HIGHER-ORDER THEORIES

- **Third-order Shear Deformation Plate Theory**
  - Displacement and strain fields
  - Equations of motion
  - Navier’s solution for bending

- **Layerwise Laminate Theory**
  - Interlaminar stress and strain continuity
  - Equations of motion
  - Numerical results

TSDT–Displacement Field (continued)

Reduction of the Displacement Field : Require that the top and bottom faces of the plate are free of shear stress

$$\sigma_{xz}(x,y,\pm\frac{h}{2},t)=\sigma_{yz}(x,y,\pm\frac{h}{2},t)=0$$

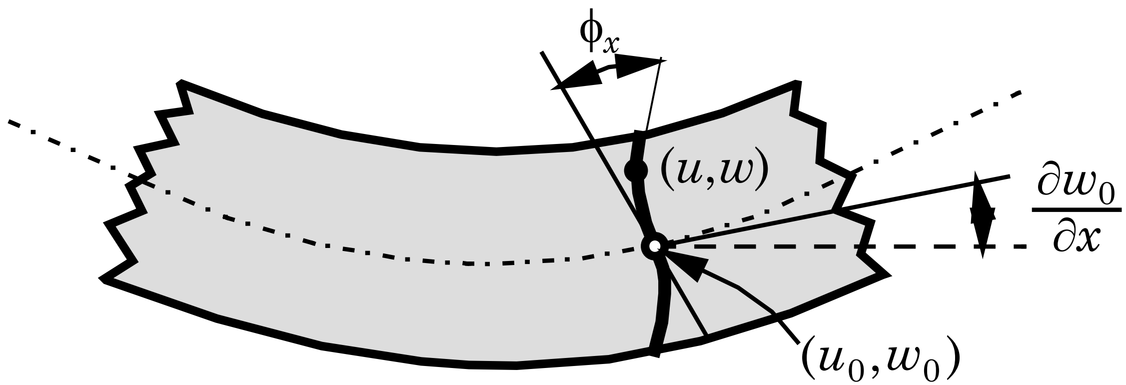
$$\Rightarrow \gamma_{xz}(x,y,\pm\frac{h}{2},t)=\gamma_{yz}(x,y,\pm\frac{h}{2},t)=0$$

$$\gamma_{xz}=\phi_x+\frac{\partial w}{\partial x}+2z\,\psi_x+3z^2\theta_x$$

$$\phi_x+\frac{\partial w}{\partial x}-h\,\psi_x+\frac{3h^2}{4}\theta_x=0,\ \ \phi_x+\frac{\partial w}{\partial x}+h\,\psi_x+\frac{3h^2}{4}\theta_x=0$$

$$\Rightarrow \boxed{\theta_x=-\frac{4}{3h^2}\left(\phi_x+\frac{\partial w}{\partial x}\right),\qquad \psi_x=0}$$

Third-Order Shear Deformation Plate Theory (TSDT) –Displacement Field



Assumed Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t) + z^2\psi_x(x,y,t) + z^3\theta_x(x,y,t)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t) + z^2\psi_y(x,y,t) + z^3\theta_y(x,y,t)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

Displacement Field of the Reddy Third-Order Laminate Plate Theory (RLPT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w}{\partial x}\right)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

Strain Field

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(0)} + z\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(1)} + z^3\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(3)}$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(0)} + z^2\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(2)}$$

Strain Field of the Reddy Third-Order Laminate Plate Theory

Strain Field (continued)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(0)} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \frac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \frac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(1)} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \quad c_1 = \frac{4}{3h^2}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(3)} = -c_1 \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(0)} = -\frac{1}{3c_1}\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(2)} = \begin{Bmatrix} \phi_x + \dfrac{\partial w}{\partial x} \\ \phi_y + \dfrac{\partial w}{\partial y} \end{Bmatrix}$$

RLPT – Equations of Motion

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 + J_1 \ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v}_0 + J_1 \ddot{\phi}_y - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y}$$

Omit subscript '0' from u, v, and w

$$\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right)$$

$$+ c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}\right) + q = I_0 \ddot{w}_0 - c_1^2 I_6\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right)$$

$$+ c_1\left[I_3\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) + J_4\left(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}\right)\right]$$

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = J_1 \ddot{u}_0 + K_2 \ddot{\phi}_x - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial x}$$

$$\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = J_1 \ddot{v}_0 + K_2 \ddot{\phi}_y - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial y}$$

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta}, \quad \bar{Q}_\alpha = Q_\alpha - c_2 R_\alpha$$

RLPT – Definition of Stress Resultants

Conventional Stress Resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z \; dz$$

Higher-Order Stress Resultants

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z^3 dz, \quad \begin{Bmatrix} R_{x} \\ R_{y} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} z^2 dz$$

Mass Inertias

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(z\right)^i \; dz \quad \left(i = 0, 1, 2, \cdots, 6\right)$$

$$J_i = I_i - c_1 I_{i+2}, \;\; K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6$$

RLPT – Boundary Conditions

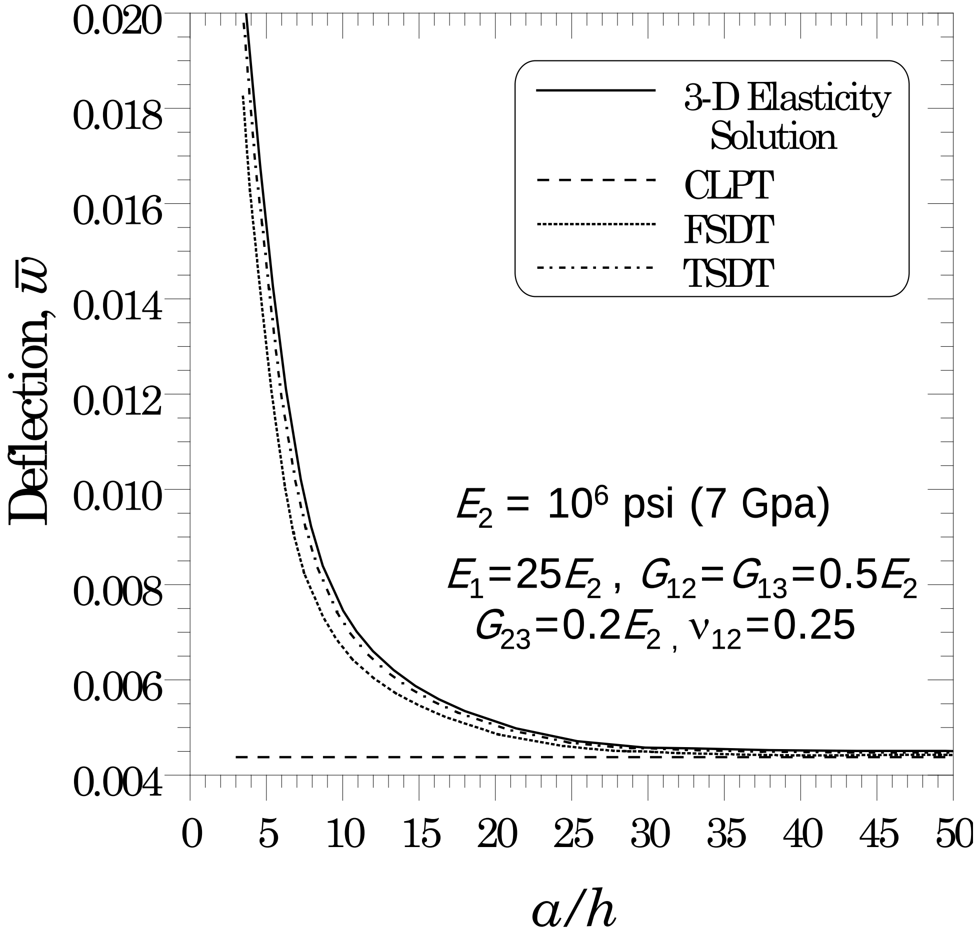
$$\text{Primary Variables}: \quad u_n,\ u_s,\ w_0,\ \frac{\partial w_0}{\partial n},\ \phi_n,\ \phi_s$$

