quad \delta\gamma_{yz} = \delta\phi_y + \frac{\partial \delta w}{\partial y} = \delta\gamma_{yz}^0$$

# PRINCIPLE OF VIRTUAL DISPLACEMENTS

$$0 = \delta W^e \equiv \delta W_I^e + \delta W_E^e$$




$$\delta W_I^e = \int_{\Omega^e} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \sigma_{xx} \left( \delta\varepsilon_{xx}^0 + z\delta\varepsilon_{xx}^1 \right) + \sigma_{yy} \left( \delta\varepsilon_{yy}^0 + z\delta\varepsilon_{yy}^1 \right) \right.\right.$$

$$+ \sigma_{xy} \left( \delta\gamma_{xy}^0 + z\delta\gamma_{xy}^1 \right) + K_s \sigma_{xz} \delta\gamma_{xz}^0$$

$$\left.\left. + K_s \sigma_{yz} \delta\gamma_{yz}^0 \right] dz \right\} dx\, dy$$

$$\delta W_E^e = - \left\{ \oint_{\Gamma^e} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \sigma_{nn} \left( \delta u_n + z\delta\phi_n \right) + \sigma_{ns} \left( \delta u_s + z\delta\phi_s \right) + \sigma_{nz} \delta w \right] dz\, ds \right.$$

$$\left. + \int_{\Omega^e} (q - kw) \delta w \; dx\, dy \right\}$$

$$0 = \int_{\Omega^e} \left[ N_{xx} \delta\varepsilon_{xx}^0 + M_{xx} \delta\varepsilon_{xx}^1 + N_{yy} \delta\varepsilon_{yy}^0 + M_{yy} \delta\varepsilon_{yy}^1 + N_{xy} \delta\gamma_{xy}^0 + M_{xy} \delta\gamma_{xy}^1 \right.$$

$$\left. + Q_x \delta\gamma_{xz}^0 + Q_y \delta\gamma_{yz}^0 - q\delta w \right] dx\, dy$$

$$- \oint_{\Gamma^e} \left( N_{nn} \delta u_n + N_{ns} \delta u_s + M_{nn} \delta\phi_n + M_{ns} \delta\phi_s + Q_n \delta w \right) ds$$

# THE FIRST-ORDER SHEAR DEFORMATION THEORY

## Equations of equilibrium

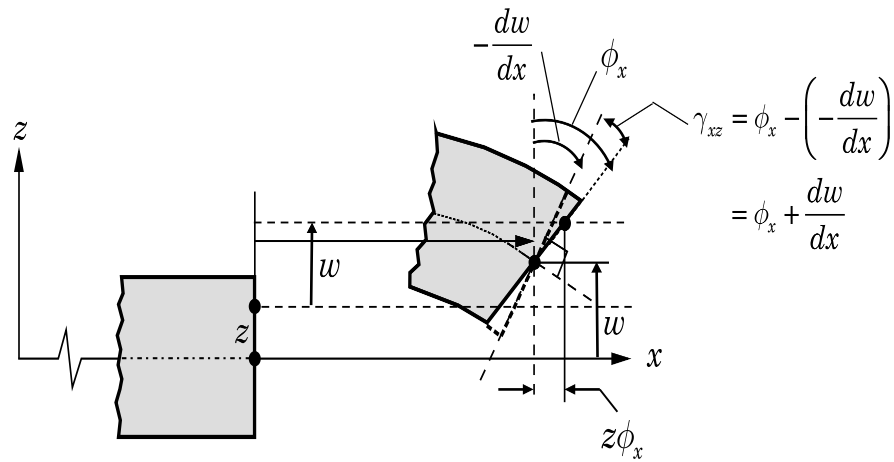
$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) - q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2}$$




## Stress resultants

$$Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz} dz = \boxed{K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)}$$

$$Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz} dz = \boxed{K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)}$$

$$M_{xx} = D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}$$

$$M_{yy} = D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y}$$

$$M_{xy} = D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

**Weak forms** $(v_1 = \delta u,\ v_2 = \delta v,\ v_3 = \delta w,\ v_4 = \delta\phi_x,\ v_5 = \delta\phi_y)$

$$0 = \int_{\Omega^e}\left(\frac{\partial \delta u}{\partial x}N_{xx} + \frac{\partial \delta u}{\partial y}N_{xy} + I_0\delta u\frac{\partial^2 u}{\partial t^2}\right)dxdy - \oint_{\Gamma^e}\delta u N_n\, ds$$

$$0 = \int_{\Omega^e}\left(\frac{\partial \delta v}{\partial x}N_{xy} + \frac{\partial \delta v}{\partial y}N_{yy} + I_0\delta v\frac{\partial^2 v}{\partial t^2}\right)dxdy - \oint_{\Gamma^e}\delta v N_{ns}\, ds$$

$$\begin{aligned}0 = \int_{\Omega^e}\Bigg[&\frac{\partial \delta w}{\partial x}\left(Q_x + N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\\ &+ \frac{\partial \delta w}{\partial y}\left(Q_y + N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right)\\ &+ I_0\delta w\frac{\partial^2 w}{\partial t^2} - \delta w q\Bigg]dxdy - \oint_{\Gamma^e}\delta w Q_n\, ds\end{aligned}$$

$$0 = \int_{\Omega^e}\left(\frac{\partial \delta\phi_x}{\partial x}M_{xy} + \delta\phi_x Q_x + I_2\delta\phi_x\frac{\partial^2 \phi_x}{\partial t^2}\right)dxdy - \oint_{\Gamma^e}\delta\phi_x M_n\, ds$$

$$0 = \int_{\Omega^e}\left(\frac{\partial \delta\phi_y}{\partial x}M_{yy} + \delta\phi_y Q_y + I_2\delta\phi_y\frac{\partial^2 \phi_y}{\partial t^2}\right)dxdy - \oint_{\Gamma^e}\delta\phi_y M_{ns}\, ds$$

# Finite Element Models of The First-order Plate Theory (FSDT) (Continued)

## Finite Element Approximation

$$u(x,y,t)=\sum_{j=1}^{m}u_j(t)\psi_j(x,y),\quad v(x,y,t)=\sum_{j=1}^{m}v_j(t)\psi_j(x,y)$$

$$w(x,y,t)=\sum_{j=1}^{n}w_j(t)\psi_j(x,y)$$

$$\phi_x(x,y,t)=\sum_{j=1}^{p}S_j^1(t)\psi_j(x,y),\quad \phi_y(x,y,t)=\sum_{j=1}^{p}S_j^2(t)\psi_j(x,y)$$

