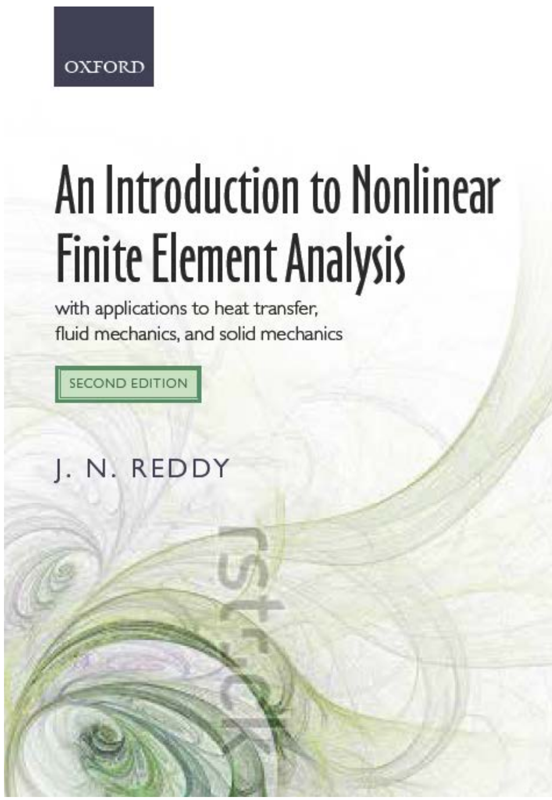
MEEN 673
Nonlinear Finite Element Analysis

Nonlinear Bending of Strait Beams

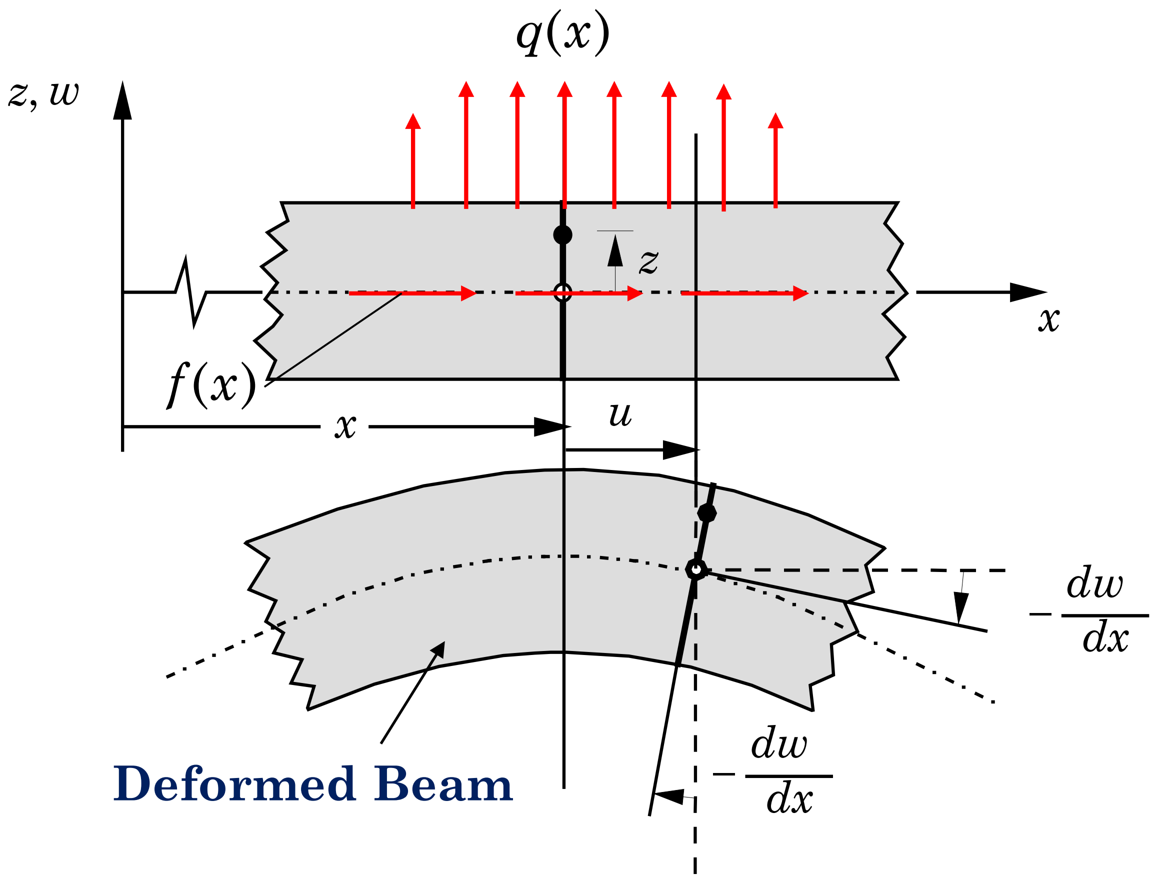
Read: Chapter 5



CONTENTS

- The Euler-Bernoulli beam theory
- The Timoshenko beam theory
  - Governing Equations
  - Weak Forms
  - Finite element models
  - Computer Implementation: calculation of element matrices
  - Numerical examples

THE EULER-BERNOULLI BEAM THEORY (development of governing equations)

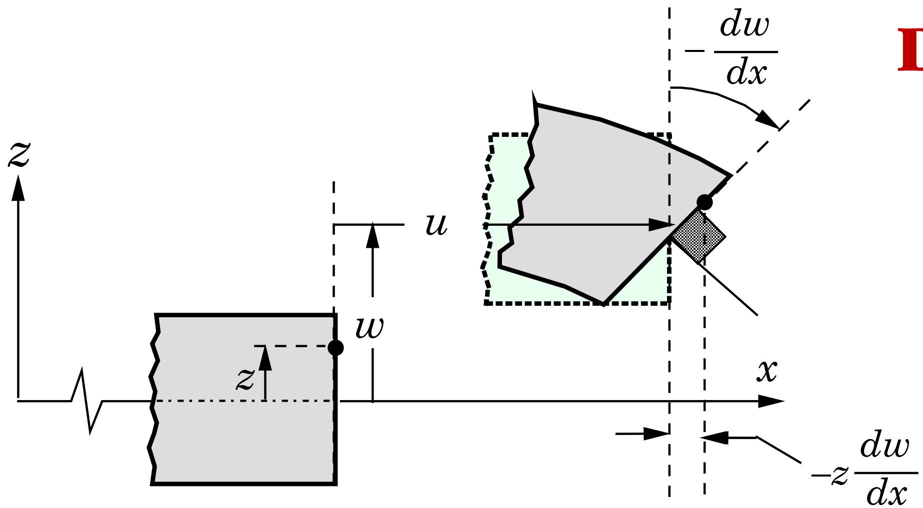


Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)

Straightness, inextensibility, and normality

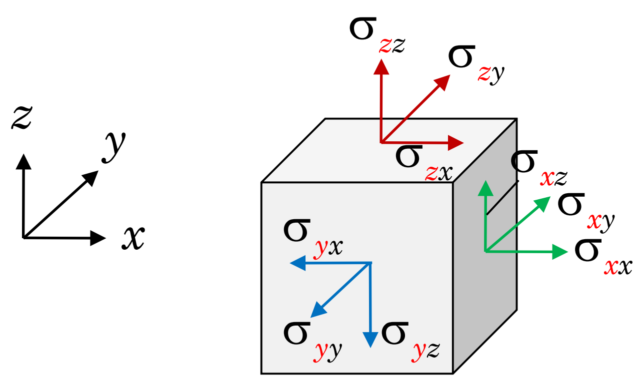
Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



Displacement field

$$\mathbf{u} = (u + z\,\theta_x)\,\hat{\mathbf{e}}_1 + w\,\hat{\mathbf{e}}_3,$$

$$\theta_x = -\frac{dw}{dx}$$



$$u_1(x,z) = u - z\frac{dw}{dx}$$

$$u_2 = 0,$$

$$u_3(x,z) = w(x)$$

Notation for stress components

Von Kármán NONLINEAR STRAINS

- **Green-Lagrange Strain Tensor Components**

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \tfrac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$$

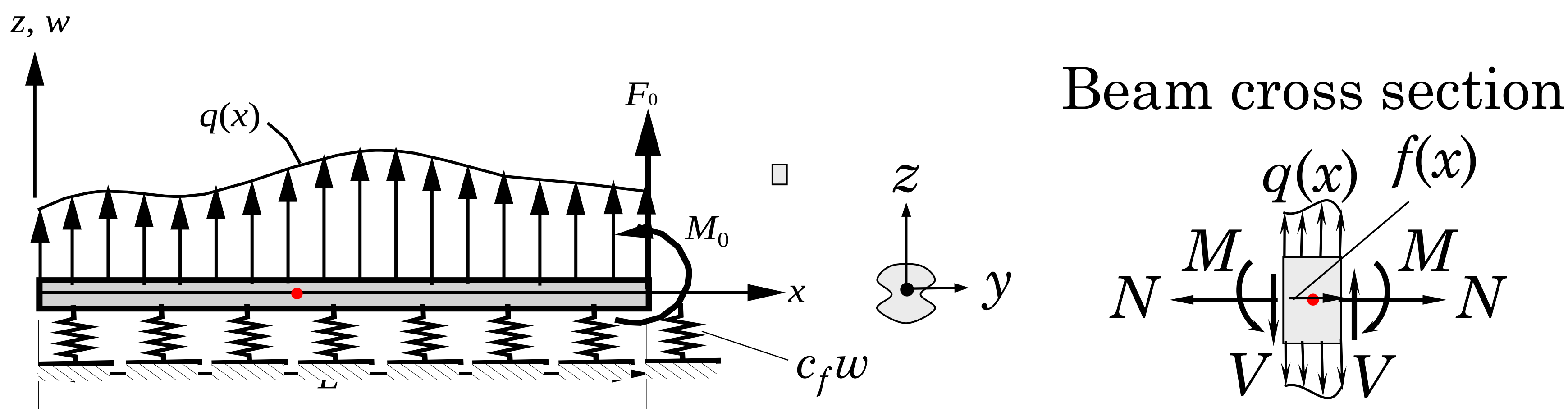
$$E_{xx} = \frac{\partial u_1}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