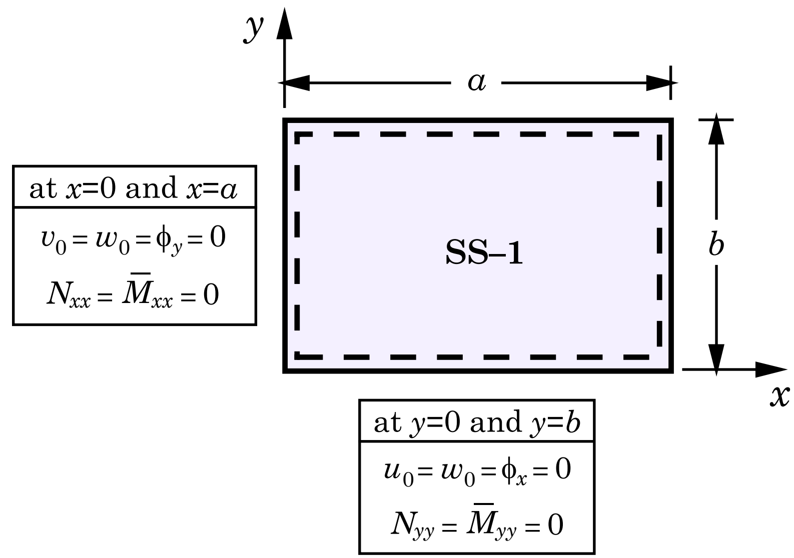
$$\text{Secondary Variables}: \quad N_{nn},\ N_{ns},\ \bar{V}_n,\ P_{nn},\ \bar{M}_{nn},\ \bar{M}_{ns}$$

$$\begin{aligned}
\bar{V}_n \equiv c_1 &\left[\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y}\right) n_x + \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y}\right) n_y\right] \\
&- c_1\left[\left(I_3\ddot{u}_0 + J_4\ddot{\phi}_x - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial x}\right) n_x + \left(I_3\ddot{v}_0 + J_4\ddot{\phi}_y - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial y}\right) n_y\right] \\
&+ \left(\bar{Q}_x n_x + \bar{Q}_y n_y\right) + \mathcal{P}(w_0) + c_1 \frac{\partial P_{ns}}{\partial s}
\end{aligned}$$

$$\mathcal{P}(w_0) = \left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) n_x + \left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) n_y$$

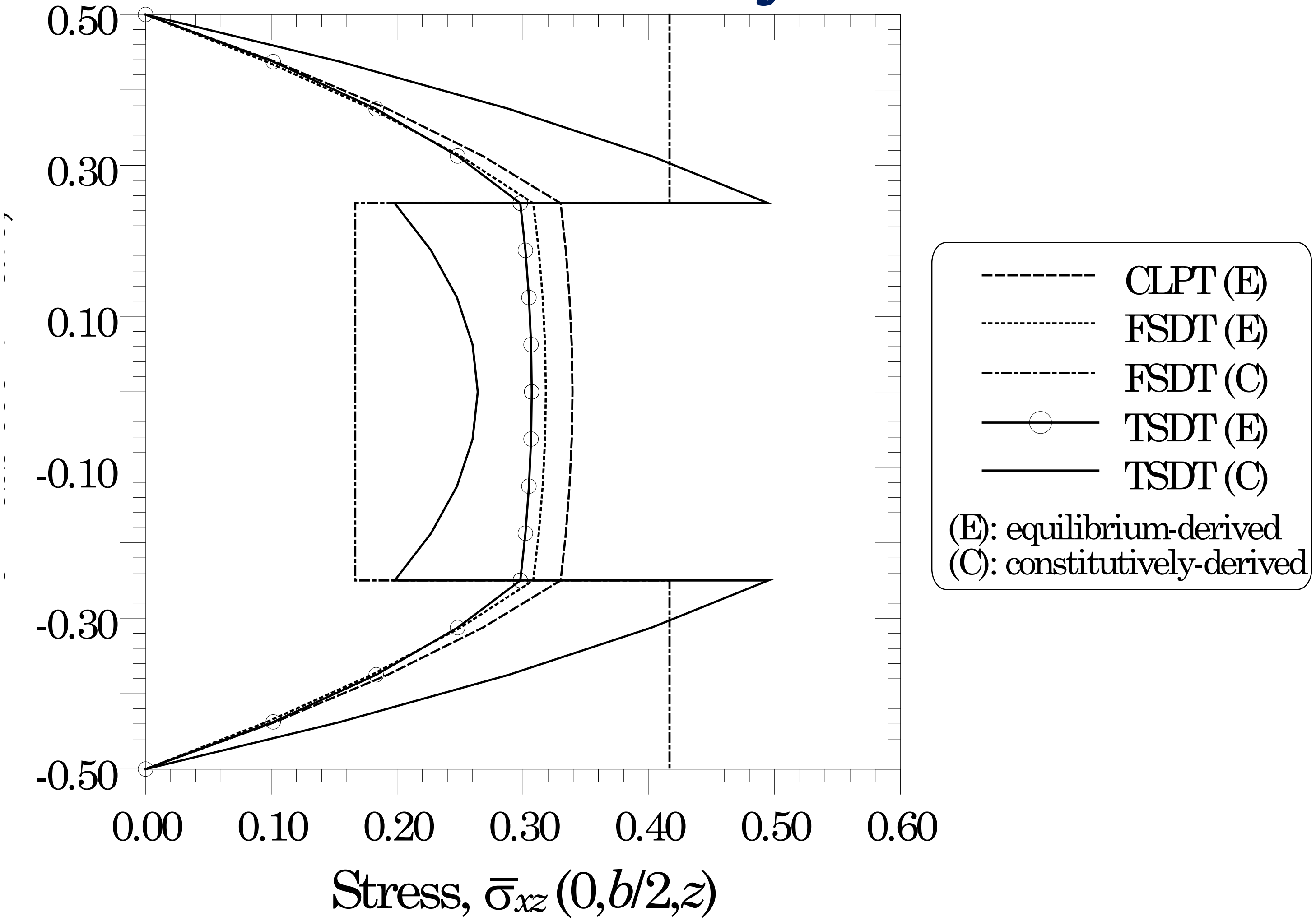
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



