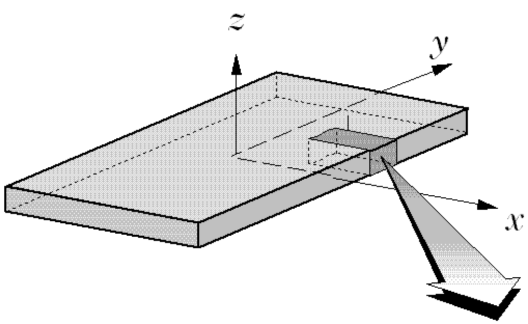
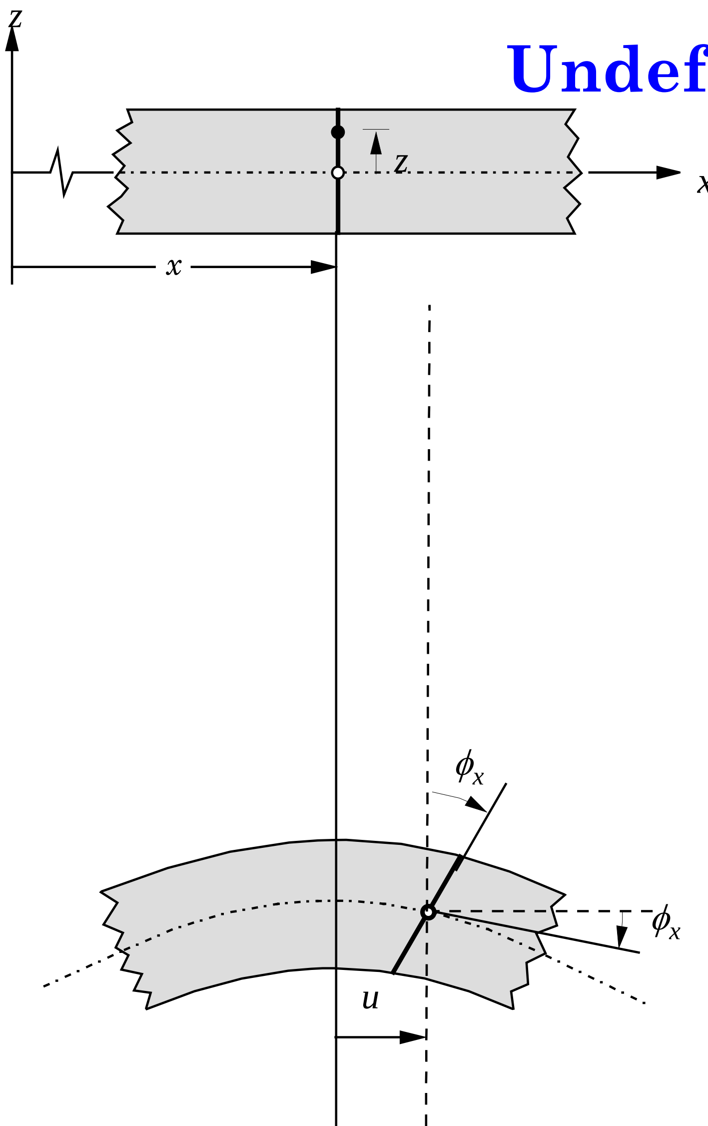
FINITE ELEMENT MODELS OF NONLINEAR PLATE BENDING

CONTENTS

- Finite element models of the FSDT
- Shear and Membrane Locking
- Numerical Examples

Kinematics of the Classical and Shear Deformation Plate Theories





Undeformed Edge

First-Order (Mindlin) Plate Theory (FST)

GOVERNING EQUATIONS OF THE FSDT
with the *von Karman Nonlinearity*

Nonlinear strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial u_m}{\partial x_1}\frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

Von Karman Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \phi_x$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \phi_y$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

EQUATIONS OF MOTION OF FSDT

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}+f_x=I_0\frac{\partial^2 u}{\partial t^2}+I_1\frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}+f_y=I_0\frac{\partial^2 v}{\partial t^2}+I_1\frac{\partial^2 \phi_y}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)$$

$$+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_{yy}\frac{\partial w}{\partial y}\right)+q=I_0\frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=I_2\frac{\partial^2 \phi_x}{\partial t^2}+I_1\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_y=I_2\frac{\partial^2 \phi_y}{\partial t^2}+I_1\frac{\partial^2 v}{\partial t^2}$$

FINITE ELEMENT MODELS OF (FSDT)

Weak Forms (from the principle of virtual displacements)

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} - \delta u\, f_x + I_0\, \delta u \frac{\partial^2 u}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta u_n\, N_{nn}\, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_{xy} + \frac{\partial \delta v}{\partial y} N_{yy} - \delta v\, f_y + I_0\, \delta v \frac{\partial^2 v}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta u_s\, N_{ns}\, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta w}{\partial x} Q_x + \frac{\partial \delta w}{\partial y} Q_y + \delta w\, N + I_0\, \delta w \frac{\partial^2 w}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta w\, Q_n ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_x}{\partial x} M_{xx} + \frac{\partial \delta \phi_x}{\partial y} M_{xy} + \delta \phi_x\, Q_x + I_2\, \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta \phi_n M_{nn} ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_y}{\partial x} M_{xy} + \frac{\partial \delta \phi_y}{\partial y} M_{yy} + \delta \phi_y Q_y + I_2\, \delta \phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta \phi_s M_{ns} ds$$

$$Q_n = Q_x n_x + Q_y n_y, \quad M_{nn} = M_{xx} n_x + M_{xy} n_y, \quad M_{ns} = M_{xy} n_x + M_{yy} n_y$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY

Stress Resultants (Nonlinear)

$$N_{xx} = A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$+ B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{xx}^T$$

$$N_{yy} = A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$+ B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{yy}^T$$

$$N_{xy} = A_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$+ B_{16}\frac{\partial \phi_x}{\partial x} + B_{26}\frac{\partial \phi_y}{\partial y} + B_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{xy}^T$$

Stress Resultants (Nonlinear)

$$M_{xx} = B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$- D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T$$

$$M_{yy} = B_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$- D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T$$

$$M_{xy} = B_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$- D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \; Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

FINITE ELEMENT MODELS OF (FSDT) (continued)

Finite element approximation

$$u \approx \sum_{j=1}^{m} u_j(t)\psi_j^{(0)}(x,y), \quad v \approx \sum_{j=1}^{m} v_j(t)\psi_j^{(0)}(x,y)$$

$$w(x,y,t) = \sum_{j=1}^{m} w_j(t)\psi_j^{(1)}(x,y)$$

$$\phi_x(x,y,t) = \sum_{j=1}^{n} S_{xj}(t)\psi_j^{(2)}(x,y), \quad \phi_y(x,y,t) = \sum_{j=1}^{n} S_{yj}(t)\psi_j^{(2)}(x,y)$$

Although, in general, different degree of interpolation can be used for various field variables, the same degree of interpolation is used for all variables:

$$\psi_j^{(0)} = \psi_j^{(1)} = \psi_j^{(2)}$$

FINITE ELEMENT MODELS OF FSDT (continued)

Semidiscrete finite element model

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{25} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{14} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{25} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{55} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \\ \ddot{\mathbf{w}} \\ \ddot{\mathbf{S}}_x \\ \ddot{\mathbf{S}}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ \mathbf{S}_x \\ \mathbf{S}_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \\ \mathbf{F}^{3} \\ \mathbf{F}^{4} \\ \mathbf{F}^{5} \end{Bmatrix}$$

$$M_{ij}^{11} = I_0 M_{ij}, \quad M_{ij}^{22} = M_{ij}^{33} = I_2 M_{ij}, \quad M_{ij} = \int_{\Omega_e} \psi_i \psi_j \, dx\, dy$$

$$K_{ij}^{11} = \int_{\Omega_e} \left(A_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx\; dy$$

$$K_{ij}^{12} = \int_{\Omega_e} A_{55} \frac{\partial \psi_i}{\partial x} \psi_j \, dx\, dy$$

$$K_{ij}^{13} = \int_{\Omega_e} A_{44} \frac{\partial \psi_i}{\partial y} \psi_j \, dx\, dy$$

Use reduced integration to avoid shear locking

FINITE ELEMENT MODELS OF FSDT (continued)

$$K_{ij}^{22} = \int_{\Omega_e} \left(D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{55} \psi_i \psi_j \right) dx\, dy$$

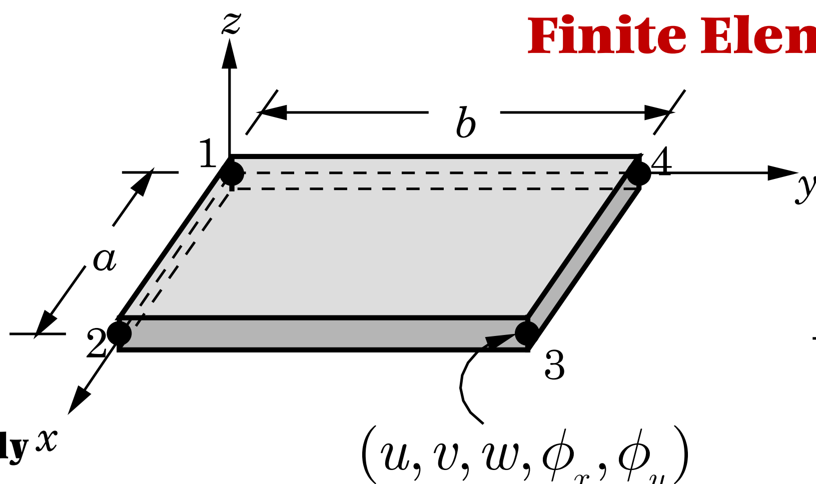
$$K_{ij}^{23} = \int_{\Omega_e} \left(D_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dx\, dy$$

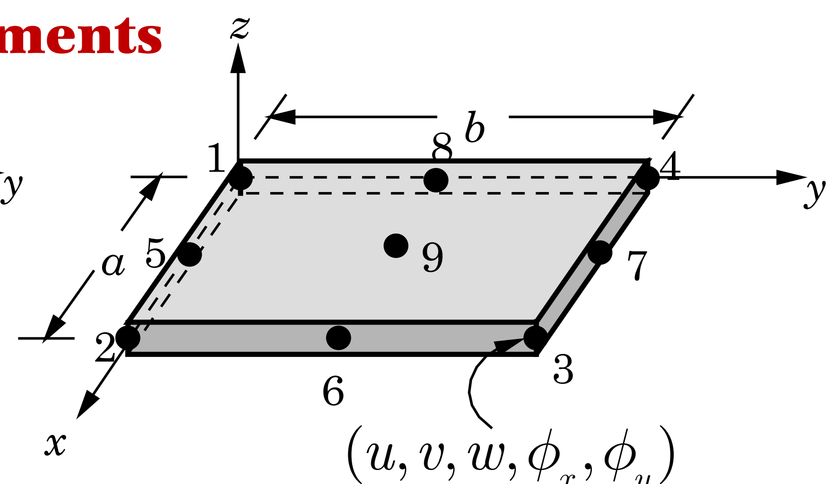
$$K_{ij}^{33} = \int_{\Omega_e} \left(D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{44} \psi_i \psi_j \right) dx\, dy$$

$$F_i^1 = \int_{\Omega_e} q \psi_i \, dx\, dy + \oint_{\Gamma_e} Q_n \psi_i \, ds$$

$$F_i^2 = \oint_{\Gamma_e} \hat{M}_{nn} \psi_i \, ds, \quad F_i^3 = \oint_{\Gamma_e} \hat{M}_{ns} \psi_i \, ds$$