- **Order-of-magnitude assumption**

$$\frac{\partial u_1}{\partial x_1} \approx O(\varepsilon), \qquad \frac{\partial u_3}{\partial x_1} \approx O(\sqrt{\varepsilon})$$

$$E_{xx} \approx \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS



➢ Displacements and strain-displacement relations

$$\mathbf{u} = (u + z\,\theta_x)\,\hat{\mathbf{e}}_1 + w\,\hat{\mathbf{e}}_3, \quad \theta_x = -\frac{dw}{dx}$$

$$u_1(x,z) = u - z\frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2 w}{dx^2},$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

Equilibrium equations

$$\frac{dN}{dx}+f=0,\quad \frac{dM}{dx}-V=0,\quad \frac{dV}{dx}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

$$\frac{dN}{dx}+f=0,\qquad \frac{d^2M}{dx^2}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

Stress resultants in terms of deflection

$$N=\int_A \sigma_{xx}\ dA=\int_A\left[E\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]dA=EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M=\int_A \sigma_{xx}\times z\ dA=\int_A\left[E\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]z\ dA=-EI\frac{d^2w}{dx^2}$$

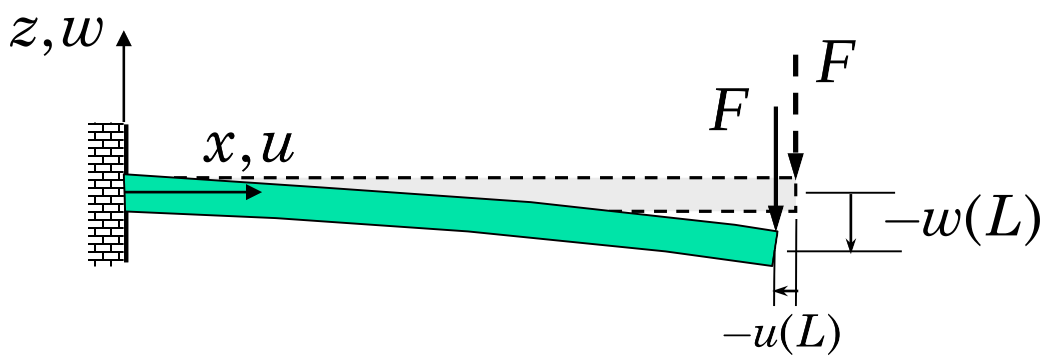
$$V=\frac{dM}{dx}=\frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right)$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

- **Equilibrium equations in terms of displacements (*u*,*w*)**

$$\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}-f=0$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left(\frac{dw}{dx}EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right)-q=0$$



- Clearly, transverse load induces both axial displacement u and transverse displacement w.

EULER-BERNOULLI BEAM THEORY (continued)

➢ Weak forms

$$0=\int_{x_a}^{x_b} v_1\left(-\frac{dN}{dx}-f\right)dx=\int_{x_a}^{x_b}\left(\frac{dv_1}{dx}N-v_1 f\right)dx-v_1(x_a)[-N(x_a)]-v_1(x_b)N(x_b)$$

$$=\int_{x_a}^{x_b}\left(\frac{dv_1}{dx}N-v_1 f\right)dx-v_1(x_a)Q_1-v_1(x_b)Q_4$$

$$N=EA\left[\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

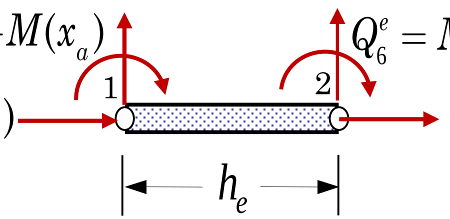
$$0=\int_{x_a}^{x_b} v_2\left[-\frac{d^2M}{dx^2}-\frac{d}{dx}\left(N\frac{dw}{dx}\right)-q\right]dx$$

$$=\int_{x_a}^{x_b}\left[EI\frac{d^2v_2}{dx^2}\frac{d^2w}{dx^2}+\frac{dv_2}{dx}\left(N\frac{dw}{dx}\right)-v_2 q\right]dx-v_2(x_a)Q_2-\left(-\frac{dv_2}{dx}\right)_{x_a}Q_3-v_2(x_b)Q_5-\left(-\frac{dv_2}{dx}\right)_{x_b}Q_6$$

$$Q_2^e=-V(x_a)\qquad Q_5^e=V(x_b)$$

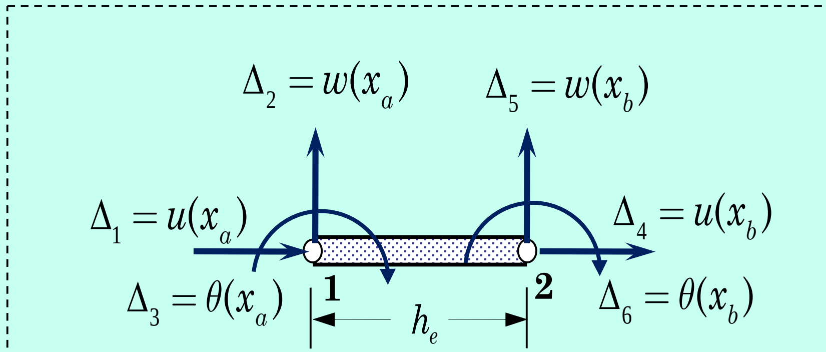
$$Q_3^e=-M(x_a)\qquad Q_6^e=M(x_b)$$

$$Q_1^e=-N(x_a)\qquad Q_4^e=N(x_b)$$

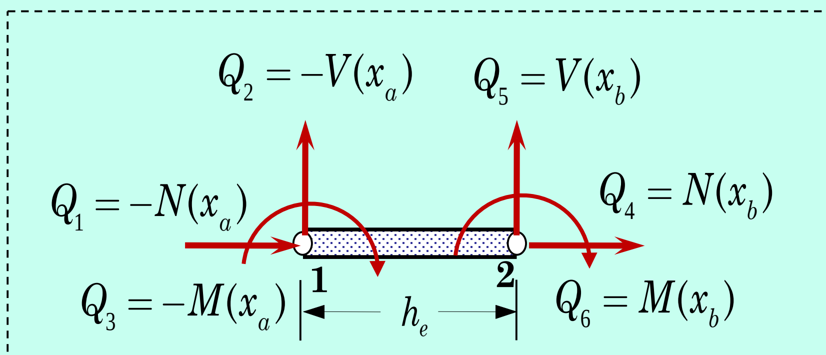


BEAM ELEMENT DEGREES OF FREEDOM

Generalized displacements



Generalized forces



FINITE ELEMENT APPROXIMATION

Primary variables (serve as the nodal variables that must be continuous across elements)

$$u,\, w,\;\; \theta = -\frac{dw}{dx}$$

$$w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x), \quad u(x) \approx \sum_{j=1}^{n} u_j \psi_j(x),$$