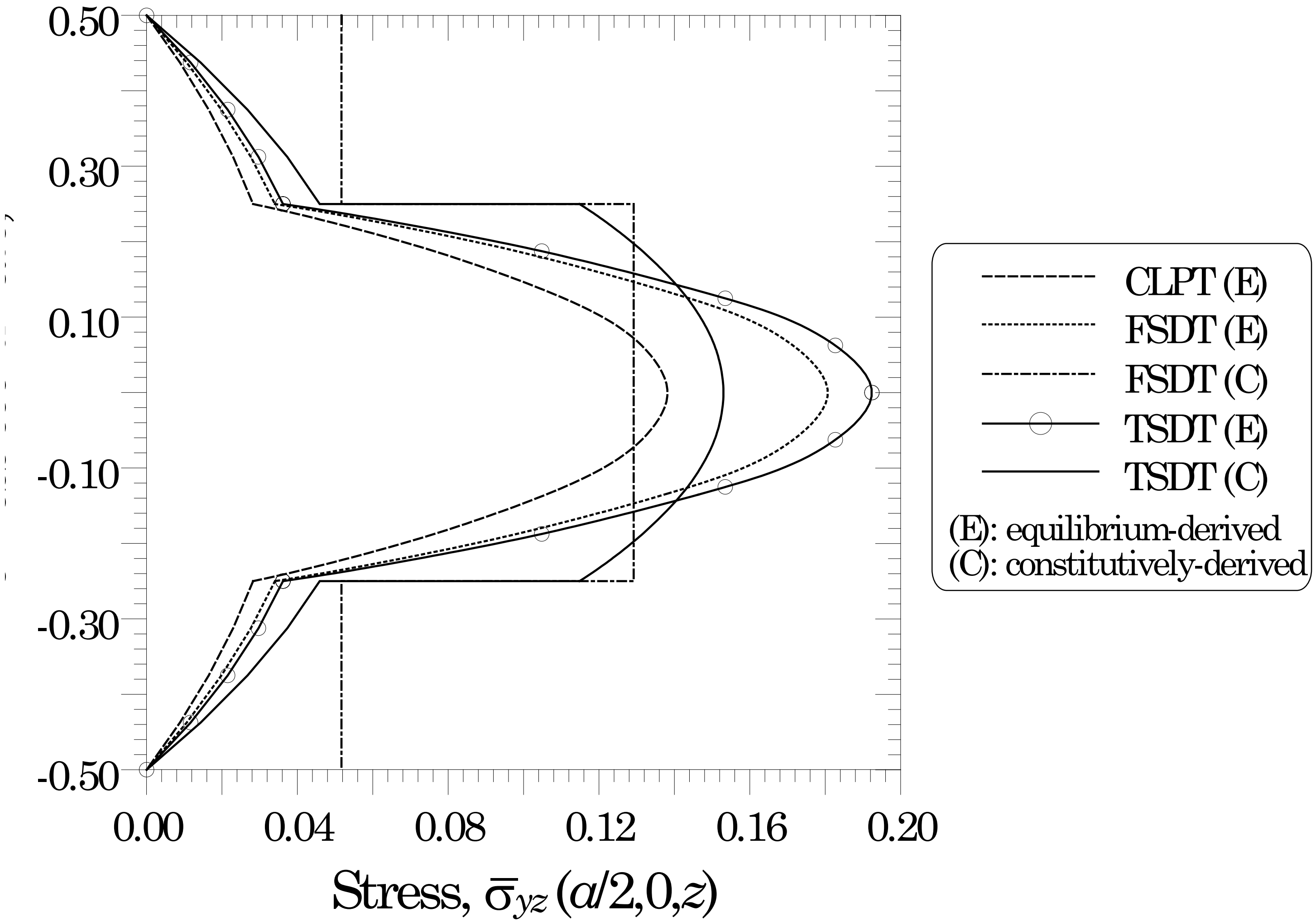


SS-1 Boundary Conditions

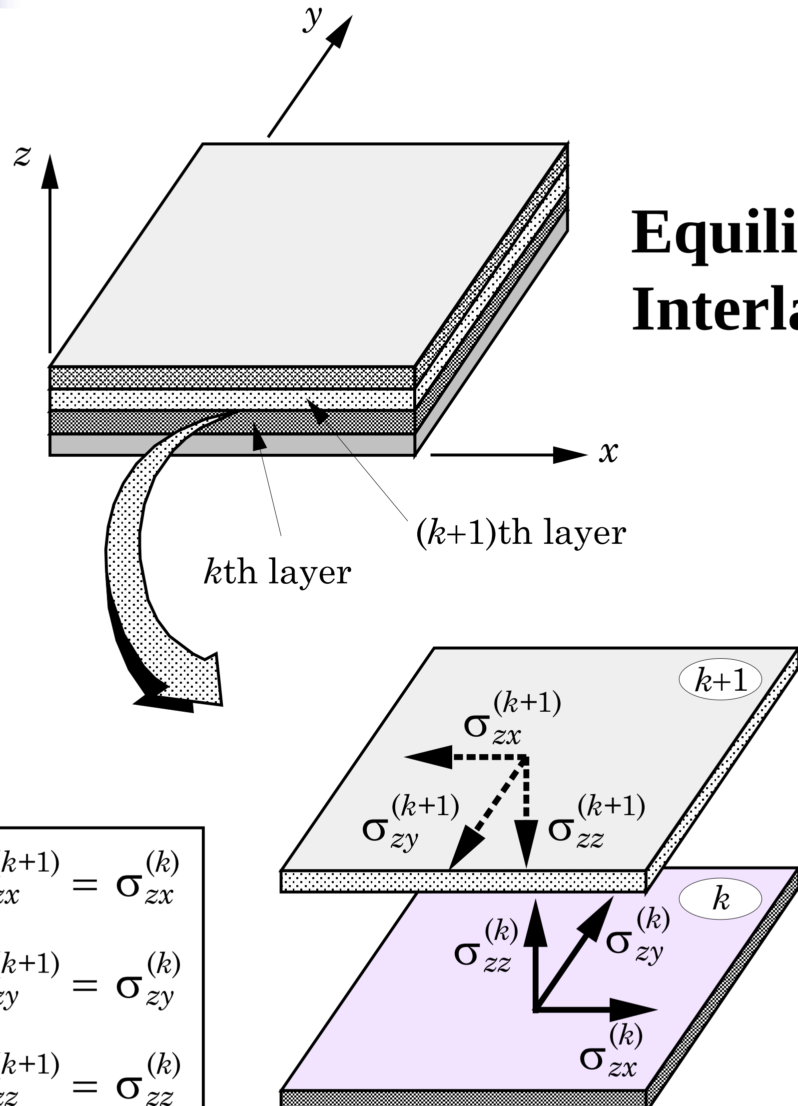
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