Finite Elements





Shear and Membrane Locking (Revisit)

Shear Locking

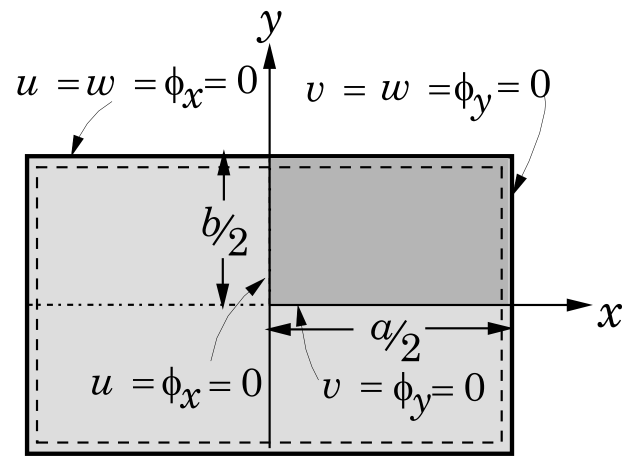
Use reduced integration to evaluate all *shear* stiffnesses (i.e., all K_{ij} that contain transverse shear terms)

Membrane Locking

Use reduced integration to evaluate all *membrane* stiffnesses (i.e., all K_{ij} that contain von Kármán nonlinear terms)

NUMERICAL EXAMPLES

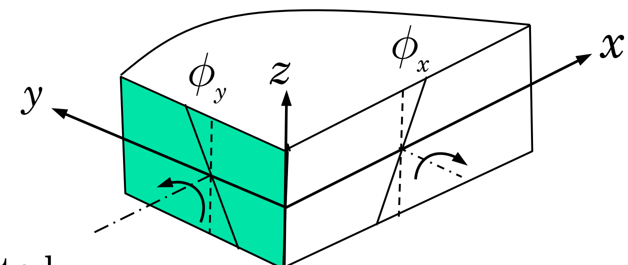
Simply Supported Plate (SS2)



$$\bar{w} = w_0(0,0)\frac{E_2 h^3}{a^4 q_0}, \quad \bar{\sigma}_{xx} = \sigma_{xx}(0,0,\frac{h}{2})\frac{h^2}{b^2 q_0}$$

$$\bar{\sigma}_{yy} = \sigma_{yy}(0,0,\frac{h}{4})\frac{h^2}{b^2 q_0}, \quad \bar{\sigma}_{xy} = \sigma_{xy}(\frac{a}{2},\frac{b}{2},-\frac{h}{2})\frac{h^2}{b^2 q_0}$$

$$\bar{\sigma}_{xz} = \sigma_{xz}(\frac{a}{2},0,-\frac{h}{2})\frac{h}{bq_0}\;, \quad \bar{\sigma}_{yz} = \sigma_{yz}(0,\frac{b}{2},\frac{h}{2})\frac{h}{bq_0}$$



$$\sigma_{xx}(A,A,\frac{h}{2}), \quad \sigma_{xy}(B,B,-\frac{h}{2}), \quad \sigma_{xz}(B,A,-\frac{h}{2})$$

Table: The Gauss point locations at which the stresses are computed in the finite element analysis of simply supported plates.

Point	2L	4L	8L	1Q9	2Q9	4Q9
A	$0.125a$	$0.0625a$	$0.0312a$	$0.1056a$	$0.0528a$	$0.0264a$
B	$0.375a$	$0.4375a$	$0.4687a$	$0.3943a$	$0.4472a$	$0.4736a$

Effect of Quadrature Rules and Shear Deformation on Deflection and Stresses

a/h	Mesh	$\bar{w} \times 10^2$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$
Finite Element Solutions†					
10	2L–F	2.4742	0.1185	0.0727	0.2627
	2L–S	4.7120	0.2350	0.1446	0.2750
	2L–R	4.8887	0.2441	0.1504	0.2750
	1Q–F	4.5304	0.2294	0.1610	0.2813
	1Q–S	4.9426	0.2630	0.1639	0.2847
	1Q–R	4.9711	0.2645	0.1652	0.2886
	4L–F	3.8835	0.2160	0.1483	0.3366
	4L–S	4.7728	0.2661	0.1850	0.3356
	4L–R	4.8137	0.2684	0.1869	0.3356
	2Q–F	4.7707	0.2699	0.1930	0.3437
	2Q–S	4.7989	0.2715	0.1939	0.3424
	2Q–R	4.8005	0.2716	0.1943	0.3425
	8L–F	4.5268	0.2590	0.1891	0.3700
	8L–S	4.7966	0.2743	0.2743	0.2014
	8L–R	4.7866	0.2737	0.2737	0.2008
	4Q–F	4.7897	0.2749	0.2044	0.3737
	4Q–S	4.7916	0.2750	0.2043	0.3735
	4Q–R	4.7917	0.2750	0.2044	0.3735
Analytical Solutions					
	[15]	4.7914	0.2762	0.2085	0.3927