## Finite Element Model

$$\begin{bmatrix}\mathbf{M}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{44} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{55}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{u}}\\ \ddot{\mathbf{v}}\\ \ddot{\mathbf{w}}\\ \ddot{\mathbf{S}}_x\\ \ddot{\mathbf{S}}_y\end{Bmatrix}+\begin{bmatrix}\mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15}\\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25}\\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35}\\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45}\\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55}\end{bmatrix}\begin{Bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\\ \mathbf{S}_x\\ \mathbf{S}_y\end{Bmatrix}=\begin{Bmatrix}\mathbf{F}^1\\ \mathbf{F}^2\\ \mathbf{F}^3\\ \mathbf{F}^4\\ \mathbf{F}^5\end{Bmatrix}$$

# Fully Dicretized Model and Iterative Scheme

## Fully Discretized  Finite Element Model

$$[\hat{K}]_{s+1}\{\Delta\}_{s+1} = \{\hat{F}\}_{s,s+1}$$

$$\begin{aligned}
[\hat{K}]_{s+1} &= [K]_{s+1} + a_3[M]_{s+1} \\
\{\hat{F}\}_{s,s+1} &= \{F\}_{s+1} + [M]_{s+1}(a_3\{\Delta\}_s + a_4\{\dot{\Delta}\}_s + a_5\{\ddot{\Delta}\}_s) \\
a_3 &= \frac{2}{\gamma(\Delta t)^2}, \quad a_4 = \frac{2}{\gamma\Delta t}, \quad a_5 = \frac{1}{\gamma} - 1
\end{aligned}$$

## Accelerations and Velocities

$$\begin{aligned}
\{\ddot{\Delta}\}_{s+1} &= a_3(\{\Delta\}_{s+1} - \{\Delta\}_s) - a_4\{\dot{\Delta}\} - a_5\{\ddot{\Delta}\}_s \\
\{\dot{\Delta}\}_{s+1} &= \{\dot{\Delta}\}_s + a_2\{\ddot{\Delta}\}_s + a_1\{\ddot{\Delta}\}_{s+1}
\end{aligned}$$

where $a_1 = \alpha\Delta t$ and $a_2 = (1-\alpha)\Delta t$.

## Newton-Raphson Iterative Scheme

$$\{\Delta\}_{r+1}^{s+1} = -[\hat{K}^T]_r^{-1}\{\hat{R}\}_r^{s+1}, \quad [\hat{K}^T]_r \equiv \left[\frac{\partial\{\hat{R}\}}{\partial\{\Delta\}}\right]_r^{s+1}$$

# Stiffness Coefficients (typical)

$$K_{ij}^{11} = \int_{\Omega^e} \left( A_{11} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \left( A_{12} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) dxdy = K_{ji}^{21}$$

$$K_{ij}^{13} = \frac{1}{2} \int_{\Omega^e} \left[ \frac{\partial \psi_i^e}{\partial x} \left( A_{11} \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{12} \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) \right.$$
$$\left. + A_{66} \frac{\partial \psi_i^e}{\partial y} \left( \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dxdy$$

$$K_{ij}^{22} = \int_{\Omega^e} \left( A_{66} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{22} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$

$$K_{ij}^{23} = \frac{1}{2} \int_{\Omega^e} \left[ \frac{\partial \psi_i^e}{\partial y} \left( A_{12} \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{22} \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) \right.$$
$$\left. + A_{66} \frac{\partial \psi_i^e}{\partial x} \left( \frac{\partial w_0}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dxdy$$

$$K_{ij}^{31} = \int_{\Omega^e} \left[ \frac{\partial \varphi_i^e}{\partial x} \left( A_{11} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) \right.$$
$$\left. + \frac{\partial \varphi_i^e}{\partial y} \left( A_{66} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{12} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) \right] dxdy$$

# Tangent Stiffness Coefficients (typical)

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta},\quad R_i^\alpha = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma}\Delta_k^\gamma - F_i^\alpha,\ \Delta_i^1 = u_i,\ \Delta_i^2 = v_i,\ \ \Delta_i^3 = w_i,\ \ \Delta_i^4 = S_i^1,\ \Delta_i^5 = S_i^2$$

$$T_{ij}^{\alpha\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma}\Delta_k^\gamma - F_i^\alpha\right) = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_k^\gamma + K_{ij}^{\alpha\beta}$$

$$\mathbf{T}^{\alpha 1} = \mathbf{K}^{\alpha 1} = (\mathbf{K}^{1\alpha})^{\mathrm{T}},\quad \mathbf{T}^{\alpha 2} = \mathbf{K}^{\alpha 2} = (\mathbf{K}^{2\alpha})^{\mathrm{T}}$$

$$\mathbf{T}^{\alpha 4} = \mathbf{K}^{\alpha 4} = (\mathbf{K}^{4\alpha})^{\mathrm{T}},\quad \mathbf{T}^{\alpha 5} = \mathbf{K}^{\alpha 5} = (\mathbf{K}^{5\alpha})^{\mathrm{T}}$$

$$T_{ij}^{13} = \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{1\gamma}}{\partial w_j}\Delta_k^\gamma + K_{ij}^{13} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{13}}{\partial w_j} w_k + K_{ij}^{13}$$

$$= \frac{1}{2}\int_{\Omega^e}\left[\frac{\partial \psi_i^{(1)}}{\partial x}\left(A_{11}\frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{12}\frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\right.$$

$$\left. + A_{66}\frac{\partial \psi_i^{(1)}}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right] dx\,dy + K_{ij}^{13}$$

$$= K_{ij}^{13} + K_{ij}^{13} = 2K_{ij}^{13} = T_{ji}^{31}$$

# Tangent Stiffness Coefficients (typical)

$$T_{ij}^{33} = K_{ij}^{33} + \sum_{\gamma=1}^{5}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{3\gamma}}{\partial w_j}\Delta_k^{\gamma} = K_{ij}^{33} + \sum_{k=1}^{n(\gamma)}\left(\frac{\partial K_{ik}^{31}}{\partial w_j}u_k + \frac{\partial K_{ik}^{32}}{\partial w_j}v_k + \frac{\partial K_{ik}^{33}}{\partial w_j}w_k\right)$$

$$= K_{ij}^{33} + \int_{\Omega^e}\left[\left(A_{11}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{12}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\frac{\partial u}{\partial x} + A_{66}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\frac{\partial u}{\partial y}\right]dx\,dy$$

$$+\int_{\Omega^e}\left[\left(A_{12}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{22}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y}\right)\frac{\partial v}{\partial y} + A_{66}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\frac{\partial v}{\partial x}\right]dx\,dy$$

$$+\int_{\Omega^e}\left\{\left[A_{11}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{22}\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + A_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right]\right.$$

$$\left.+\frac{1}{2}(A_{12}+A_{66})\left[\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)\right]\right\}dx\,dy$$