LAYERWISE LAMINATE THEORY

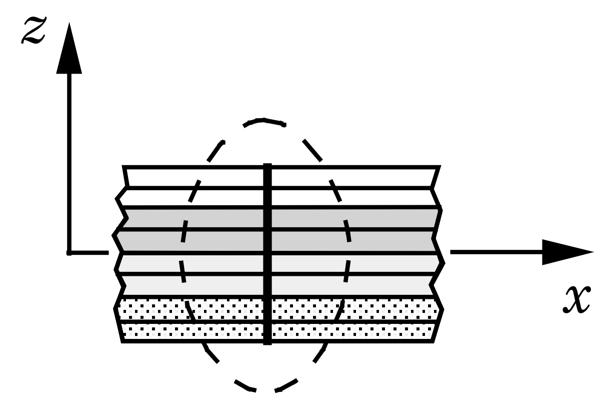


Equilibrium of Interlaminar Stresses

$$\sigma_{zx}^{(k+1)} = \sigma_{zx}^{(k)}$$

$$\sigma_{zy}^{(k+1)} = \sigma_{zy}^{(k)}$$

$$\sigma_{zz}^{(k+1)} = \sigma_{zz}^{(k)}$$



INTERLAMINAR STRESS AND STRAIN CONTINUITY

Equilibrium Requirements

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(k)} \neq \begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(k+1)} , \quad \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k)} = \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k+1)}$$

$$\begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k)} = \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k+1)} \rightarrow \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k)} \neq \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k+1)}$$

Single-Layer Theories

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(k)} \neq \begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(k+1)} , \quad \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k)} \neq \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k+1)}$$

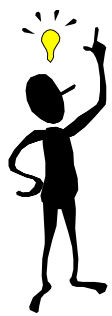
$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}^{(k)} = \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}^{(k+1)} , \quad \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k)} = \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k+1)}$$

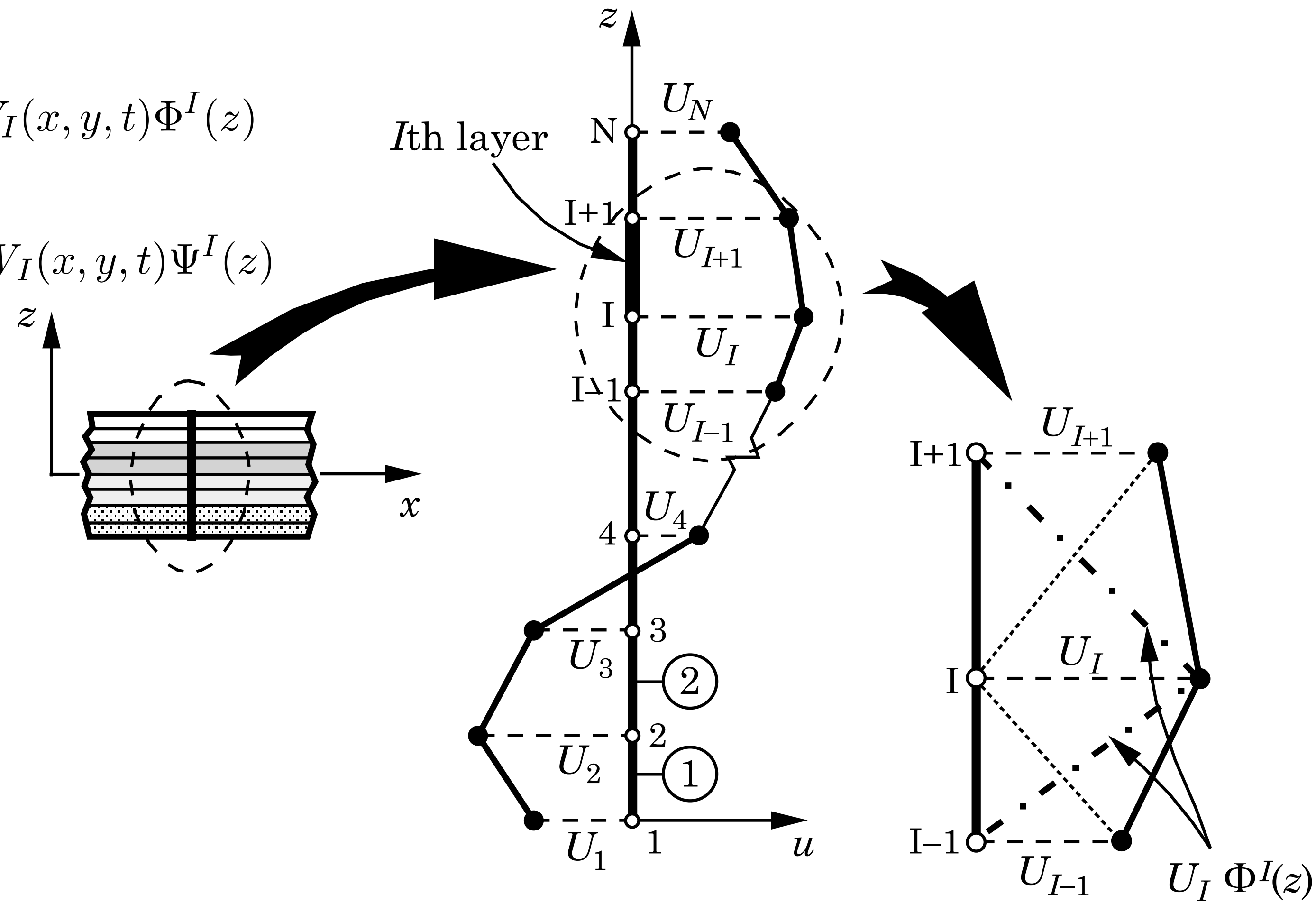
LAYERWISE KINEMATIC MODEL

$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$





Layerwise Displacement Field, Governing Equations, and FEM Approximation

$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

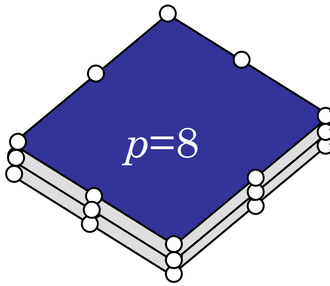
$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$

$$\frac{\partial N_{xx}^I}{\partial x} + \frac{\partial N_{xy}^I}{\partial y} - Q_x^I = \sum_{J=1}^{N} I^{IJ} \frac{\partial^2 U_J}{\partial t^2}$$

$$\frac{\partial N_{xy}^I}{\partial x} + \frac{\partial N_{yy}^I}{\partial y} - Q_y^I = \sum_{J=1}^{N} I^{IJ} \frac{\partial^2 V_J}{\partial t^2}$$

$$\frac{\partial \tilde{Q}_x^I}{\partial x} + \frac{\partial \tilde{Q}_y^I}{\partial y} - \tilde{Q}_z^I + \tilde{\mathcal{N}}^I + q_b\delta_{I1} + q_t\delta_{IM} = \sum_{J=1}^{M} \tilde{I}^{IJ} \frac{\partial^2 W_J}{\partial t^2}$$