Square plate under UDL

F – full integration

S – Selective integration

R- Reduced integration

L – Linear element

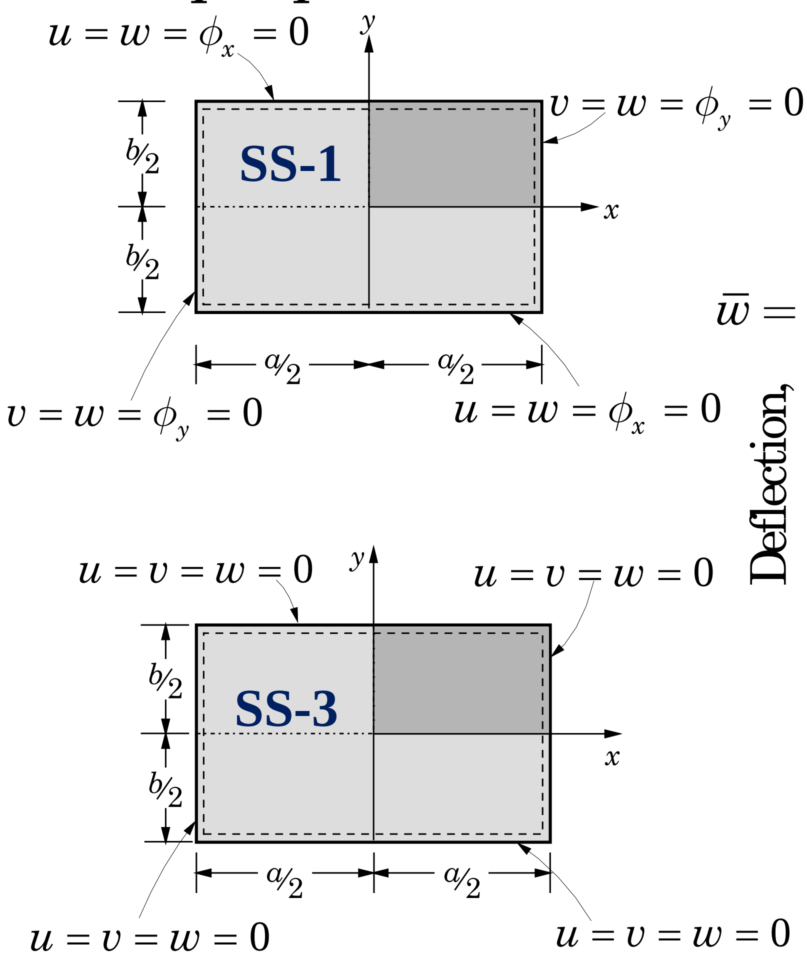
Q- Qudratic element

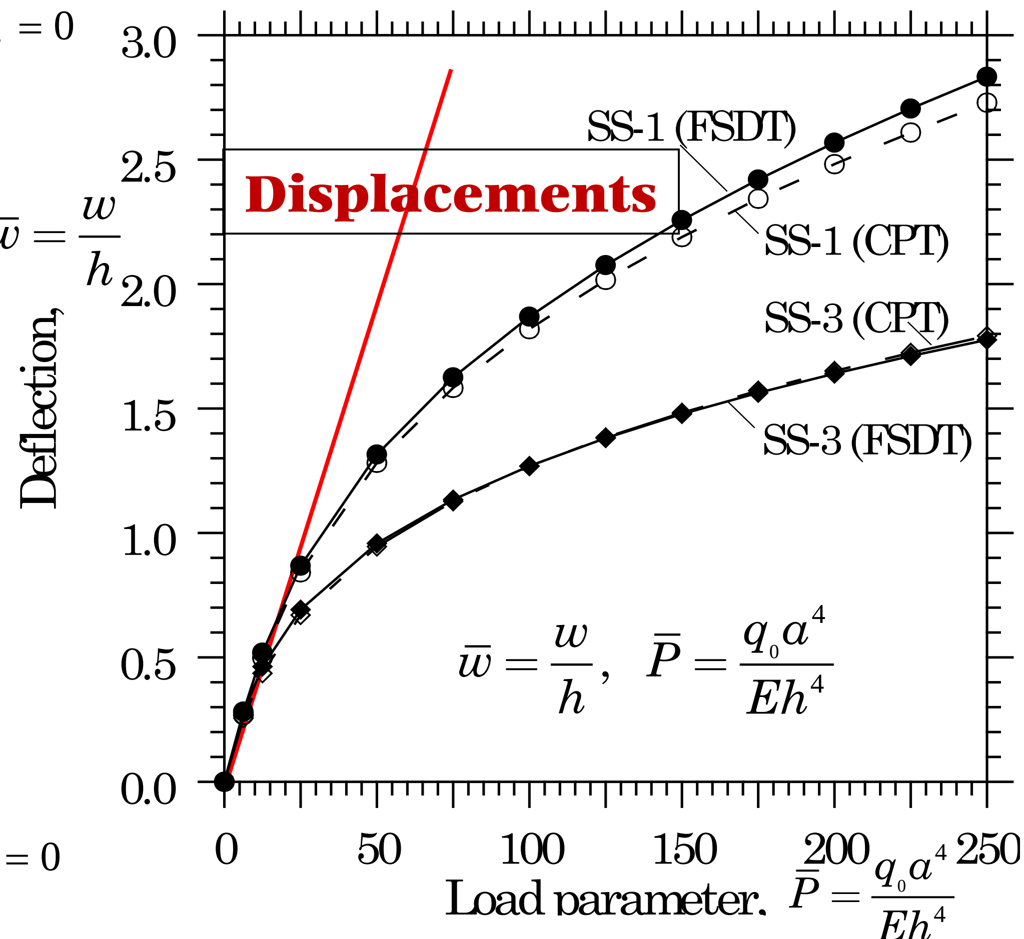
Effect of Quadrature Rules and Shear Deformation on Deflection and Stresses