# Tangent Stiffness Coefficients (typical)

$$T_{ij}^{33} = K_{ij}^{33} + \int_{\Omega^e} \left\{ A_{11}\left[\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\frac{\partial v}{\partial y} + \frac{1}{2}(A_{12}+A_{66})\left(\frac{\partial w}{\partial y}\right)^2 \right\} \frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} dx\,dy$$

$$+\int_{\Omega^e} \left\{ A_{22}\left[\frac{\partial v}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2\right] + A_{12}\frac{\partial u}{\partial x} + \frac{1}{2}(A_{12}+A_{66})\left(\frac{\partial w}{\partial x}\right)^2 \right\} \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} dx\,dy$$

$$+\int_{\Omega^e} \left[ A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + \frac{1}{2}(A_{12}+A_{66})\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right) dx\,dy$$

$$T_{ij}^{33} = \int_{\Omega^e} \left( A_{55}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + A_{44}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + k\psi_i^{(2)}\psi_j^{(2)} \right) dx\,dy$$

$$+\int_{\Omega^e} \left\{ N_{xx}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + N_{yy}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + N_{xy}\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + N_{xy}\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \right.$$

$$+(A_{12}+A_{66})\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + \frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x}\right)$$

$$\left. +\left[A_{11}\left(\frac{\partial w}{\partial x}\right)^2 + A_{66}\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial \psi_i^{(2)}}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + \left[A_{66}\left(\frac{\partial w}{\partial x}\right)^2 + A_{22}\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial \psi_i^{(2)}}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} \right\} dx\,dy$$

# Shear and Membrane Locking (Revisit)

## Shear Locking

Use reduced integration to evaluate all *shear* stiffnesses (i.e., all $K_{ij}$ that contain transverse shear terms)

## Membrane Locking

Use reduced integration to evaluate all *membrane* stiffnesses (i.e., all $K_{ij}$ that contain von Kármán nonlinear terms)

# SOME ELEMENTS OF THE PLATE PROGRAM

## Read material properties and generate plate material stiffnesses

```
READ(IN,*)E1,E2,G12,G13,G23,PR12
READ(IN,*)CF,SCF,THI
PR21=(E2/E1)*PR12
WRITE(IT,425) E1,E2,G12,G13,G23,PR12,PR21,
                   CF,SCF,THI
C
Q11 = E1/(1-PR12*PR21)
Q12 = (PR12*E2)/(1-PR12*PR21)
Q22 = E2/(1-PR12*PR21)
Q44 = G23
Q55 = G13
Q66 = G12
```

```
A11 = THI*Q11
A12 = THI*Q12
A22 = THI*Q22
A44 = THI*Q44*SCF
A55 = THI*Q55*SCF
A66 = THI*Q66
```

Error calculation, convergence check, and other nonlinear analysis aspects remain the same as in 2D

$A_{ij}$ and $D_{ij}$ must be transferred to the subroutine ELMATRCS2D through common block.

```
D11 = THI*THI*THI*Q11/12.0
D12 = THI*THI*THI*Q12/12.0
D22 = THI*THI*THI*Q22/12.0
D66 = THI*THI*THI*Q66/12.0
```

# SOME ELEMENTS OF THE PLATE PROGRAM

**Initialize** (before the do-loops on numerical integration): $K_{ij}^{\alpha\beta}$, $F_i^{\alpha}$, $T_{ij}^{\alpha\beta}$

**Define** (inside the do-loops on numerical integration). For example, the linear stiffnesses coefficients are [F0=F0+DQ(NL); DQ(NL) array of load increments ]

```
DO I=1,NPE
ELF3(I)=ELF3(I)+F0*SFL(I)*CNST
DO J=1,NPE
   S00=SFL(I)*SFL(J)*CNST
   S11=GDSFL(1,I)*GDSFL(1,J)*CNST
   S22=GDSFL(2,I)*GDSFL(2,J)*CNST
   S12=GDSFL(1,I)*GDSFL(2,J)*CNST
   S21=GDSFL(2,I)*GDSFL(1,J)*CNST
ELK11(I,J)=ELK11(I,J)+A11*S11+A66*S22
ELK12(I,J)=ELK12(I,J)+A12*S12+A66*S21
ELK21(I,J)=ELK21(I,J)+A12*S21+A66*S12
ELK22(I,J)=ELK22(I,J)+A66*S11+A22*S22
ELK33(I,J)=ELK33(I,J)+CF*S00
ELK44(I,J)=ELK44(I,J)+D11*S11+D66*S22
ELK45(I,J)=ELK45(I,J)+D12*S12+D66*S21
ELK54(I,J)=ELK54(I,J)+D12*S21+D66*S12
ELK55(I,J)=ELK55(I,J)+D66*S11+D22*S22
ENDDO
ENDDO
```

Similarly, define the shear and nonlinear coefficients in the reduced integration loop; compute the residual vector and tangent stiffness coefficients.

# REARRANGE THE ELEMENT COEFFICIENTS

```
      II=1
      DO 220 I=1,NPE
        ELF(II)   =ELF1(I)
        ELF(II+1)=ELF2(I)
        ELF(II+2)=ELF3(I)
        ELF(II+3)=ELF4(I)
        ELF(II+4)=ELF5(I)
        JJ=1
        DO 210 J=1,NPE
          ELK(II,JJ)     = ELK11(I,J)
          ELK(II,JJ+1)   = ELK12(I,J)
          ELK(II,JJ+2)   = ELK13(I,J)
          ELK(II,JJ+3)   = ELK14(I,J)
          ELK(II,JJ+4)   = ELK15(I,J)
          ELK(II+1,JJ)   = ELK21(I,J)
          ELK(II+2,JJ)   = ELK31(I,J)
          ELK(II+3,JJ)   = ELK41(I,J)
          ELK(II+4,JJ)   = ELK51(I,J)
          ELK(II+1,JJ+1) = ELK22(I,J)
          ELK(II+1,JJ+2) = ELK23(I,J)
          ELK(II+1,JJ+3) = ELK24(I,J)
          ELK(II+1,JJ+4) = ELK25(I,J)
          ELK(II+2,JJ+1) = ELK32(I,J)
          ELK(II+3,JJ+1) = ELK42(I,J)
          ELK(II+4,JJ+1) = ELK52(I,J)
          ELK(II+2,JJ+2) = ELK33(I,J)
          ELK(II+2,JJ+3) = ELK34(I,J)
          ELK(II+2,JJ+4) = ELK35(I,J)
          ELK(II+3,JJ+2) = ELK43(I,J)
          ELK(II+4,JJ+2) = ELK53(I,J)
          ELK(II+3,JJ+3) = ELK44(I,J)
          ELK(II+3,JJ+4) = ELK45(I,J)
          ELK(II+4,JJ+3) = ELK54(I,J)
          ELK(II+4,JJ+4) = ELK55(I,J)
210     JJ=NDF*J+1
220   II=NDF*I+1
```

The same applies to the tangent stiffness coefficients

# Post-Computation of Stress Components

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix},\quad \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\end{Bmatrix}=\begin{bmatrix}Q_{55} & 0\\ 0 & Q_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}$$

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}},\, Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\, Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}},\, Q_{66}=G_{12},\, Q_{44}=G_{23},\, Q_{55}=G_{13},$$