Finite element approximation



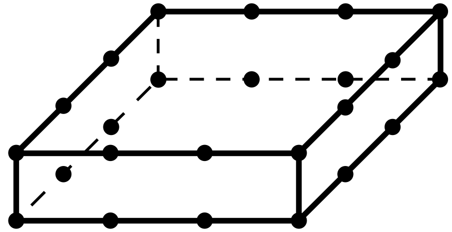
$$U_I(x,y,t) = \sum_{j=1}^{p} U_I^j(t)\psi_j(x,y)$$

$$V_I(x,y,t) = \sum_{j=1}^{p} V_I^j(t)\psi_j(x,y)$$

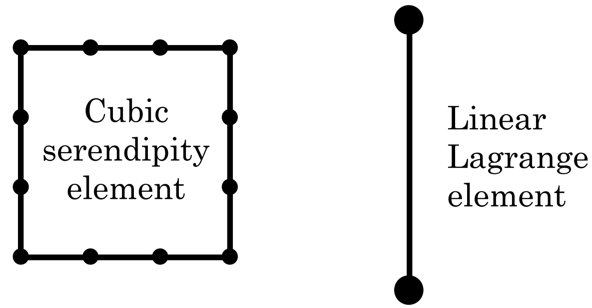
$$W_I(x,y,t) = \sum_{j=1}^{q} W_I^j(t)\varphi_j(x,y)$$

Layerwise Kinematic Model

Conventional 3D   Layerwise 2D + 1D

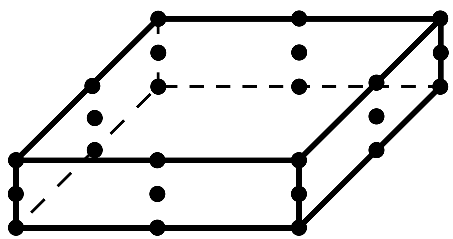


(1a)

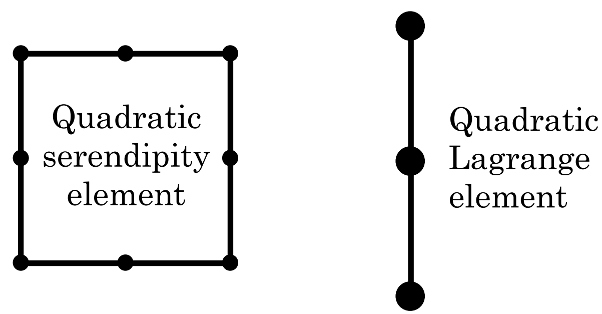


(in-plane)   (through thickness)

(1b)



(2a)



(in-plane)   (through thickness)

(2b)

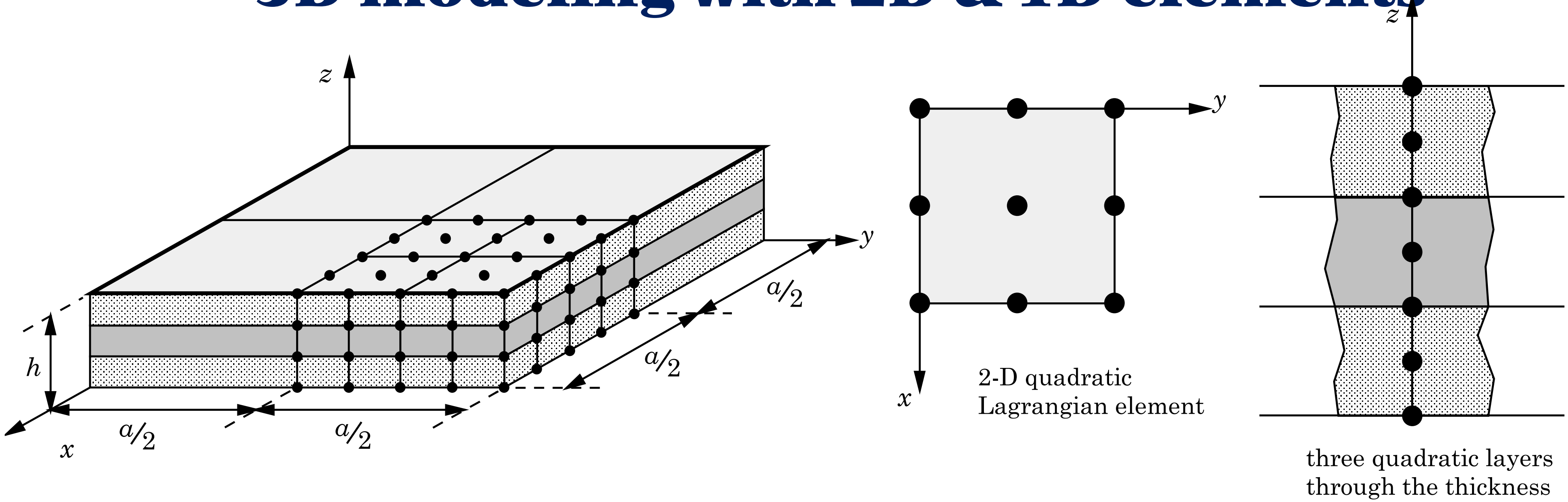
Table: Comparison of the number of operations needed to form the element stiffness matrices for equivalent elements in the conventional 3-D format and the layerwise 2-D format. Full quadrature is used in all.

Element Type†	Multipli.	Addition	Assignments
1a (3-D)	1,116,000	677,000	511,000
1b (LWPT)	423,000	370,000	106,000
2a (3-D)	1,182,000	819,000	374,000
2b (LWPT)	284,000	270,000	69,000

† *Element* 1*a*: 72 degrees of freedom, 24-node 3-D isoparametric hexahedron with cubic in-plane interpolation and linear transverse interpolation.
Element 1*b*: 72 degrees of freedom, E12–L1 layerwise element.
Element 2*a*: 81 degrees of freedom, 27-node 3-D isoparametric hexahedron with quadratic interpolation in all three directions.
Element 2*b*: 81 degrees of freedom, E9–Q1 layerwise element.