a/h	Mesh	$\bar{w} \times 10^2$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$
Finite Element Solutions					
100	2L–F	0.0469	0.0024	0.0014	0.2635
	2L–S	4.4645	0.2350	0.1446	0.2750
	2L–R	4.6412	0.2441	0.1504	0.2750
	1Q–F	4.0028	0.2040	0.1591	0.2733
	1Q–S	4.7196	0.2629	0.1643	0.2837
	1Q–R	4.7483	0.2645	0.1652	0.2886
	4L–F	0.1819	0.0108	0.0071	0.3462
	4L–S	4.5481	0.2661	0.1850	0.3356
	4L–R	4.5890	0.2684	0.1869	0.3356
	2Q–F	4.4822	0.2644	0.1893	0.3485
	2Q–S	4.5799	0.2715	0.1941	0.3414
	2Q–R	4.5815	0.2716	0.1943	0.3425
	8L–F	0.6497	0.0401	0.0275	0.3847
	8L–S	4.5664	0.2737	0.2008	0.3691
	8L–R	4.5764	0.2743	0.2014	0.3691
	4Q–F	4.5530	0.2741	0.2020	0.3749
	4Q–S	4.5728	0.2750	0.2044	0.3734
	4Q–R	4.5729	0.2750	0.2044	0.3735
Analytical Solutions					
	[15]	4.5698	0.2762	0.2085	0.3927