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\end{Bmatrix}-z\begin{Bmatrix}\dfrac{\partial \phi_x}{\partial x}\\ \dfrac{\partial \phi_y}{\partial y}\\ \dfrac{\partial \phi_x}{\partial y}+\dfrac{\partial \phi_y}{\partial x}\end{Bmatrix},\quad \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\begin{Bmatrix}\phi_x+\dfrac{\partial w}{\partial x}\\ \phi_y+\dfrac{\partial w}{\partial y}\end{Bmatrix}$$

# Post-Computation of Stress Components

```
      SUBROUTINE STRESS (NPE,NDF,NGPR,ELXY,ELU)
         IMPLICIT REAL*8 (A-H,O-Z)
         COMMON/STR/Q11,Q22,Q12,Q44,Q55,Q66,THI
         COMMON/SHP/SF(9),GDSF(2,9)
         DIMENSION GAUSSPT(4,4),ELXY(9,2),ELU(45)
         DATA GAUSSPT/4*0.0D0,-.57735027D0,.57735027D0,2*0.0D0,
        1  -.77459667D0,0.0D0,.77459667D0,0.0D0,-.86113631D0,
        2  -.33998104D0,.33998104D0,0.86113631D0/
C
         DO 40 NI=1,NGPR
         DO 40 NJ=1,NGPR
         XI=GAUSSPT(NI,NGPR)
         ETA=GAUSSPT(NJ,NGPR)
         CALL INTERPLN2D (NPE,XI,ETA,DET,ELXY)
         X=0.0
         Y=0.0
```

# Subroutine STRESS (continued)

Initialize here

```
DO 20 I=1,NPE
    L=(I-1)*NDF+1
    X=X+SF(I)*ELXY(I,1)
    Y=Y+SF(I)*ELXY(I,2)
    DUX=DUX+GDSF(1,I)*ELU(L)
    DUY=DUY+GDSF(2,I)*ELU(L)
    DVX=DVX+GDSF(1,I)*ELU(L+1)
    DVY=DVY+GDSF(2,I)*ELU(L+1)
    DWX=DWX+GDSF(1,I)*ELU(L+2)
    DWY=DWY+GDSF(2,I)*ELU(L+2)
    PHIX=PHIX+SF(I)*ELU(L+3)
    PHIY=PHIY+SF(I)*ELU(L+4)
    DPXX=DPXX+GDSF(1,I)*ELU(L+3)
    DPXY=DPXY+GDSF(2,I)*ELU(L+3)
    DPYX=DPYX+GDSF(1,I)*ELU(L+4)
 20 DPYY=DPYY+GDSF(2,I)*ELU(L+4)
```

```
PHIX=0.0
PHIY=0.0
DUX=0.0
DUY=0.0
DVX=0.0
DVY=0.0
DWX=0.0
DWY=0.0
DPXX=0.0
DPXY=0.0
DPYX=0.0
DPYY=0.0
```

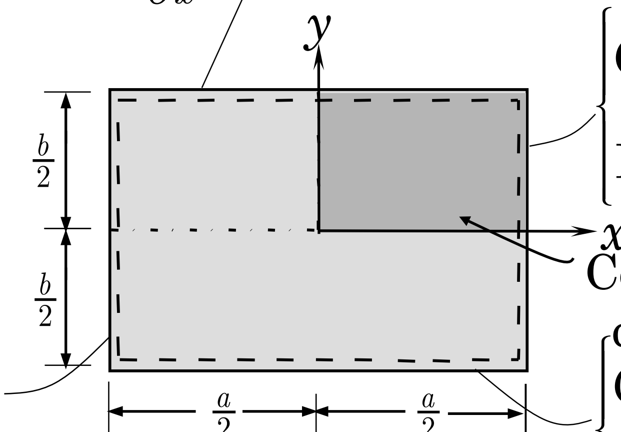
## Subroutine STRESS (continued)

```
      EX0 = DUX +0.5*DWX*DWX
      EY0 = DVY +0.5*DWY*DWY
      EXY0= DUY+DVX +DWX*DWY
      EX1 = DPXX
      EY1 = DPYY
      EXY1=DPXY+DPYX
      EXZ = PHIX+DWX
      EYZ = PHIY+DWY
C  Write the statements for stress components SXXT, SXXB,
C   etc. and write them out for each Gauss point
C   *** your task ***
     WRITE(IT,50) X,Y,SXXT,SYYT,SXYT,SXZ,SYZ
     WRITE(IT,50)       SXXB,SYYB,SXYB
40 CONTINUE
     RETURN
50 FORMAT (5X,8E12.4)
     END
```

# TYPICAL SIMPLY SUPPORT CONDITIONS for Pure Bending case

CPT: $w = \dfrac{\partial w}{\partial x} = 0$; FSDT: $w = \phi_x = 0$

$\begin{cases} \text{CPT:} \quad w = \dfrac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{cases}$




$\left.\begin{matrix} \text{CPT:} \quad w = \dfrac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{matrix}\right\}$

$\begin{cases} \text{CPT:} \quad w = \dfrac{\partial w}{\partial x} = 0 \\ \text{FSDT:} \quad w = \phi_x = 0 \end{cases}$

Symmetry conditions: $\begin{cases} \text{CPT:} \quad \dfrac{\partial w}{\partial x} = 0 \text{ at } x = 0; \ \dfrac{\partial w}{\partial y} = 0 \text{ at } y = 0 \\ \text{FSDT:} \ \phi_x = 0 \text{ at } x = 0; \quad \phi_y = 0 \text{ at } y = 0 \end{cases}$

# The effect of reduced integration, thickness, and mesh refinement

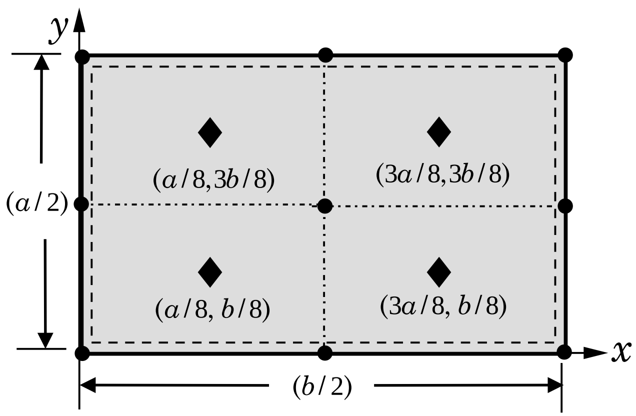
on the linear center deflections and stresses of a simply supported, isotropic ($\nu = 0.25$) square plate under a uniform transverse load of intensity $q_0$.