Hermite cubic polynomials

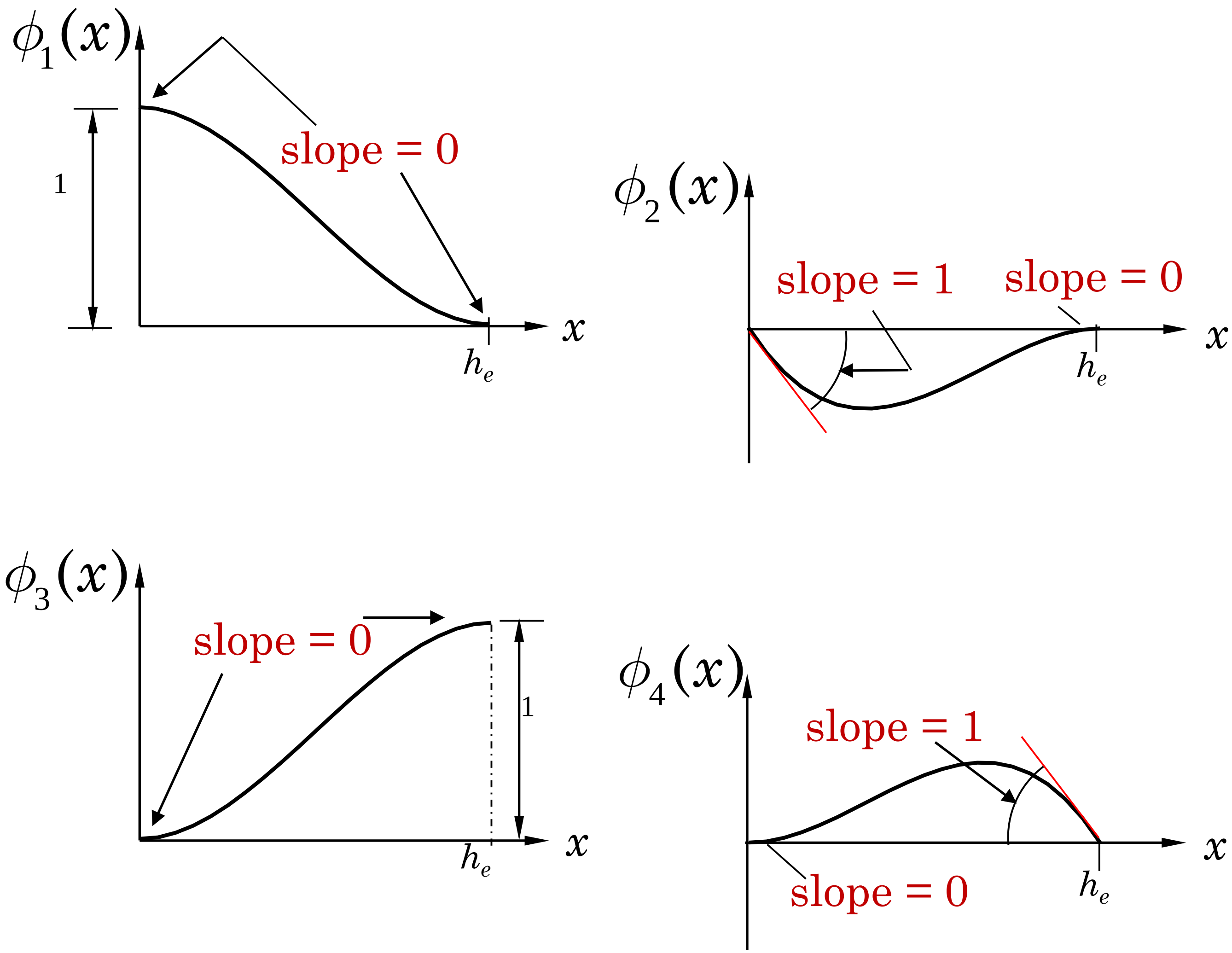
$$\phi_1^e = 1 - 3\left(\frac{x - x_a}{h_e}\right)^2 + 2\left(\frac{x - x_a}{h_e}\right)^3$$

$$\phi_2^e = -(x - x_a)\left(1 - \frac{x - x_a}{h_e}\right)^2$$

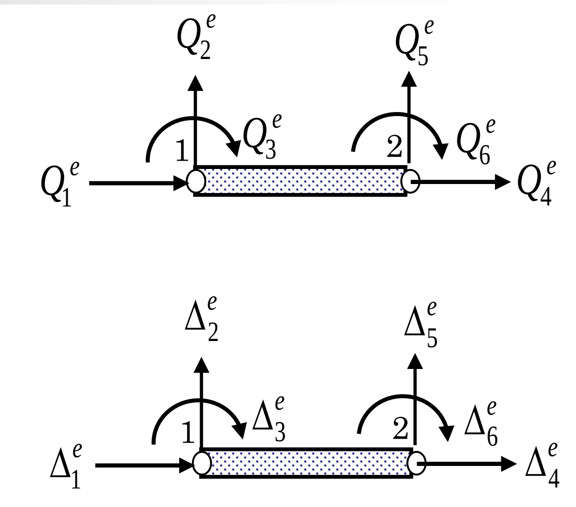
$$\phi_3^e = 3\left(\frac{x - x_a}{h_e}\right)^2 - 2\left(\frac{x - x_a}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_a)\left[\left(\frac{x - x_a}{h_e}\right)^2 - \frac{x - x_a}{h_e}\right]$$

HERMITE CUBIC INTERPOLATION FUNCTIONS $\phi_i(x)$



FINITE ELEMENT MODEL



➤ Finite Element Equations

$$u(x) \approx \sum_{j=1}^{2} u_j \psi_j(x), \quad w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x)$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{\Delta\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} EA \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} dx, \quad K_{ij}^{12} = \frac{1}{2}\int_{x_a}^{x_b} EA \frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\phi_j}{dx} dx,$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} EA \frac{dw}{dx}\frac{d\phi_i}{dx}\frac{d\psi_j}{dx} dx, \quad F_i^1 = \int_{x_a}^{x_b} f\psi_i \, dx + \psi_i(x_a)Q_1 + \psi_i(x_b)Q_4$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} EI \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2} dx + \int_{x_a}^{x_b} EA \left(\frac{dw}{dx}\right)^2 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx,$$

$$F_i^2 = \int_{x_a}^{x_b} q\psi_i \, dx + \phi_i(x_a)Q_2 + \phi_i(x_b)Q_5 + \left(-\frac{d\phi_i}{dx}\right)_{x_a} Q_3 + \left(-\frac{d\phi_i}{dx}\right)_{x_b} Q_6$$

MEMBRANE LOCKING

Membrane strain

$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

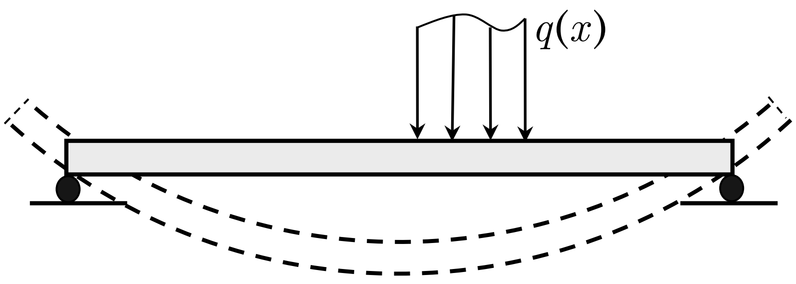
$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

Remedy

$$\Rightarrow \quad \text{make } \left(\frac{dw}{dx}\right)^2 \text{ to behave like a constant}$$

Beam on roller supports



SOLUTION OF NONLINEAR EQUATIONS

Direct Iteration

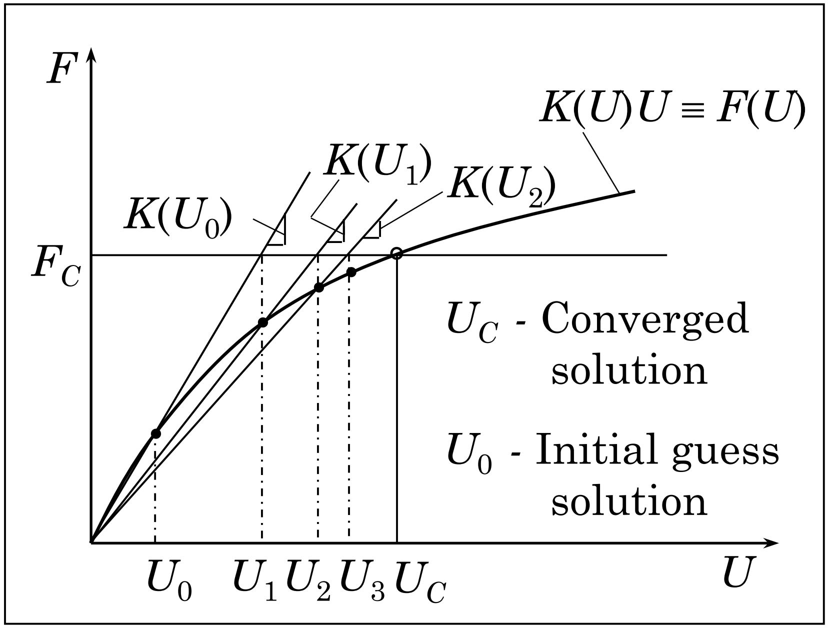
Non-Linear Finite Element Model

$$[K^e(\Delta^e)]\{\Delta^e\} = \{F^e\} \Rightarrow \text{assembled } [K(U)]\{U\} = \{F\}$$

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$[K(\{U\}^r)]\{U\}^{r+1} = \{F\}$$



SOLUTION OF NONLINEAR EQUATIONS (continued)

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$[K(\{U\}^r)]\{U\}^{r+1} = \{F\}$$

Convergence Criterion

$$\varepsilon = \sqrt{\frac{\sum_{I=1}^{NEQ} \left(U_I^r - U_I^{r+1}\right)^2}{\sum_{I=1}^{NEQ} \left(U_I^{r+1}\right)^2}} \leq \text{specified tolerance}$$

SOLUTION OF NONLINEAR EQUATIONS

Newton's Iteration Method

Taylor's series

$$\text{Residual,} \quad \{R\} \equiv [K(\{U\}^r)]\{U\}^{r+1} - \{F\}^r$$

$$\{R(U^{r+1})\} = \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + \frac{1}{2!}(U^{r+1} - U^r)^2\left[\frac{\partial^2 R}{\partial U^2}\right]^r + \cdots$$

$$\approx \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + O(\delta U)^2, \quad \boxed{\delta U = U^{r+1} - U^r}$$