Bending of Simply Supported Plates

Isotropic plate under uniformly distributed transverse load





$a = b = 10$ in, $h = 1$ in, $E = 7.8 \times 10^6$ psi, $\nu = 0.3$

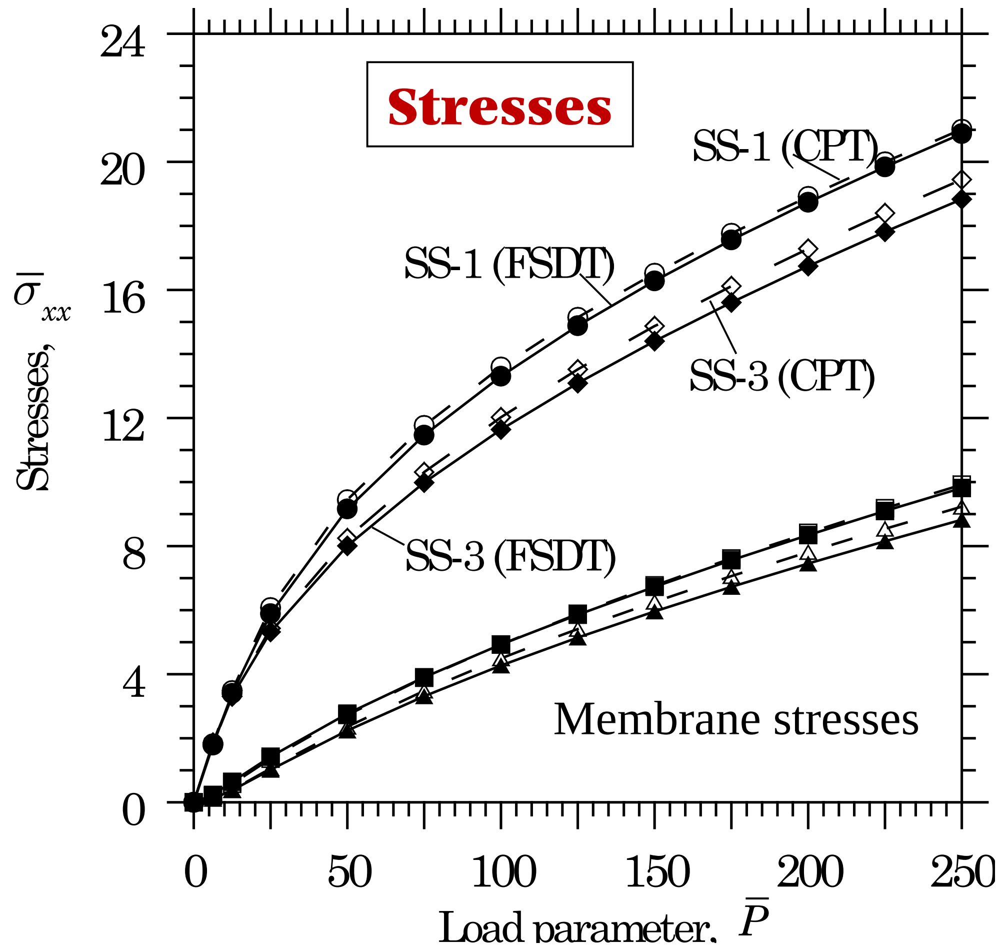
Bending of Simply Supported Plates

Isotropic plate under uniformly distributed transverse load

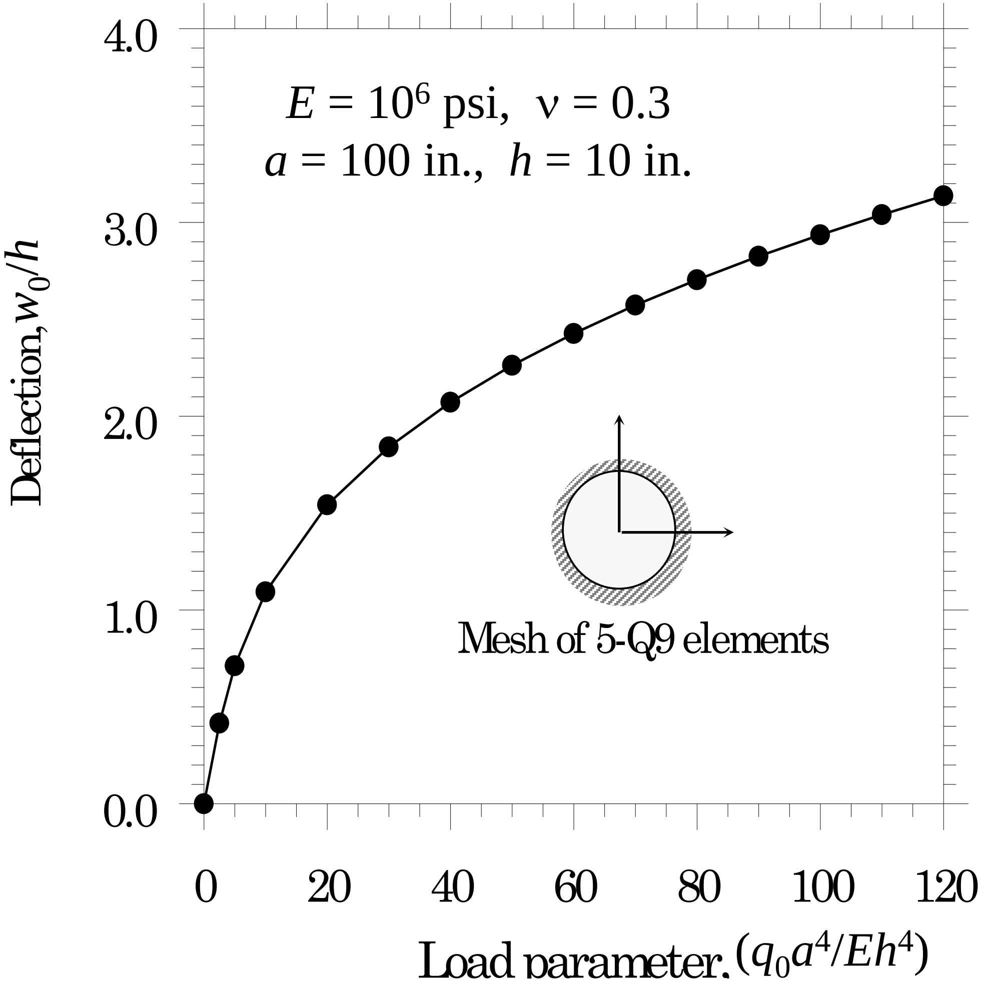
$$\bar{\sigma}_{xx} = \sigma_{xx}\left(\frac{a^2}{Eh^2}\right), \quad \bar{P} = \frac{q_0 a^4}{Eh^4}$$

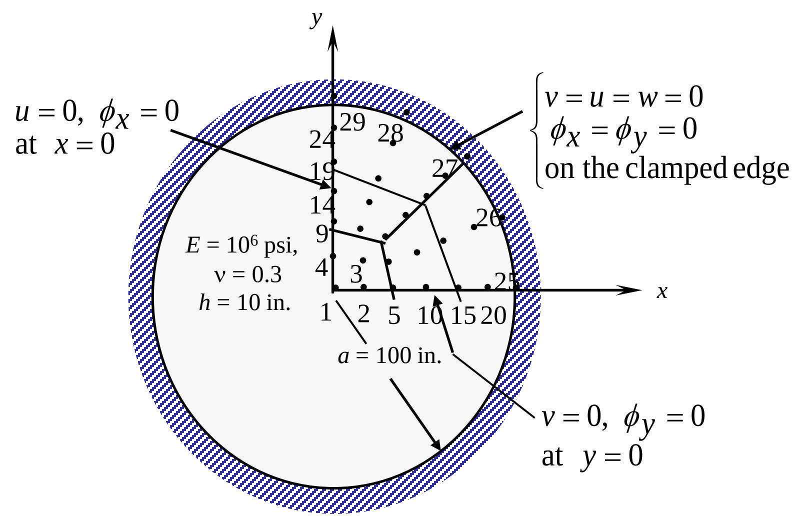
$$a = b = 10 \text{ in}, \; h = 1 \text{ in},$$