**F – full integration**
**M – Mixed integration**

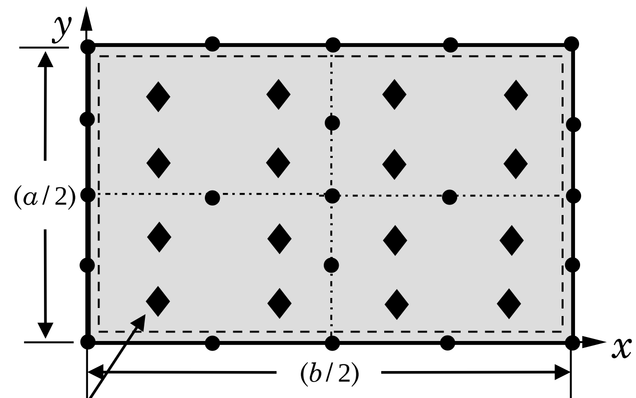
| | | 1 × 1 linear | | 2 × 2 linear | | 4 × 4 linear | | 2 × 2 quadratic | | Exact‡ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a/h$ | Integ. | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{w}$ | $\bar{\sigma}_x$ |
| 10 | F | 0.964 | 0.018 | 2.474 | 0.119 | 3.883 | 0.216 | 4.770 | 0.290 | 4.791 | 0.276 |
| | M | 3.950 | 0.095 | 4.712 | 0.235 | 4.773 | 0.266 | 4.799 | 0.272 | | |
| 20 | F | 0.270 | 0.005 | 0.957 | 0.048 | 2.363 | 0.138 | 4.570 | 0.268 | 4.625 | 0.276 |
| | M | 3.669 | 0.095 | 4.524 | 0.235 | 4.603 | 0.266 | 4.633 | 0.272 | | |
| 40 | F | 0.070 | 0.001 | 0.279 | 0.014 | 0.944 | 0.056 | 4.505 | 0.270 | 4.584 | 0.276 |
| | M | 3.599 | 0.095 | 4.375 | 0.235 | 4.560 | 0.266 | 4.592 | 0.271 | | |
| 50 | F | 0.005 | 0.000 | 0.182 | 0.009 | 0.652 | 0.039 | 4.496 | 0.267 | 4.579 | 0.276 |
| | M | 3.590 | 0.095 | 4.472 | 0.235 | 4.555 | 0.266 | 4.587 | 0.271 | | |
| 100 | F | 0.011 | 0.000 | 0.047 | 0.002 | 0.182 | 0.011 | 4.482 | 0.266 | 4.572 | 0.276 |
| | M | 3.579 | 0.095 | 4.465 | 0.235 | 4.548 | 0.266 | 4.580 | 0.272 | | |
| CPT(N) | | 5.643 | 0.260 | 4.857 | 0.274 | 4.643 | 0.276 | — | — | 4.570 | 0.276 |
| CPT(C) | | 4.638 | 0.262 | 4.574 | 0.272 | 4.570 | 0.275 | — | — | 4.570 | 0.276 |

†$\bar{w} = wEh^3 \times 10^2/q_0a^4$, $\bar{\sigma}_x = \sigma_x(A, A, \pm h)h^2/q_0a^2$, $A = \frac{1}{4}a$ (1 × 1 linear), $\frac{1}{8}a$ (2 × 2 linear), $\frac{1}{16}a$ (4 × 4 linear), $0.05283a$ (2 × 2 quadratic).

# Gauss Point Locations (based on reduced Integration Gauss points) for Stress Computation




$2 \times 2$ Mesh of
4-node (linear) elements




$2 \times 2$ Mesh of
9-node (quadratic) elements

$$\left(\frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}},\ \frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}}\right) = (0.05283a, 0.05283b)$$

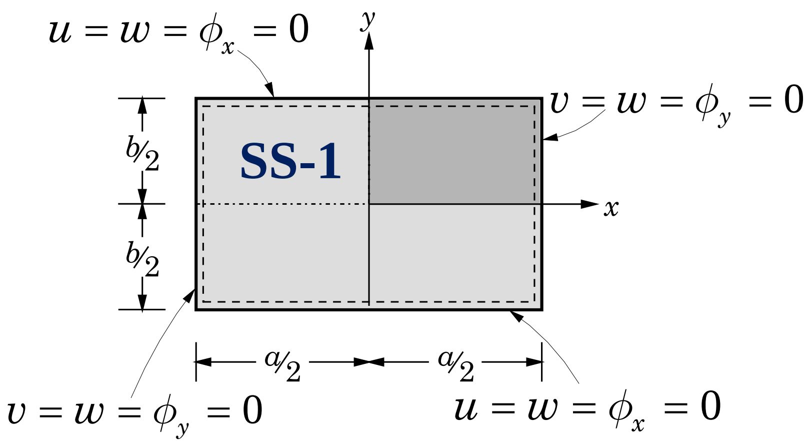
# REMARKS

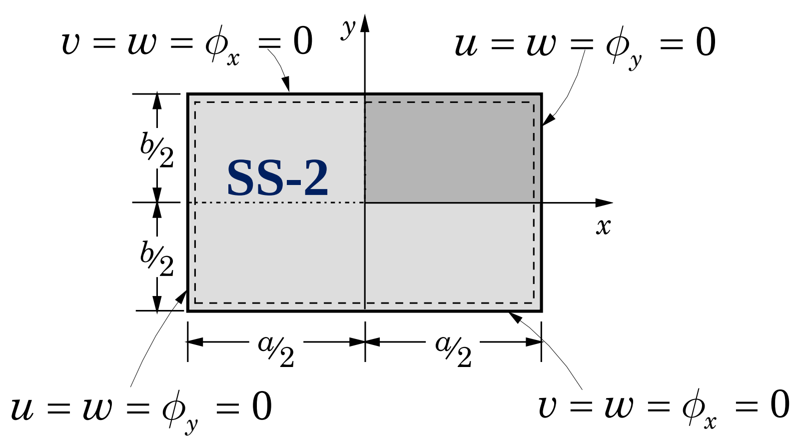
The nine-node element gives virtually the same results for full (3×3 Gauss rule) and mixed (3×3 and 2×2 Gauss rules for bending and shear terms, respectively) integrations. However, the results obtained using the mixed integration are closest to the exact solution.