Requiring the residual $\{R\}^{r+1}$ to be zero at the $r+1^{\text{st}}$ iteration, we have

$$\boxed{[K^{\tan}(\{U\}^r)]\{\delta U\} = -\{R\}^r = \{F\}^r - [K(U^r)]^r\{U\}^r}$$

The tangent matrix at the element level is

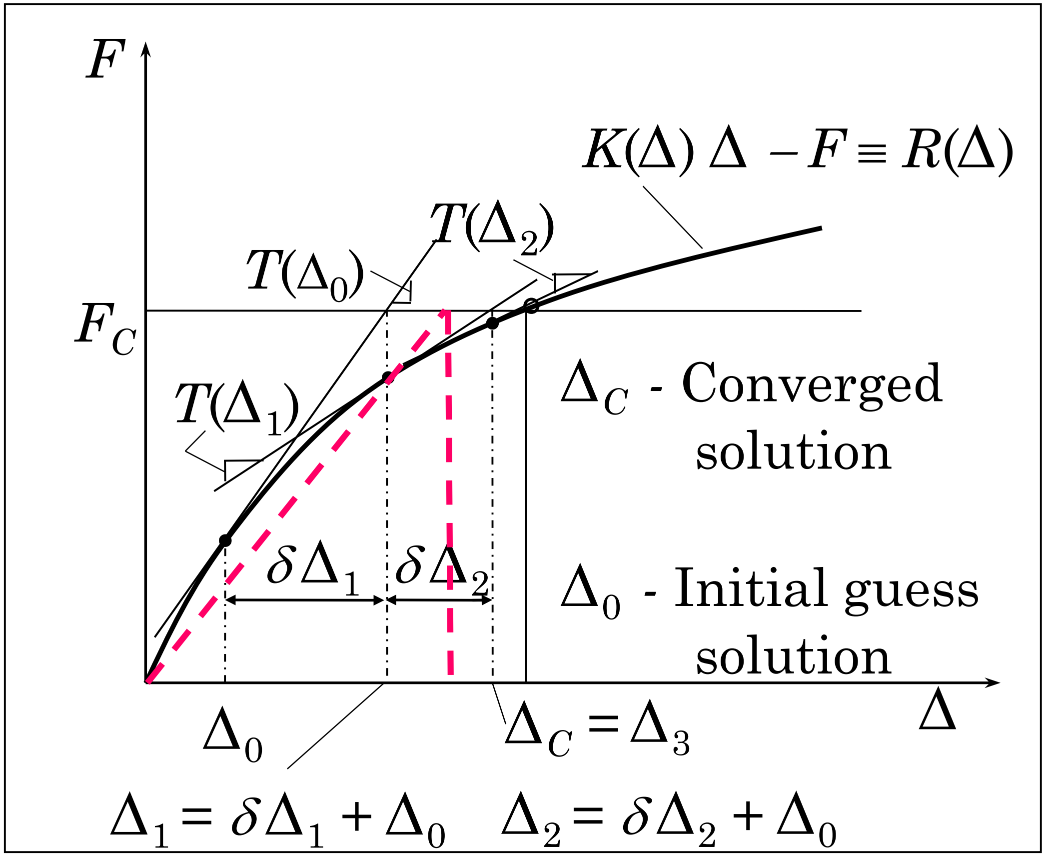
$$\left(K_{ij}^{\alpha\beta}\right)^{\tan} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n} K_{ip}^{\alpha\lambda}\Delta_p^\gamma - F_i^\alpha\right)$$

SOLUTION OF NONLINEAR EQUATIONS

Newton's Iteration (continued)

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n_\beta} K_{ip}^{\alpha\gamma}\Delta_p^\gamma - F_i^\alpha\right) = K_{ij}^{\alpha\beta} + \sum_{\gamma=1}^{2}\sum_{p=1}^{n}\frac{\partial K_{ip}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_p^\gamma \equiv T_{ij}^{\alpha\beta}$$

$$[T(\{\Delta\}^r)]\{\delta\Delta\} = \{F\}^r - [K(\Delta^r)]^r\{\Delta\}^r, \;\; \{\Delta\}^{r+1} = \{\Delta\}^r + \{\delta\Delta\}$$



Summary of the N-R Method

$$[T(\{\Delta\}^{(r-1)}][\delta\Delta\}^{r} = -\{R(\{\Delta\}^{(r-1)})\}$$

$$\{\Delta\}^{r} = \{\Delta\}^{(r-1)} + \{\delta\Delta\}$$

Computation of tangent stiffness matrix

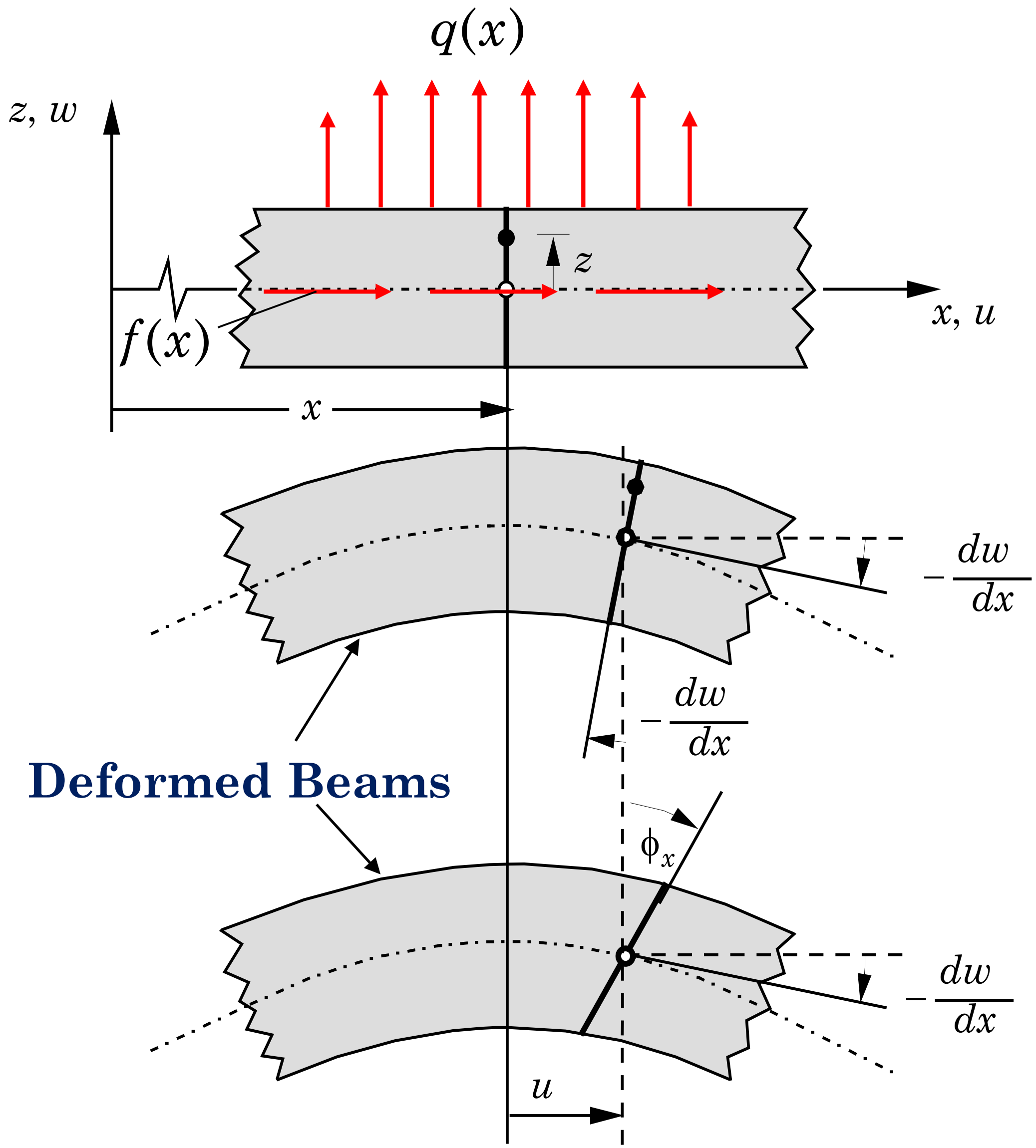
$$R_i^{\alpha} = \sum_{\gamma=1}^{2}\sum_{p=1} K_{ip}^{\alpha\gamma}\Delta_p^{\gamma} - F_i^{\alpha} = \sum_{p=1}^{n} K_{ip}^{\alpha 1} u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2}\overline{\Delta}_P - F_i^{\alpha}$$

$$T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^{\alpha}}{\partial \Delta_j^{\beta}}\right) = K_{ij}^{\alpha\beta} + \sum_{p=1}^{n} \frac{\partial}{\partial \Delta_j^{\beta}}\left(K_{ip}^{\alpha 1}\right) u_p + \sum_{P=1}^{4} \frac{\partial}{\partial \Delta_j^{\beta}}\left(K_{iP}^{\alpha 2}\right)\overline{\Delta}_P$$

$$T_{ij}^{11} = K_{ij}^{11} + \sum_{p=1}^{n} \frac{\partial K_{ip}^{11}}{\partial u_j} u_p + \sum_{P=1}^{4} \frac{\partial K_{iP}^{12}}{\partial u_j}\overline{\Delta}_P$$

$$= K_{ij}^{11} + \sum_{p=1}^{n} 0 \cdot u_p + \sum_{P=1}^{4} 0 \cdot \overline{\Delta}_P$$