$$E = 7.8\times 10^6 \text{ psi}, \; \nu = 0.3$$

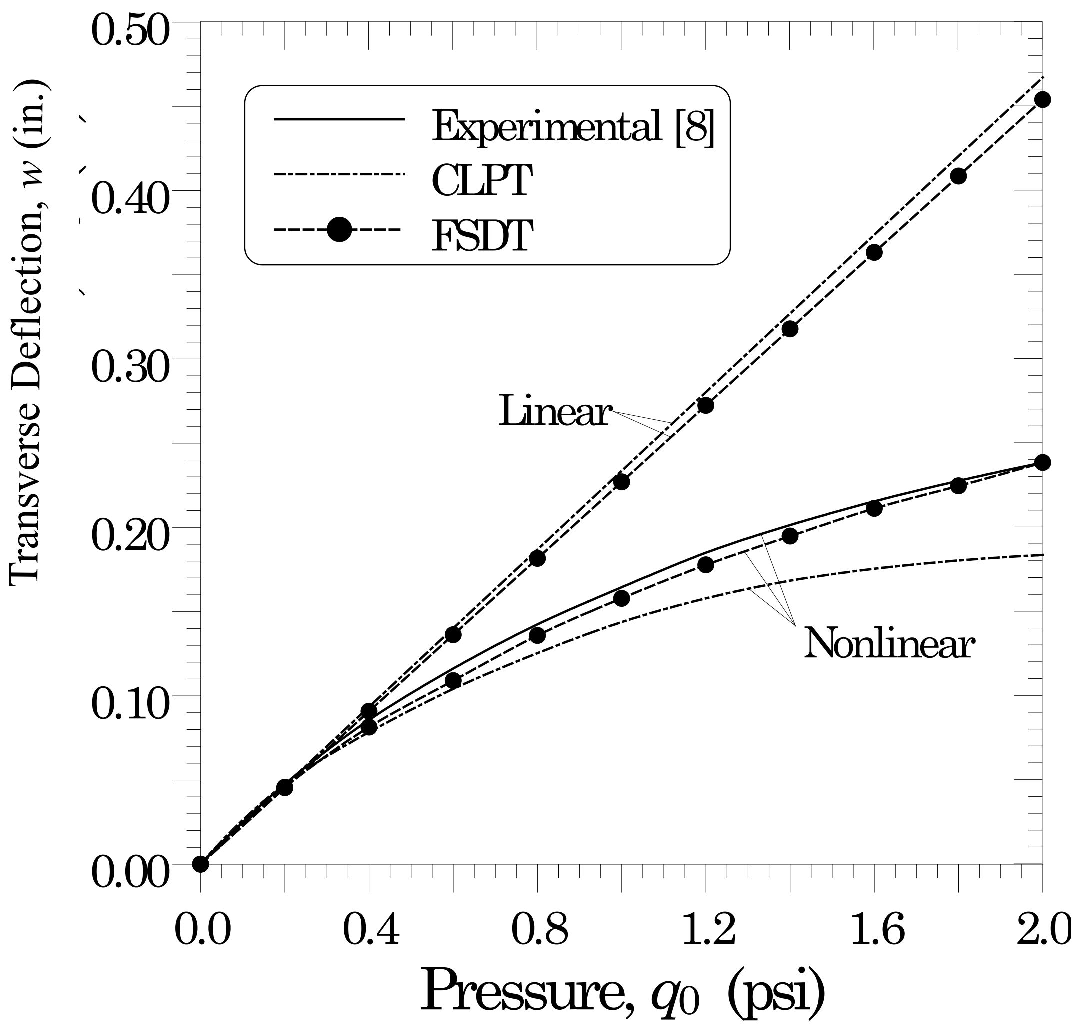


Clamped Circular Plate under UDL





Simply Supported (SS2) Orthotropic* Plate



Geometry and Material Properties

$a = b = 12$ in, $h = 0.138$ in

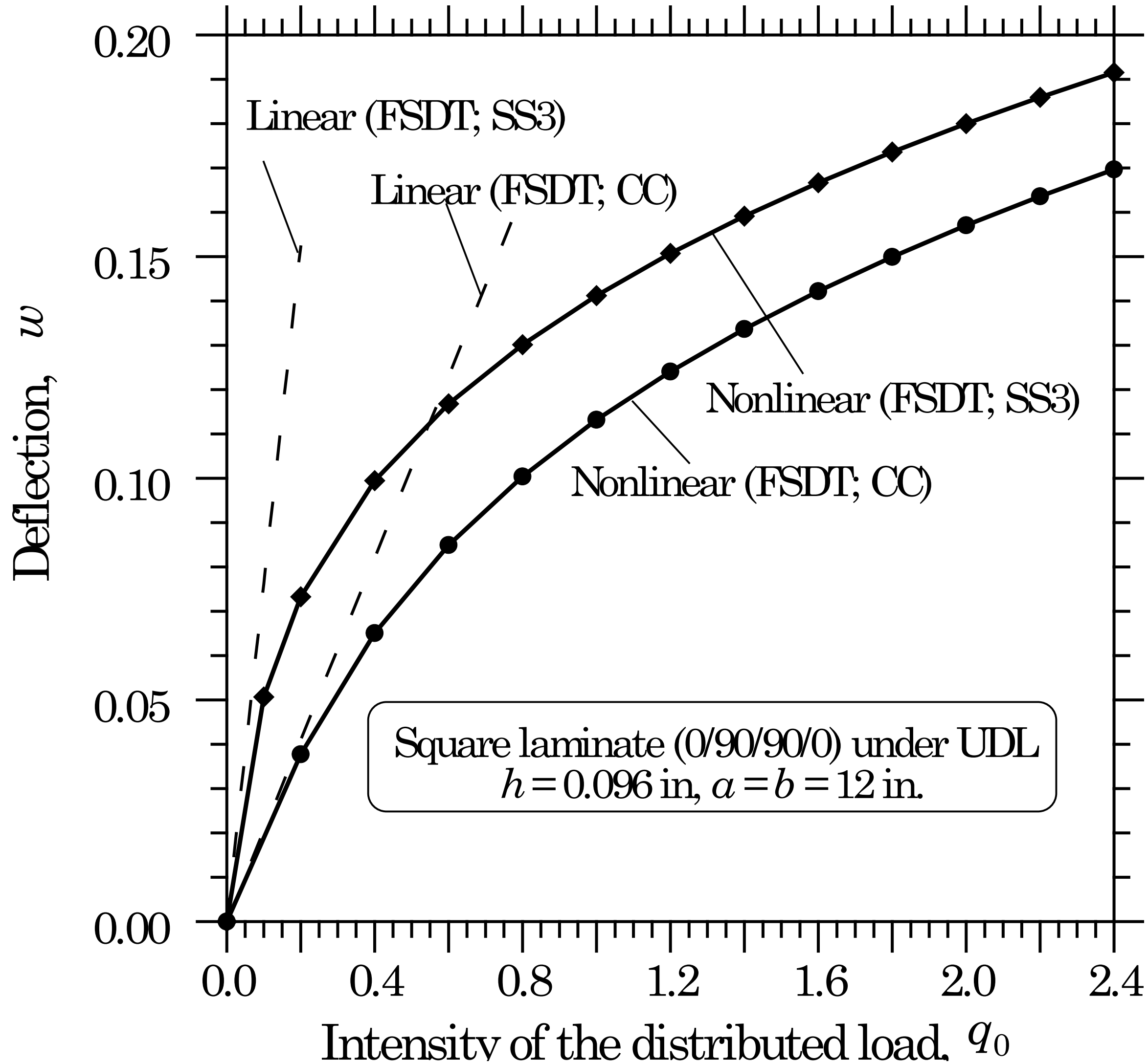
$E_1 = 3\times10^6$ psi, $E_2 = 1.28\times10^6$ psi