Layerwise Kinematic Model

3D modeling with 2D & 1D elements



$$E_1 = 25 \times 10^6 \text{ psi}, \quad E_2 = E_3 = 10^6 \text{ psi}$$

$$G_{12} = 0.5{\times}10^6 \text{ psi}, \; G_{13} = G_{23} = 0.2{\times}10^6 \text{ psi}, \; \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

$$u(x, a/2, z) = u(a/2, y, z) = 0$$
$$v(a/2, y, z) = u(x, a/2, z) = 0$$
$$w(x, a, z) = u(a, y, z) = 0$$

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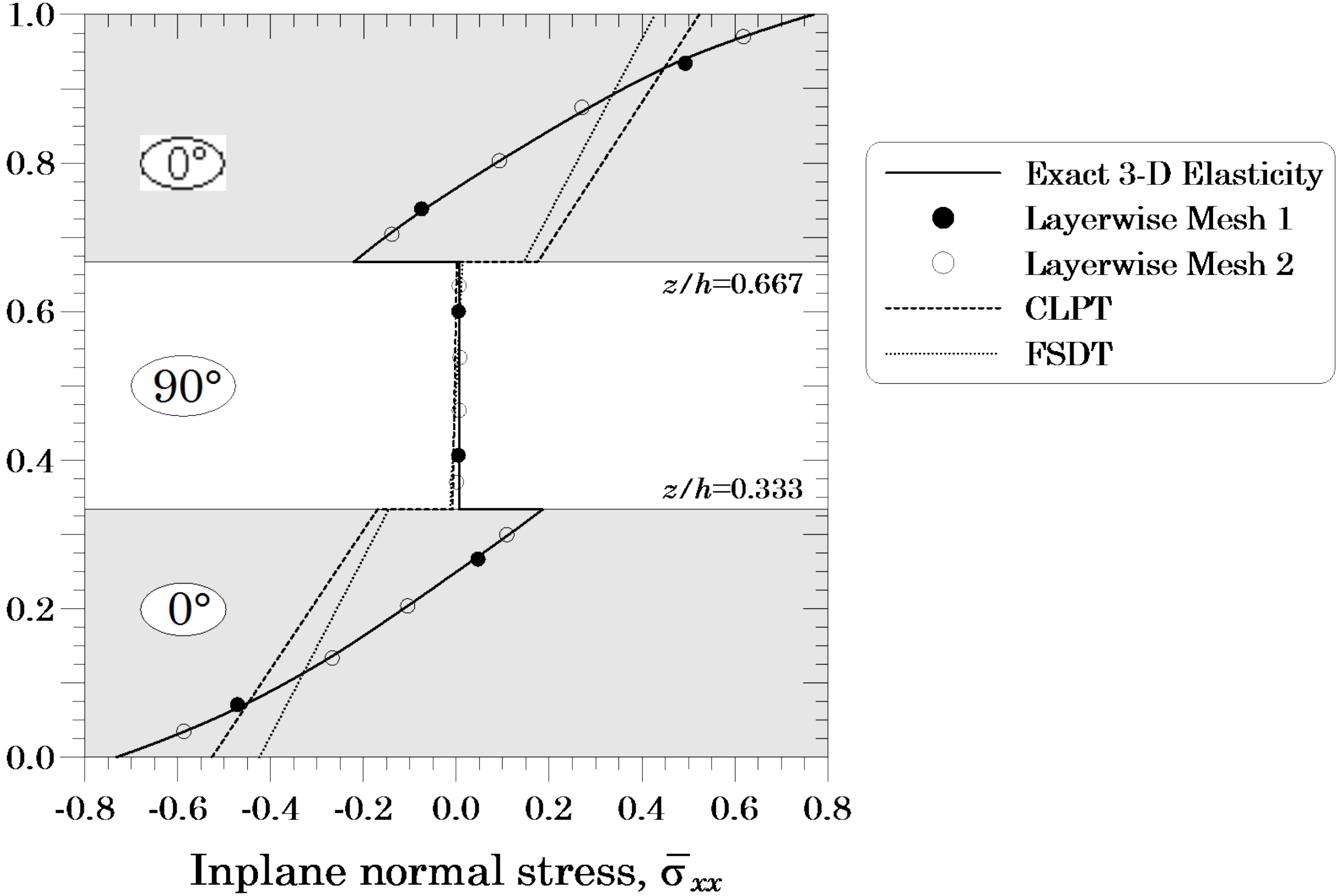
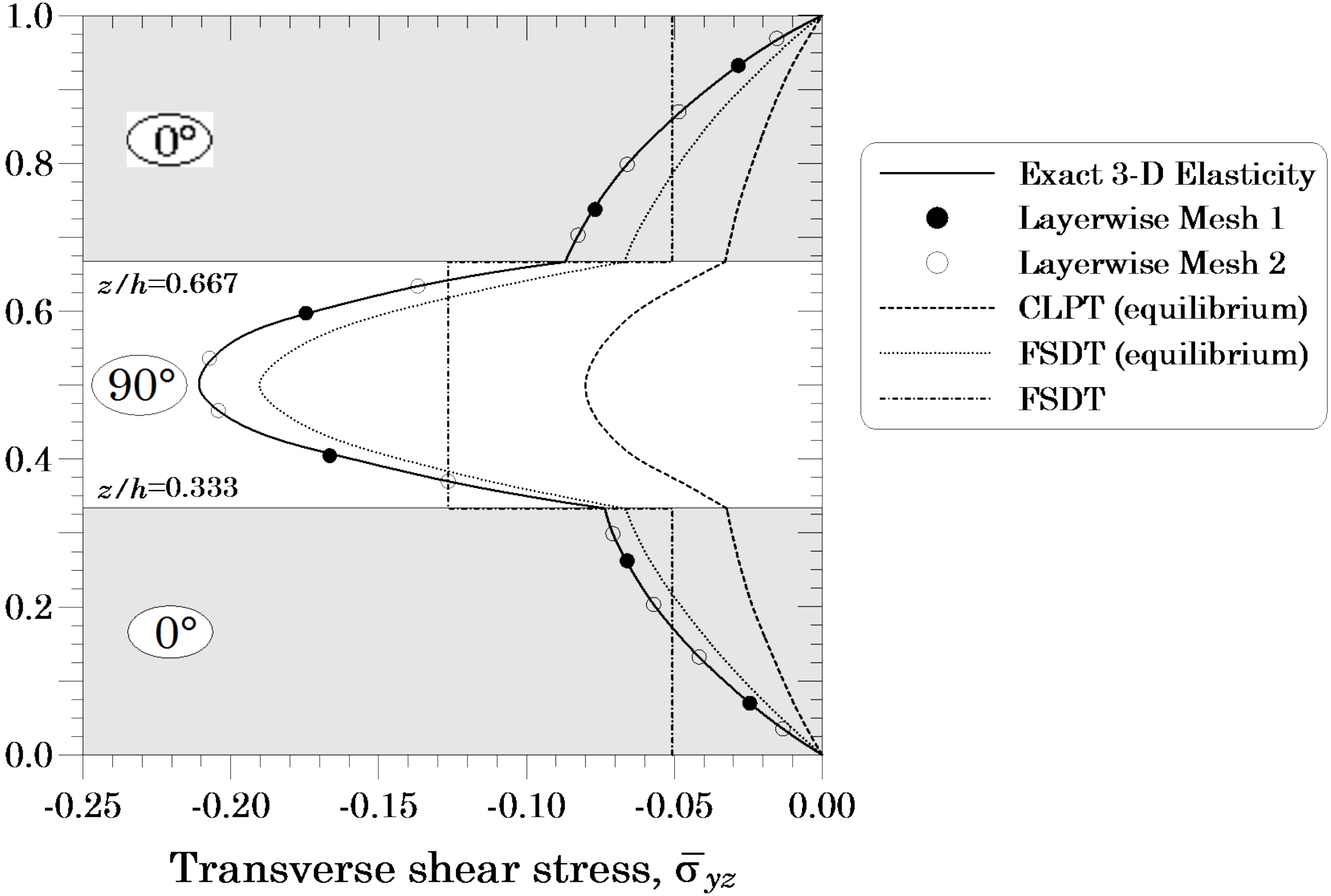
Validation of the Layerwise Theory