THE TIMOSHENKO BEAM THEORY



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)

Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)

Straightness and inextensibility

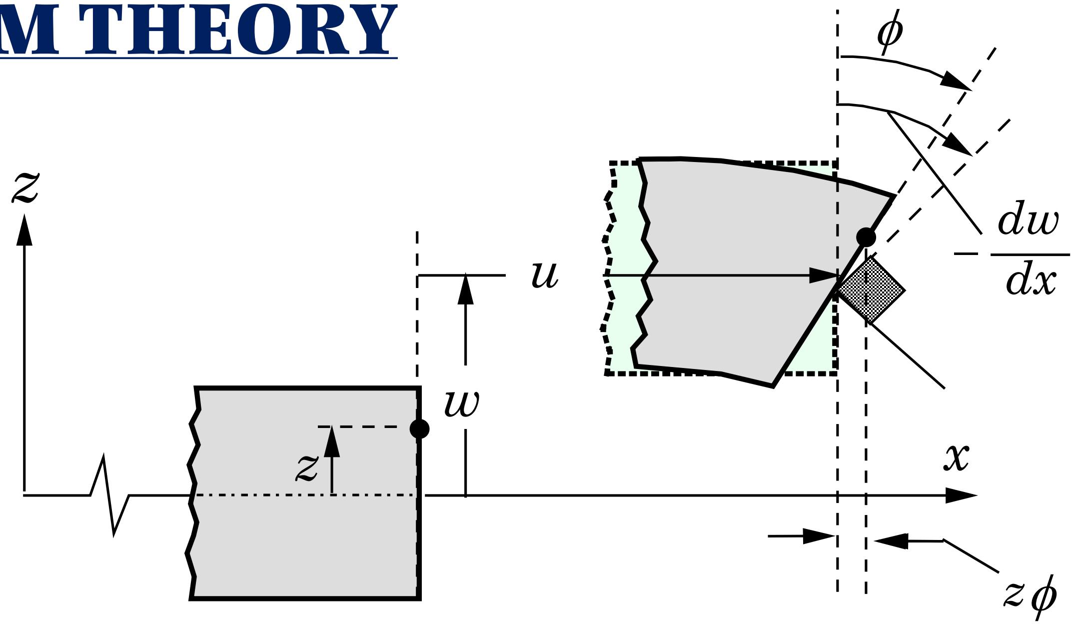
KINEMATICS OF THE TIMOSHENKO BEAM THEORY

Displacement field

$$\mathbf{u} = (u + z\,\phi_x)\,\hat{\mathbf{e}}_1 + w\,\hat{\mathbf{e}}_3$$

$$u_1(x,z) = u(x) + z\phi(x),$$

$$u_2 = 0, \quad u_3(x,z) = w(x)$$



$$E_{xx} \approx \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2$$

$$= \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\,\frac{d\phi_x}{dx}$$

$$2E_{xz} \approx 2\varepsilon_{xz} = \gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}$$

$$= \phi_x + \frac{dw}{dx}$$

Constitutive Equations

$$\sigma_{xx} = E\,\varepsilon_{xx}, \quad \sigma_{xz} = G\,\gamma_{xz}$$

TIMOSHENKO BEAM THEORY (continued)

Equilibrium Equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0$$

Beam Constitutive Equations

$$N = \int_A \sigma_{xx}\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] dA = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M = \int_A \sigma_{xx} z\, dA = \int_A E\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx}\right] z\, dA = EI\frac{d\phi_x}{dx}$$

$$V = K_s \int_A \sigma_{xz}\, dA = GK_s\left(\phi_x + \frac{dw}{dx}\right)\int_A dA = GAK_s\left(\phi_x + \frac{dw}{dx}\right)$$

WEAK FORMS OF TBT

Weak Form of Eq. (1) $v_1 \sim u$

$$0 = \int_{x_a}^{x_b} v_1 \left(-\frac{dN}{dx} - f \right) dx = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N - v_1 f \right) dx$$

$$- v_1(x_a)[-N(x_a)] - v_1(x_b) N(x_b)$$

$$= \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N - v_1 f \right) dx - v_1(x_a) Q_1 - v_1(x_b) Q_4$$

$$0 = \int_{x_a}^{x_b} \left\{ EA \frac{dv_1}{dx} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] - v_1 f \right\} dx$$

$$- v_1(x_a) Q_1 - v_1(x_b) Q_4$$

WEAK FORMS OF TBT (continued)

Weak Form of Eq. (2) $v_2 \sim w$

$$0 = \int_{x_a}^{x_b} v_2 \left\{ -\frac{d}{dx}\left[GAK_s\left(\phi_x + \frac{dw}{dx}\right)\right] - \frac{d}{dx}\left(N\frac{dw}{dx}\right) - q \right\} dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{dv_2}{dx}\left[GAK_s\left(\phi_x + \frac{dw}{dx}\right)\right] + \frac{dv_2}{dx} N \frac{dw}{dx} - v_2 q \right\} dx$$

$$- \left\{ v_2 \cdot \left[GAK_s\left(\phi_x + \frac{dw}{dx}\right) + N\frac{dw}{dx}\right]\right\}_{x_a}^{x_b}$$

$$0 = \int_{x_a}^{x_b} \left\{ \frac{dv_2}{dx}\left[GAK_s\left(\phi_x + \frac{dw}{dx}\right)\right] + \frac{dv_2}{dx} N \frac{dw}{dx} - v_2 q \right\} dx - v_2(x_a)\cdot Q_2 - v_2(x_b)\cdot Q_5$$

$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

WEAK FORMS OF TBT
(continued)

Weak Form of Eq. (3) $v_3 \sim \phi_x$

$$0=\int_{x_a}^{x_b} v_3\left[-\frac{d}{dx}\left(EI\frac{d\phi_x}{dx}\right)+GAK_s\left(\phi_x+\frac{dw}{dx}\right)\right]dx$$

$$=\int_{x_a}^{x_b}\left[\frac{dv_3}{dx}\left(EI\frac{d\phi_x}{dx}\right)+GAK_s v_3\left(\phi_x+\frac{dw}{dx}\right)\right]dx-\left[v_3\cdot EI\frac{d\phi_x}{dx}\right]_{x_a}^{x_b}$$

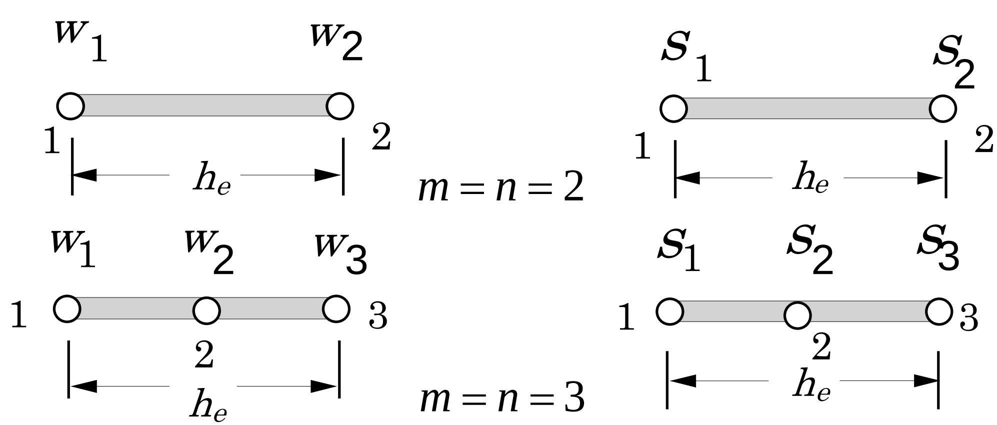
$$0=\int_{x_a}^{x_b}\left[\frac{dv_3}{dx}\left(EI\frac{d\phi_x}{dx}\right)+GAK_s v_3\left(\phi_x+\frac{dw}{dx}\right)\right]dx$$
$$-v_3(x_a)\cdot Q_3-v_3(x_b)\cdot Q_6$$

FINITE ELEMENT MODELS OF TIMOSHENKO BEAMS

Finite Element Approximation

$$u \approx \sum_{j=1}^{m} u_j \psi_j^{(1)}(x), \quad w \approx \sum_{j=1}^{n} w_j \psi_j^{(2)}(x), \quad \phi \approx \sum_{j=1}^{p} S_j \psi_j^{(3)}(x)$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix}$$



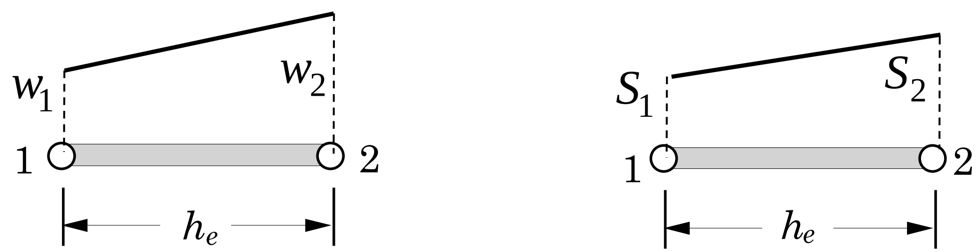
SHEAR LOCKING IN TIMOSHENKO BEAMS

(1) Thick beam experiences shear deformation, $\phi_x \neq -\dfrac{dw}{dx}$

(2) Shear deformation is negligible in thin beams, $\phi_x = -\dfrac{dw}{dx}$

Linear interpolation of both $w,\ \phi_x$

$$w(x) \approx w_1\psi_1(x) + w_2\psi_2(x), \qquad \phi_x(x) \approx S_1\psi_1(x) + S_2\psi_2(x)$$



In the thin beam limit it is not possible for the element to realize the requirement

$$\phi_x = -\frac{dw}{dx}$$

SHEAR LOCKING - REMEDY

In the thin beam limit, ϕ should become constant so that it matches dw/dx. However, if ϕ is a constant then the bending energy becomes zero. If we can mimic the two states (constant and linear) in the formulation, we can overcome the problem. Numerical integration of the coefficients allows us to evaluate both ϕ and $d\phi/dx$ as constants. The terms **highlighted** should be evaluated using "reduced integration".