$G_{12} = G_{23} = G_{13} = 0.37\times10^6$ psi

$\nu_{12} = \nu_{23} = \nu_{13} = 0.32$

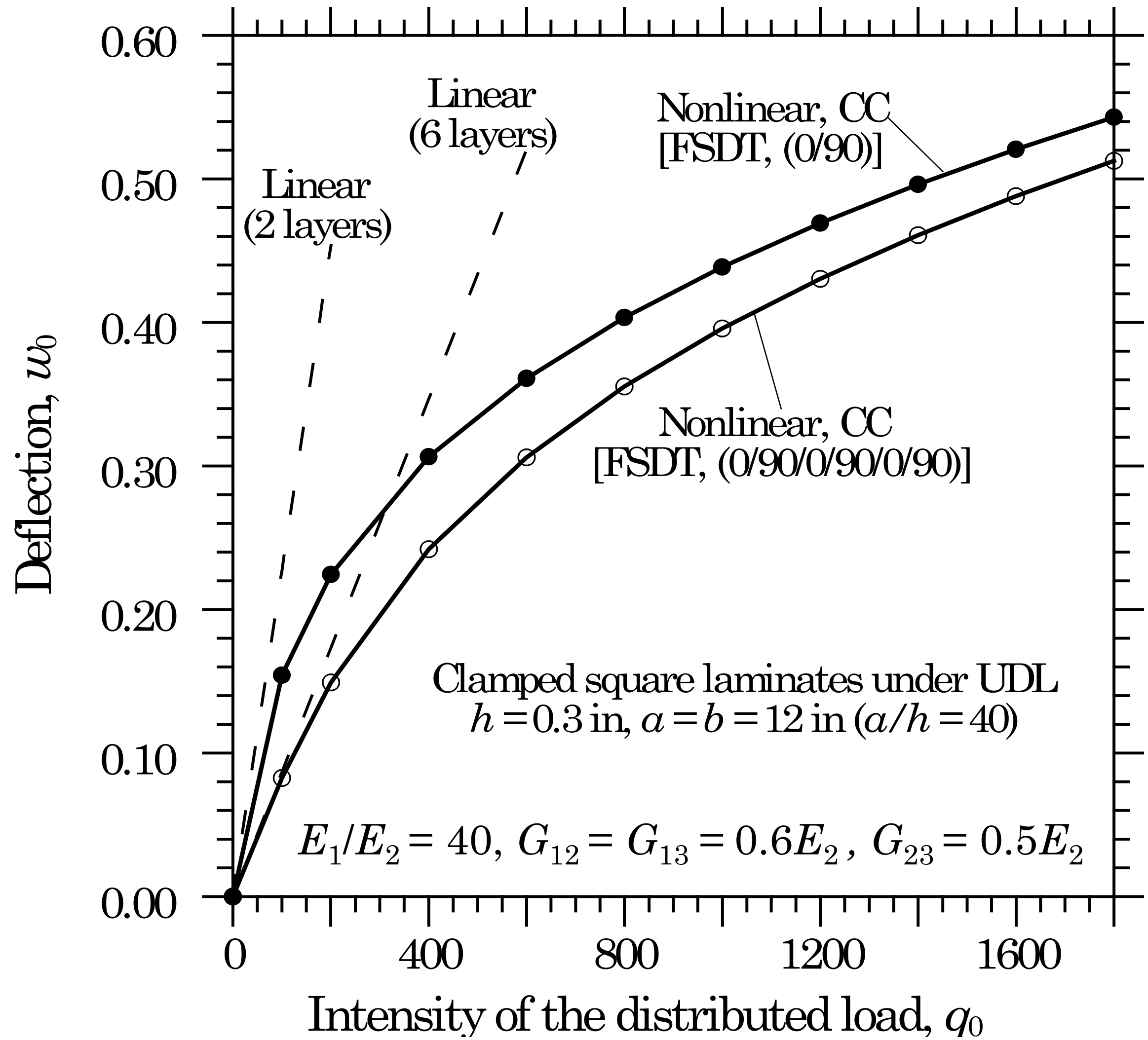
[8] Zaghloul, S. A. and Kennedy, J. B., ``Nonlinear Behavior of Symmetrically Laminated Plates," *Journal of Applied Mechanics*, 42, 234-236, 1975.

DEFLECTION VS. LOAD PARAMETER FOR (0/90/90/0) LAMINATE UDL



$E_1 = 1.8282 \times 10^6$ psi,
$E_2 = 1.8315 \times 10^6$ psi,
$G_{12} = G_{13} = G_{23}$
$= 0.3125 \times 10^6$ psi,
$\nu_{12} = 0.2395$

DEFLECTION VS. LOAD PARAMETER FOR TWO- AND SIX-LAYER CROSS-PLY LAMINATES UDL



SUMMARY

In this lecture we have covered the following topics:

- **Governing equations of FSDT**
- **Finite element models of FSDT**
- **Shear and membrane locking**
- **Numerical examples**