Verification of the Layerwise Theory



Variable Kinematic Model for Global-Local Analysis

Composite displacement field:

$$u_i(x,y,z) = u_i^{ESL}(x,y,z) + u_i^{LWT}(x,y,z)$$

ESL Displacement field:

$$u_1^{ESL}(x,y,z) = u_0(x,y) + z\phi_x(x,y)$$
$$u_2^{ESL}(x,y,z) = v_0(x,y) + z\phi_y(x,y)$$
$$u_3^{ESL}(x,y,z) = w_0(x,y)$$

LWT Displacement field:

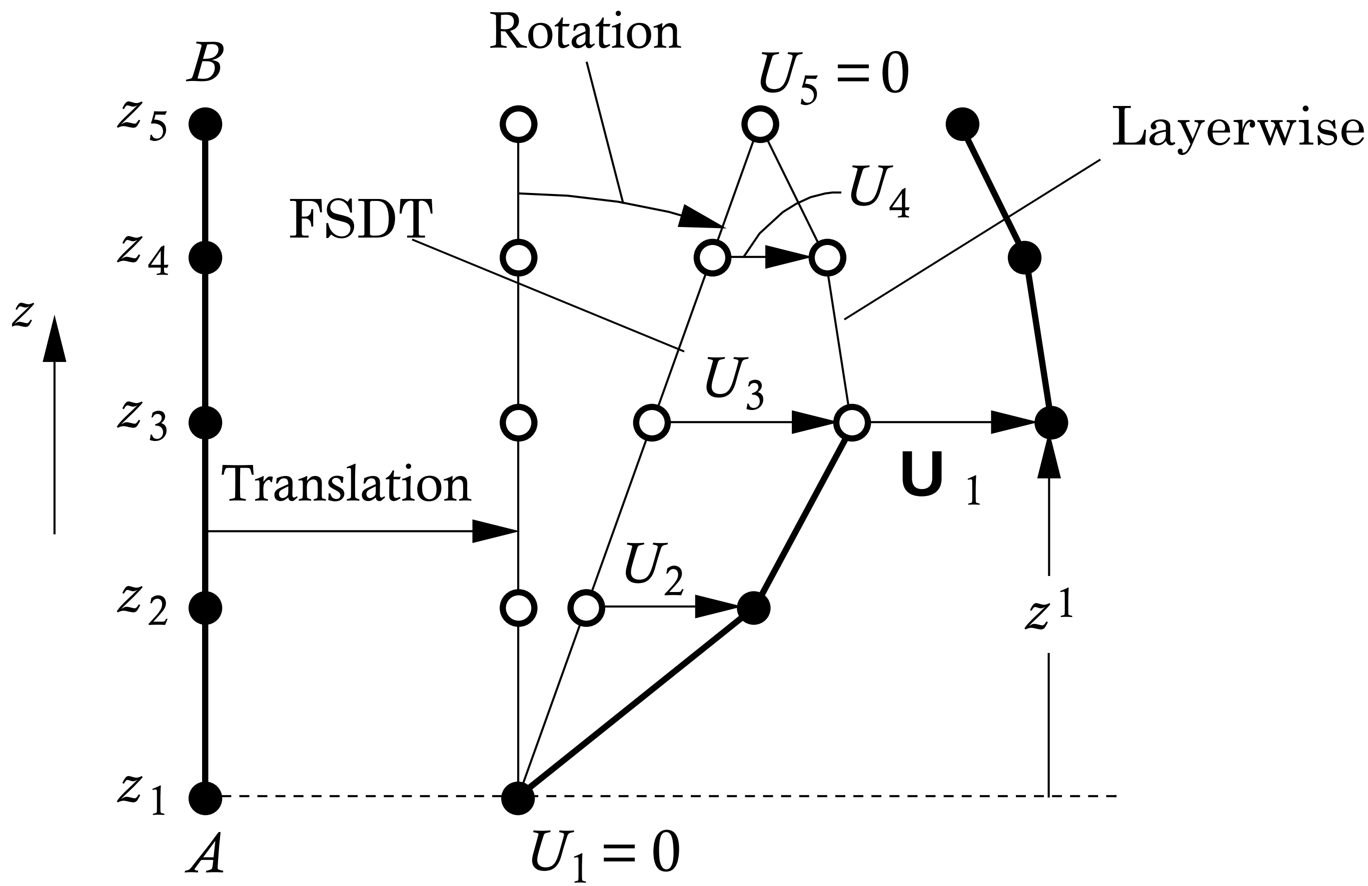
$$u_1^{LWT}(x,y,z) = \sum_{I=1}^{N} U_I(x,y)\Phi^I(z)$$

$$u_2^{LWT}(x,y,z) = \sum_{I=1}^{N} V_I(x,y)\Phi^I(z)$$

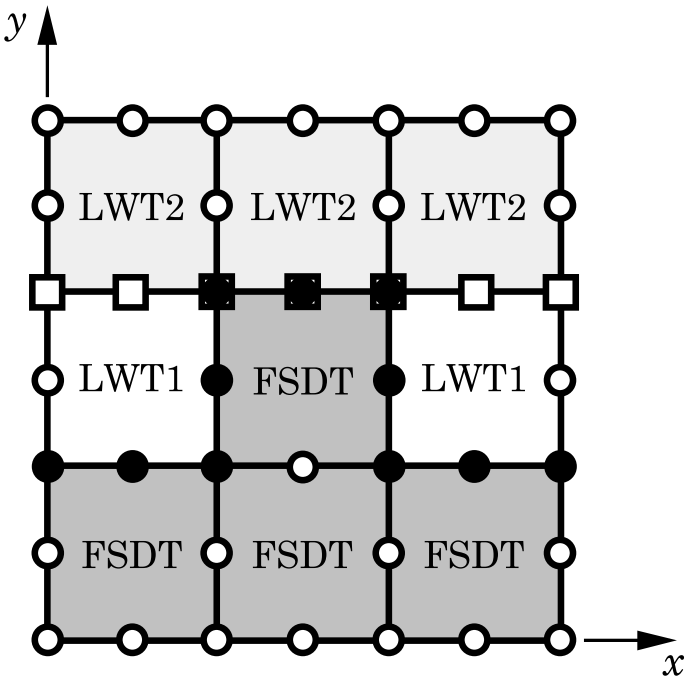
$$u_3^{LWT}(x,y,z) = \sum_{I=1}^{M} W_I(x,y)\Psi^I(z)$$