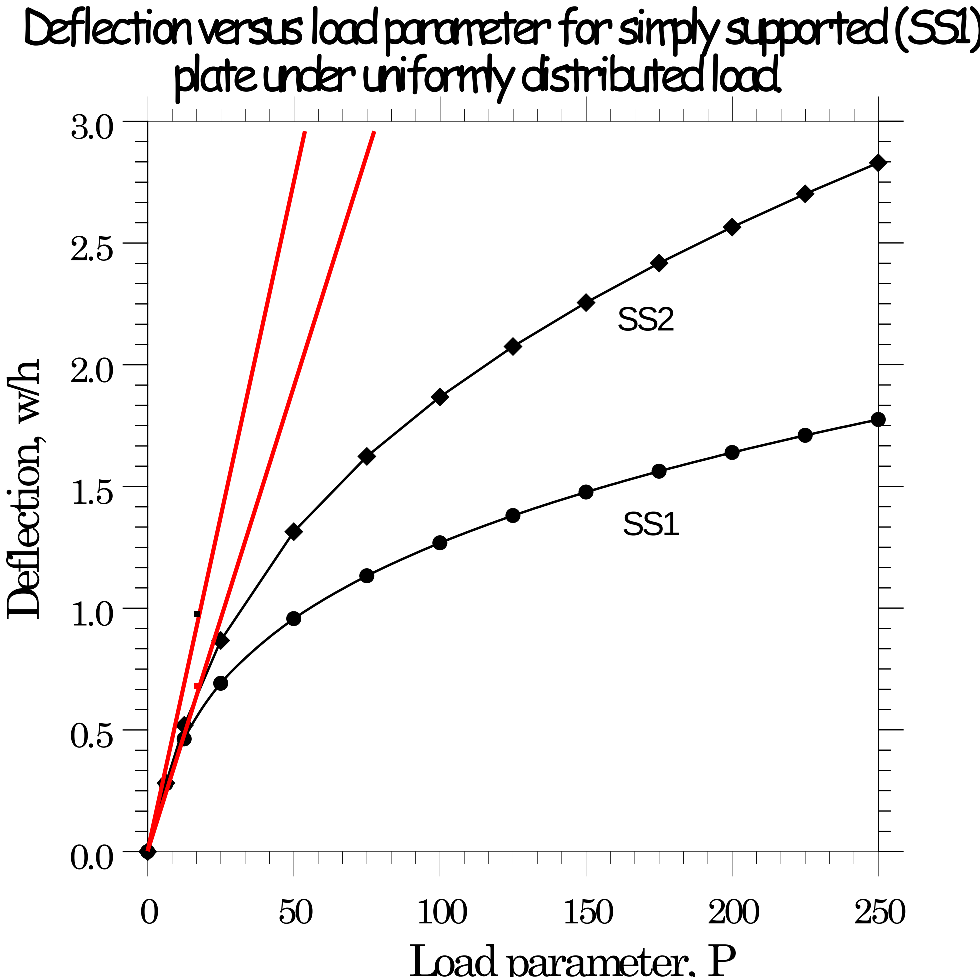
Full integration gives less accurate results than mixed integration, and the error increases with an increase in side-to-thickness ratio ($a/h$). This implies that mixed integration is essential for thin plates, especially when modeled by lower-order elements.

Full integration results in smaller errors for quadratic elements and refined meshes than for linear elements and/or coarser meshes.

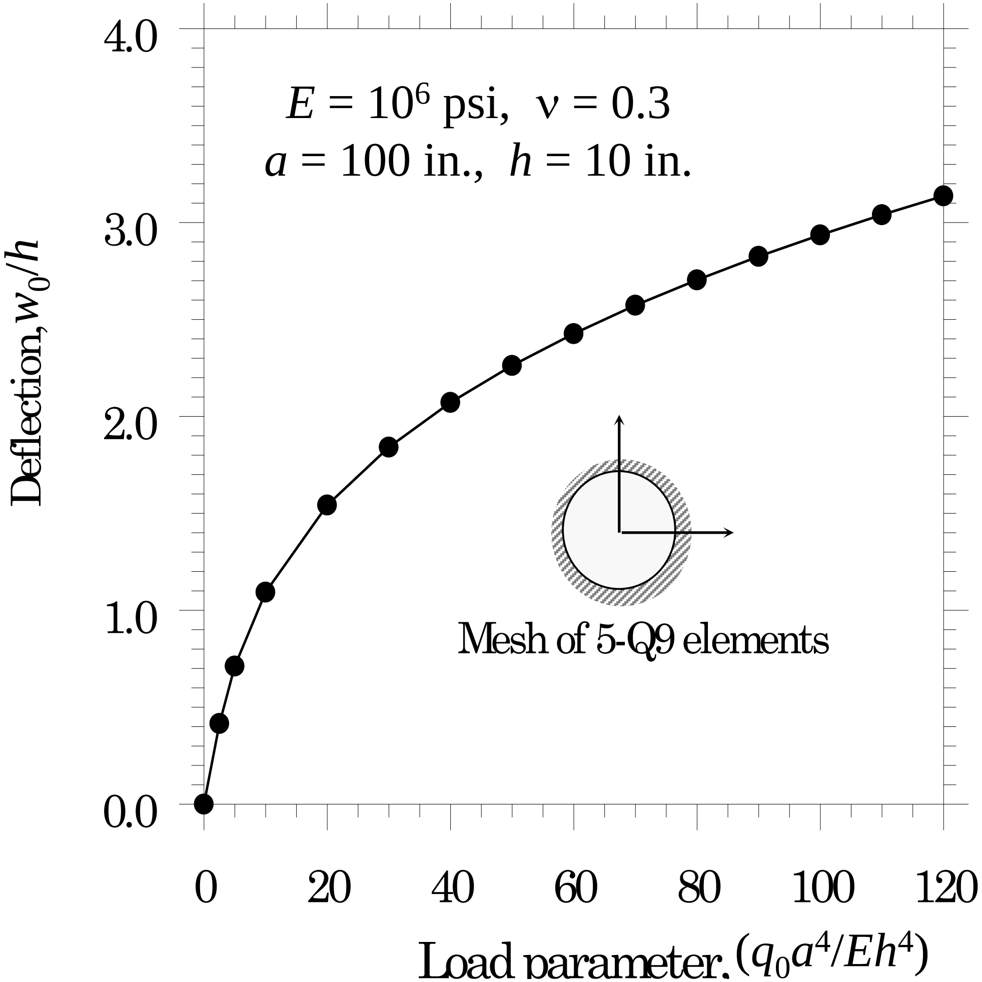
# Nonlinear Analysis of Simply Supported Plate (SS-1)