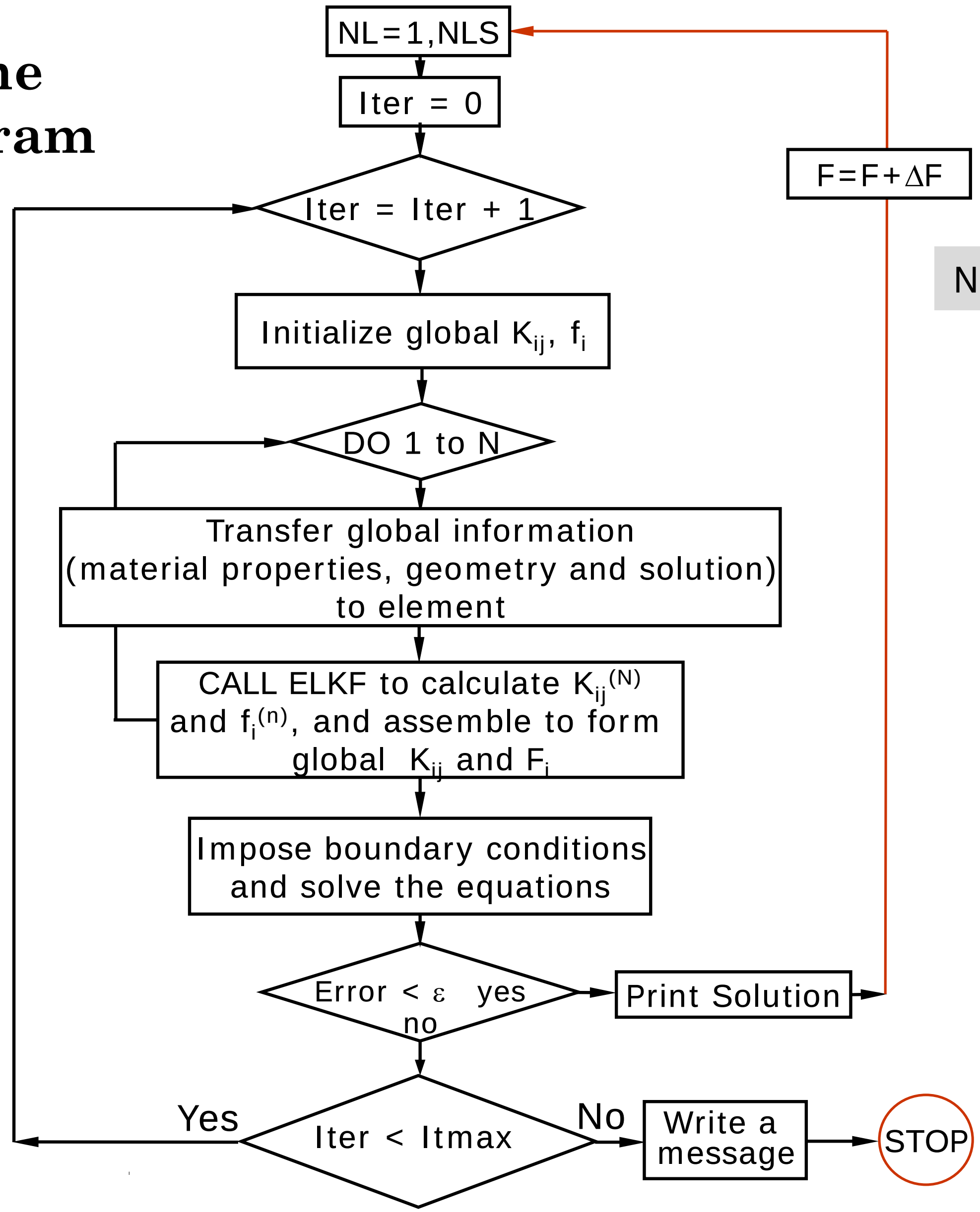
$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + \ldots \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} \; dx = K_{ji}^{32}$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left[EI \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + GAK_s \psi_i^{(3)} \psi_j^{(3)} \right] dx$$

GENERAL LOGIC IN A COMPUTER PROGRAM for the nonlinear analysis

Logic in the MAIN program



NLS = no. of load steps

CALCULATION OF BEAM PARAMETERS AND INITIALIZATIONS

```
      IF(MODEL.GE.2)THEN
C
C     Define the beam stiffness coefficients, EA, EI, GAKs, from the
C     geometric and material parameters read in the main program
C     (should be passed to this subroutine)
C
C      Initialize arrays
C
         DO 20 I=1,NPE
                 ELF1(I)=0.0
                 ELF2(I)=0.0
                 ELF3(I)=0.0
         DO 20 J=1,NPE
                 ELK11(I,J)=0.0
                 ELK12(I,J)=0.0
                 ELK13(I,J)=0.0
                 ELK21(I,J)=0.0
                 ELK22(I,J)=0.0
                 ELK23(I,J)=0.0
                 ELK31(I,J)=0.0
                 ELK32(I,J)=0.0
                 ELK33(I,J)=0.0
```

CALCULATION OF BEAM PARAMETERS AND INITIALIZATIONS

```
                 IF(NONLIN.GT.1)THEN
                   TAN12(I,J)=0.0
                   TAN13(I,J)=0.0
                   TAN22(I,J)=0.0
                   TAN23(I,J)=0.0
                   TAN32(I,J)=0.0
                   TAN33(I,J)=0.0
                 ENDIF
   20         CONTINUE
            ENDIF
C
C     Full integration of the coefficients
C
      DO 100 NI=1,NGP
      XI=GAUSPT(NI,NGP)
      CALL INTERPLN1D(ELX,GJ,IEL,MODEL,NPE,XI)
      X=ELX(1)+0.5*(1.0+XI)*EL
      CNST=GJ*GAUSWT(NI,NGP)

C     DEFINE AXX, BXX, CXX, DXX, FX, and so on as needed to define
C     the element force and stiffness coefficients
```

CALCULATION OF ELEMENT MATRICES

(see Box 5.2.2 of the textbook)

```
C     The EULER-BERNOULLI beam element (MODEL=2) - LINEAR
C
      IF(MODEL.EQ.2)THEN
        DO 50 I=1,NPE
        I0=2*I-1
        ELF1(I)=ELF1(I)+F0*FX*SFL(I)*CNST
        ELF2(I)=ELF2(I)+F0*QX*SFH(I0)*CNST
        ELF3(I)=ELF3(I)+F0*QX*SFH(I0+1)*CNST
        DO 50 J=1,NPE
        J0=2*J-1
        S11=GDSFL(I)*GDSFL(J)*CNST
        H22=GDDSFH(I0)*GDDSFH(J0)*CNST
        H23=GDDSFH(I0)*GDDSFH(J0+1)*CNST
        H32=GDDSFH(I0+1)*GDDSFH(J0)*CNST
        H33=GDDSFH(I0+1)*GDDSFH(J0+1)*CNST
        ELK11(I,J)=ELK11(I,J)+AXX*S11
        ELK22(I,J)=ELK22(I,J)+DXX*H22
        ELK23(I,J)=ELK23(I,J)+DXX*H23
        ELK32(I,J)=ELK32(I,J)+DXX*H32
        ELK33(I,J)=ELK33(I,J)+DXX*H33
50     CONTINUE
      ENDIF
```

MODEL = Type of physical problem
=1, 2nd order eqn. in 1 variable
=2, EBT
>2, TBT

SFL(I) = ψ_i; SFH(I0) = ϕ_i; GDSFL = $\dfrac{d\psi_i}{dx}$; GDDSFH = $\dfrac{d^2\phi_i}{dx^2}$

$$F_i^1 = \int_{x_a}^{x_b} f(x)\,\psi_i\,dx$$

$$F_i^2 = \int_{x_a}^{x_b} q(x)\,\phi_i\,dx$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\,dx,$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} D_{xx}\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx$$

CALCULATION OF ELEMENT MATRICES

```
C
C     The TIMOSHENKO beam element (MODEL=3) - LINEAR
C
      IF(MODEL.GT.2)THEN
        DO 60 I=1,NPE
        ELF1(I)=ELF1(I)+F0*FX*SFL(I)*CNST
        ELF2(I)=ELF2(I)+F0*QX*SFL(I)*CNST
        DO 60 J=1,NPE
        S11=GDSFL(I)*GDSFL(J)*CNST
        ELK11(I,J)=ELK11(I,J)+AXX*S11
        ELK33(I,J)=ELK33(I,J)+DXX*S11
 60     CONTINUE
      ENDIF

100  CONTINUE   ! (loop on NI =1, NGP ends here)

C
C     Define shear and nonlinear coefficients for the two beam theories as
C     appropriate in the reduced integration do-loop; define ELK and TAN coefficients
C
```

REARRANGE ELEMENT COEFFICIENTS

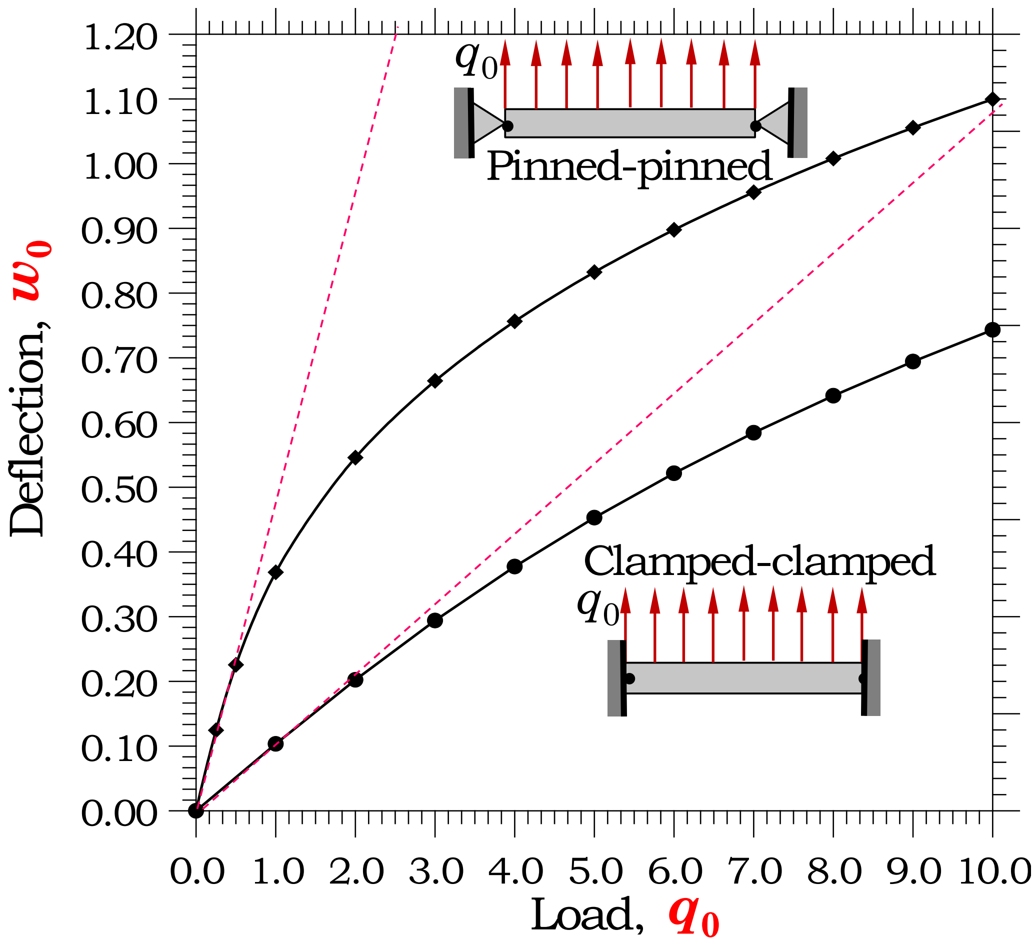
```
      IF(MODEL.GT.1)THEN
         II=1
         DO 220 I=1,NPE
            ELF(II)    =ELF1(I)
            ELF(II+1)=ELF2(I)
            ELF(II+2)=ELF3(I)
            JJ=1
            DO 210 J=1,NPE
               ELK(II,JJ)       = ELK11(I,J)
               ELK(II,JJ+1)   = ELK12(I,J)
               ELK(II,JJ+2)   = ELK13(I,J)
               ELK(II+1,JJ)   = ELK21(I,J)
               ELK(II+2,JJ)   = ELK31(I,J)
               ELK(II+1,JJ+1) = ELK22(I,J)
               ELK(II+1,JJ+2) = ELK23(I,J)
               ELK(II+2,JJ+1) = ELK32(I,J)
               ELK(II+2,JJ+2) = ELK33(I,J)
210         JJ=NDF*J+1
220      II=NDF*I+1
      ENDIF
```

COMPUTATION OF RESIDUAL VECTOR AND TANGENT MATRIX

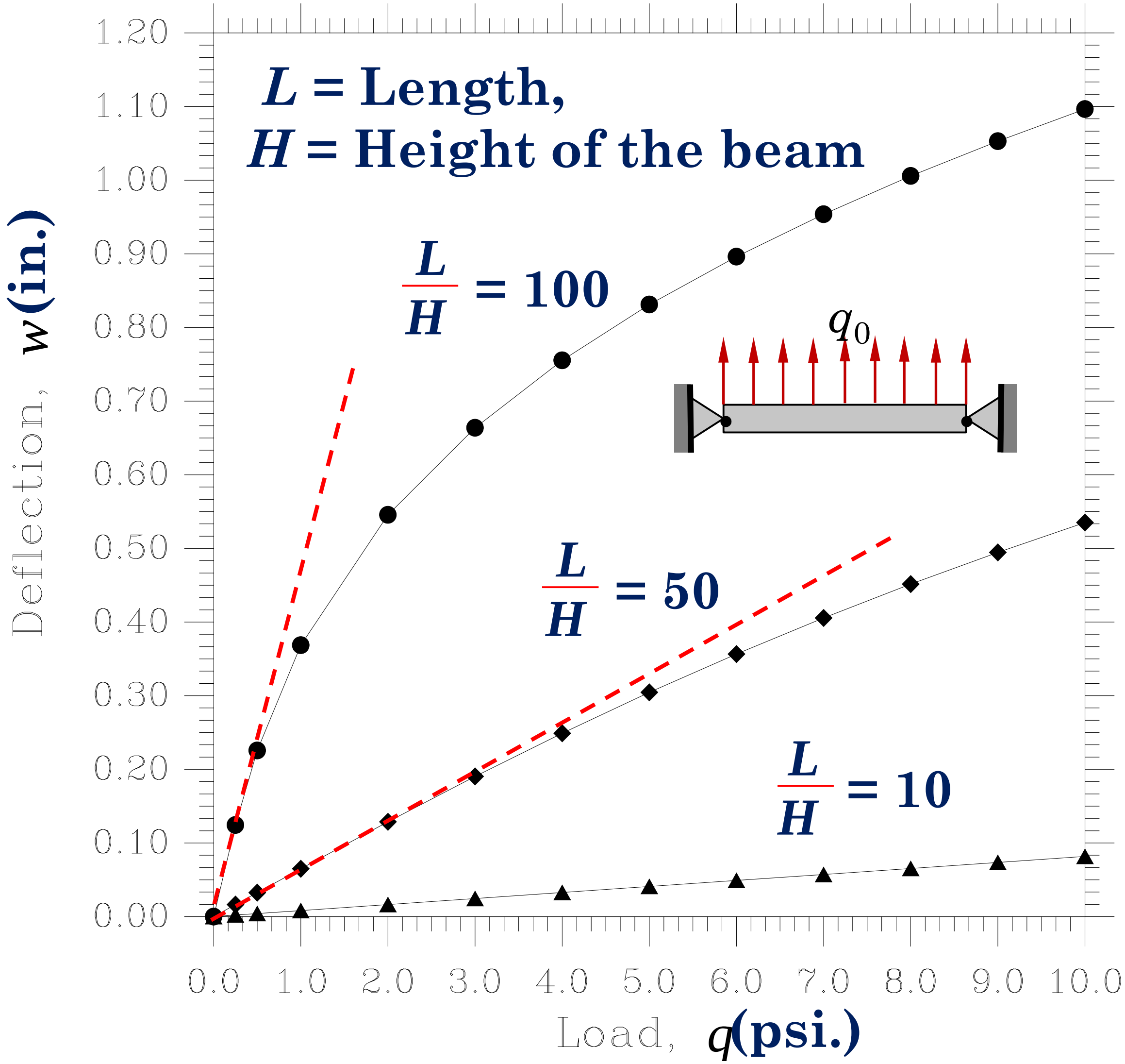
```
C     Compute the residual vector and tangent coefficient matrix for
      the Newton iteration method (only)
C
         IF(NONLIN.GT.1)THEN
            DO 230 I=1,NET
            DO 230 J=1,NET
  230    ELF(I)=ELF(I)-ELK(I,J)*ELU(J)
         II=1
         DO 260 I=1,NPE
         JJ=1
         DO 250 J=1,NPE
         ELK(II,JJ+1)     = ELK(II,JJ+1)    +TAN12(I,J)
         ELK(II+1,JJ+1) = ELK(II+1,JJ+1)+TAN22(I,J)
         IF(MODEL.EQ.2)THEN
            ELK(II,JJ+2)     = ELK(II,JJ+2)    +TAN13(I,J)
            ELK(II+1,JJ+2) = ELK(II+1,JJ+2)+TAN23(I,J)
            ELK(II+2,JJ+1) = ELK(II+2,JJ+1)+TAN32(I,J)
            ELK(II+2,JJ+2) = ELK(II+2,JJ+2)+TAN33(I,J)
         ENDIF
  250   JJ=NDF*J+1
  260   II=NDF*I+1
ENDIF
```