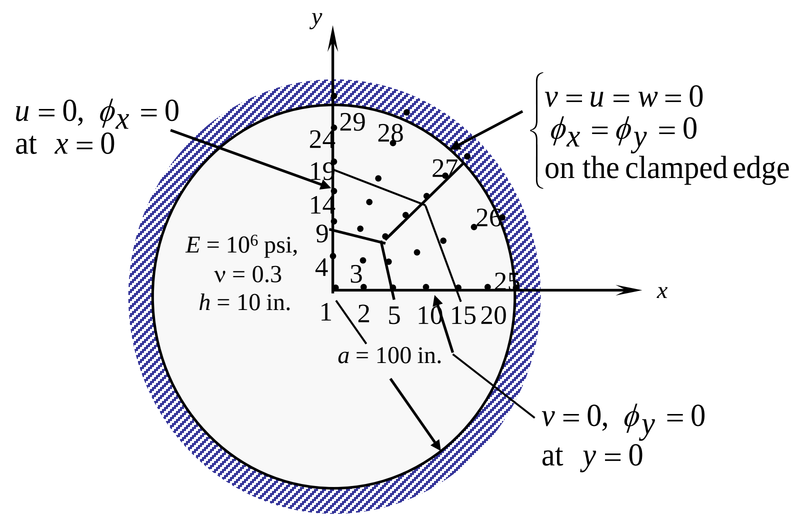


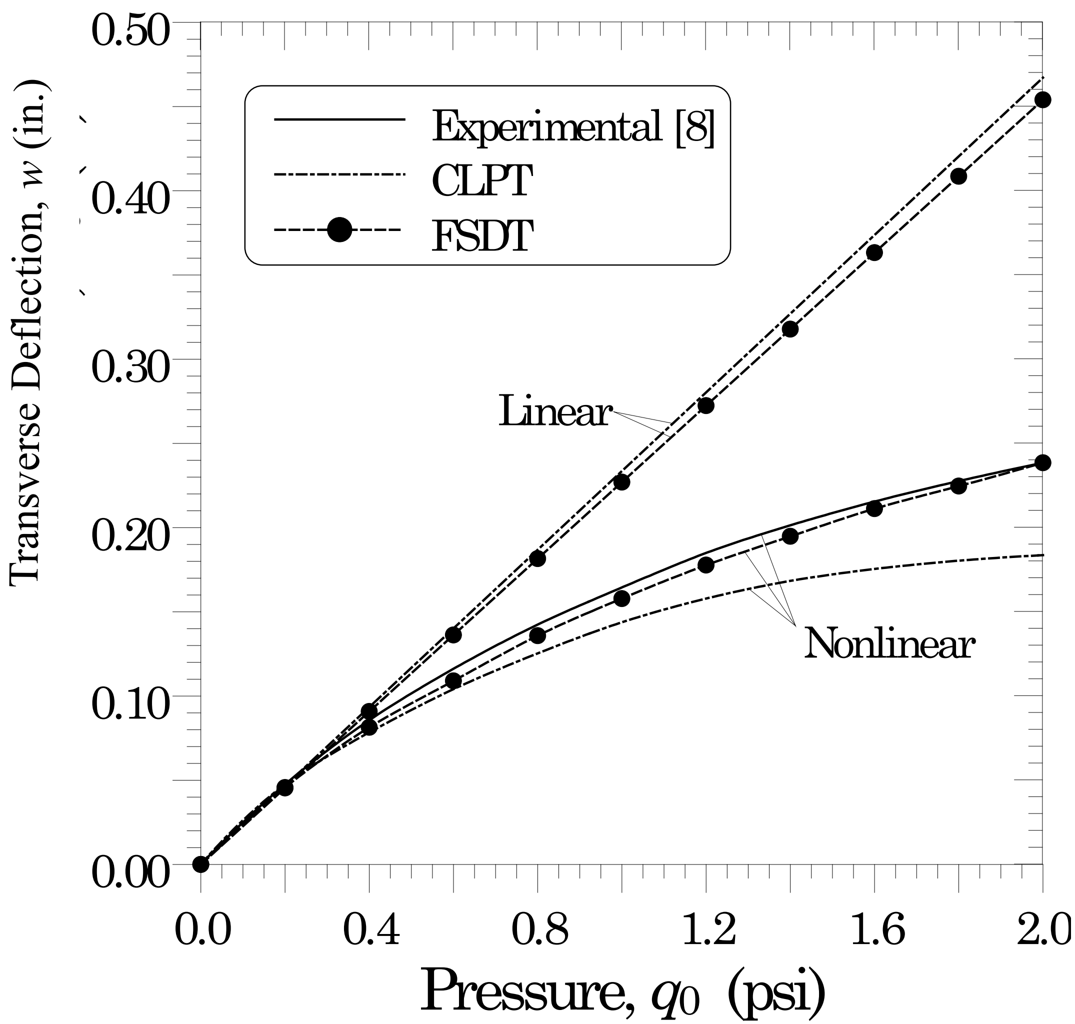



# Clamped Circular Plate under UDL



# Simply Supported (SS2) Orthotropic* Plate




Geometry and Material Properties

$a = b = 12$ in, $h = 0.138$ in

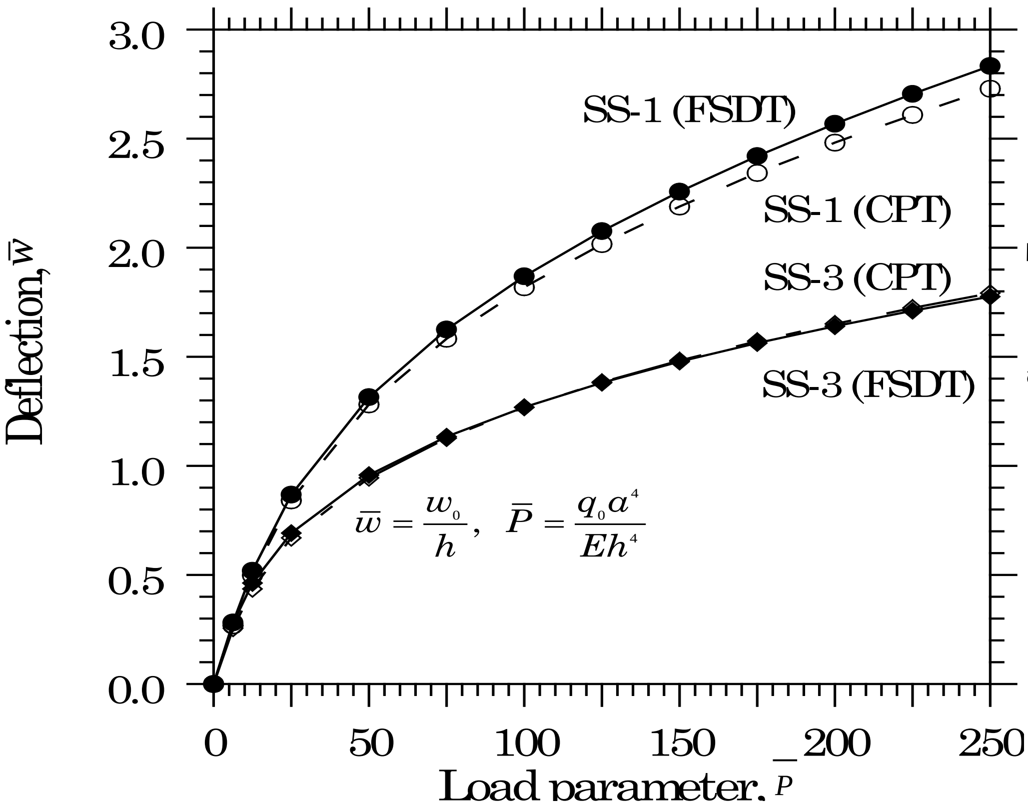
$E_1 = 3\times10^6$ psi, $E_2 = 1.28\times10^6$ psi