NUMERICAL EXAMPLES

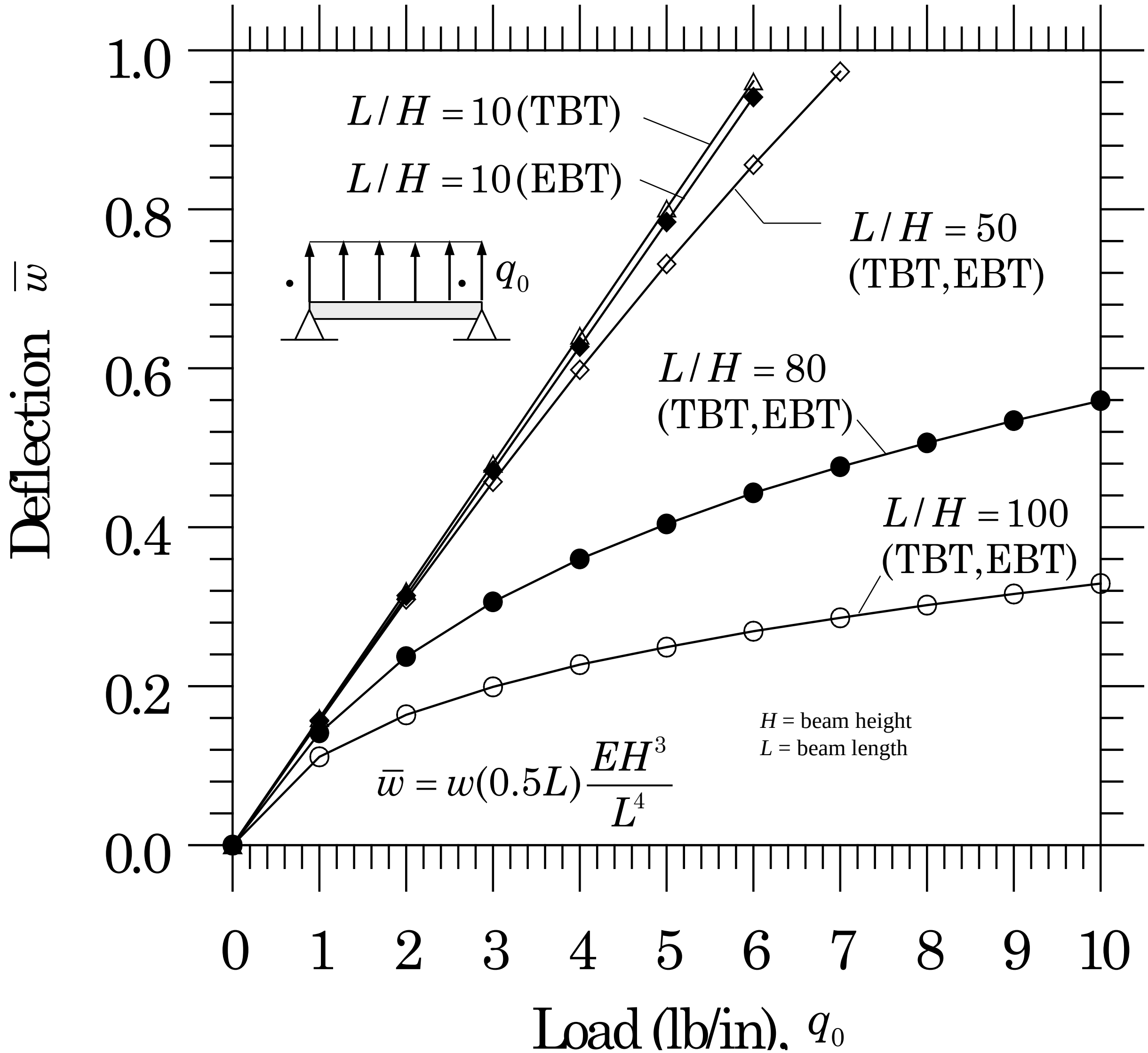
Pinned-pinned beam (EBT)



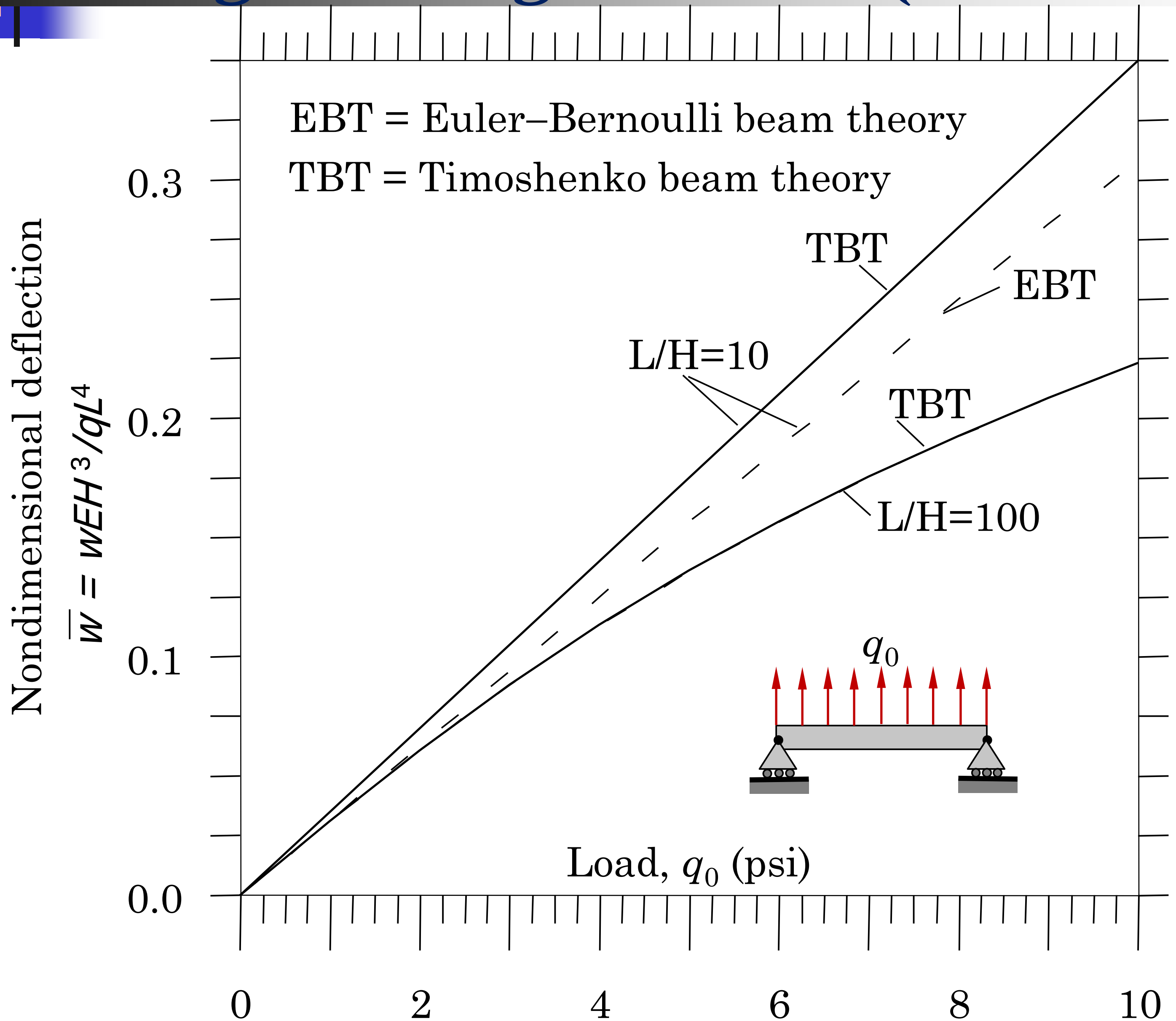
Pinned-pinned beam (TBT)



Pinned-pinned beam (EBT, TBT)



Hinged-Hinged beam (EBT and TBT)



SUMMARY

In this lecture we have covered the following topics:

- Derived the governing equations of the **Euler-Bernoulli beam theory**
- Derived the governing equations of the **Timoshenko beam theory**
- Developed Weak forms of EBT and TBT
- Developed Finite element models of EBT and TBT
- Discussed **membrane locking** (due to the geometric nonlinearity)
- Discussed **shear locking** in Timoshenko beam finite element
- Discussed examples