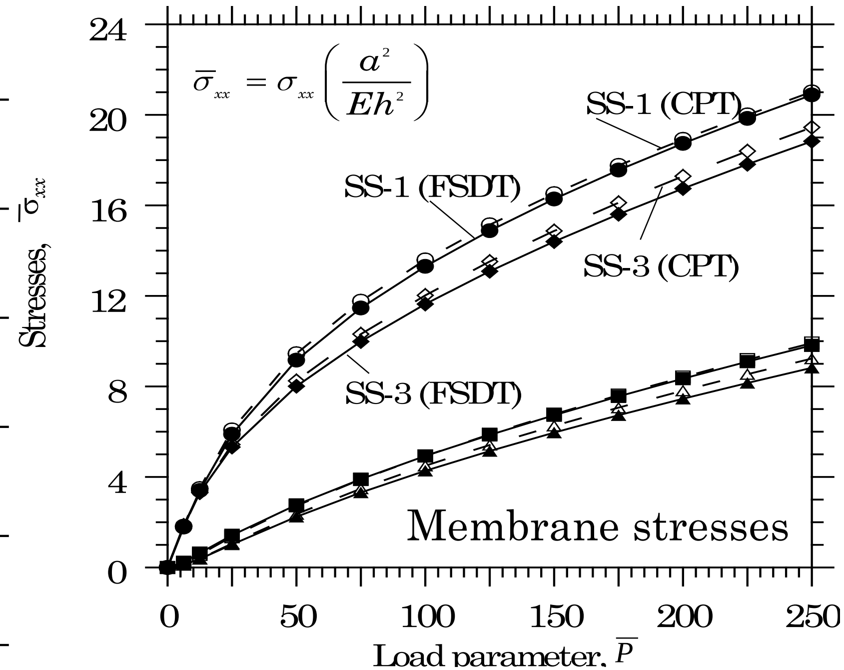
$G_{12} = G_{23} = G_{13} = 0.37\times10^6$ psi

$\nu_{12} = \nu_{23} = \nu_{13} = 0.32$

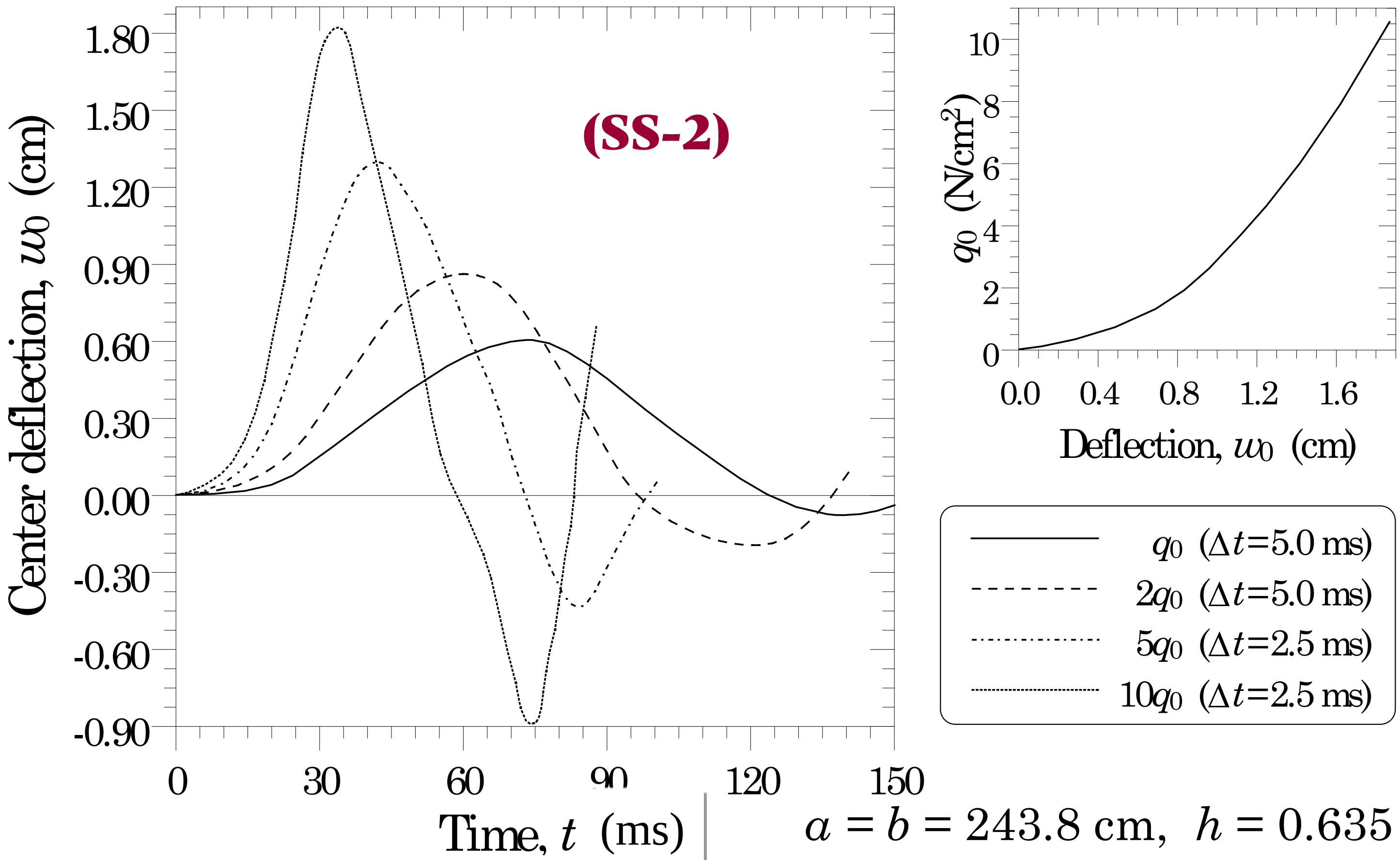

[8] Zaghloul, S. A. and Kennedy, J. B., ``Nonlinear Behavior of Symmetrically Laminated Plates," *Journal of Applied Mechanics*, 42, 234-236, 1975.

# Deflection vs. load parameter for plates under uniformly distributed load



# Center Deflection vs. Time for a Simply Supported Isotropic Plate Under Suddenly Applied Uniformly Distributed Pressure Load




$a = b = 243.8$ cm, $h = 0.635$ cm,

$\rho = 2.547\times 10^{-6}$ N-s$^2$/cm$^4$,

$E_1 = E_2 = 7.031 \times 10^5$ N/cm$^2$, $\nu_{12} = 0.25$

$q_0 = 4.882 \times 10^{-4}$ N/cm$^2$, $\Delta t = 0.005$ s $= 5$ms

# SUMMARY

**In this lecture we have covered the following topics:**

- **Governing Equations of FSDT**
- **Finite element models of FSDT**
- **Tangent stiffness coefficients**
- **Shear and membrane locking**
- **Programming aspects (including stress computation)**
- **Numerical examples**