Laywerwise Kinematic Model

(continued)



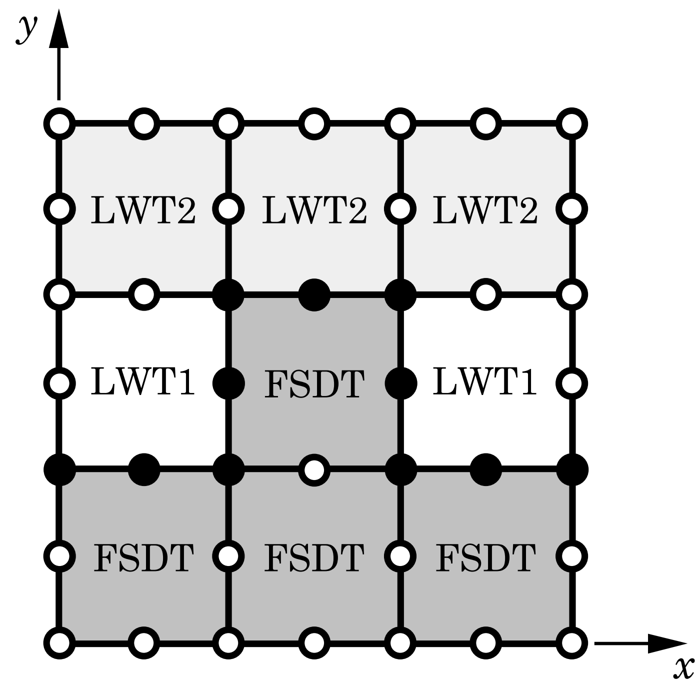
Sub-region Continuity of the Solution



At nodes ●, set $U_j = V_j = 0$, $j = 1,2,..,n$

At nodes □, set $W_j = 0$, $j = 1,2,..,n$

(a) Enforcing **strict** subregion compatibility

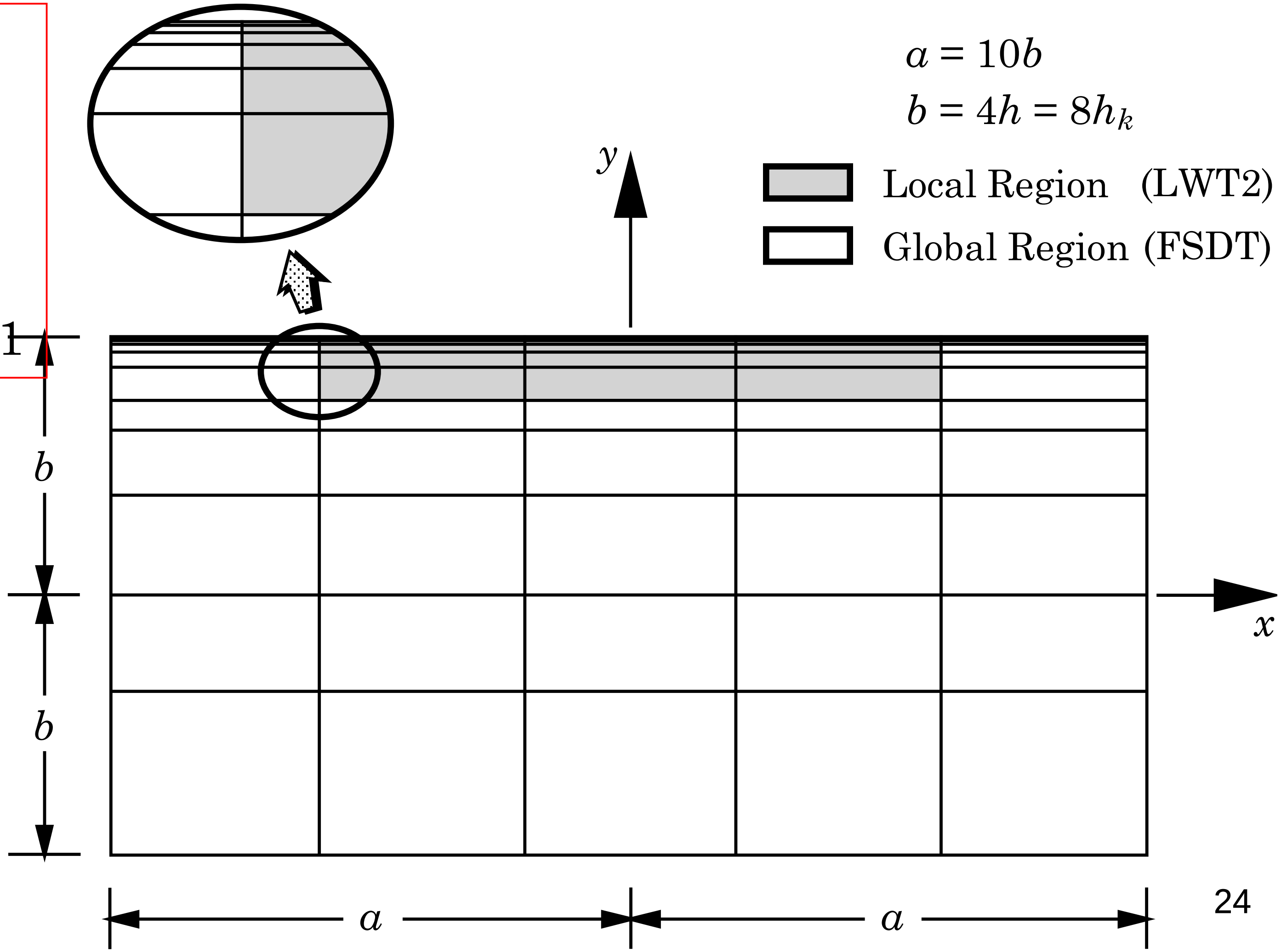


At nodes ●, set $U_j = V_j = 0$, $j = 1,2,..,n$

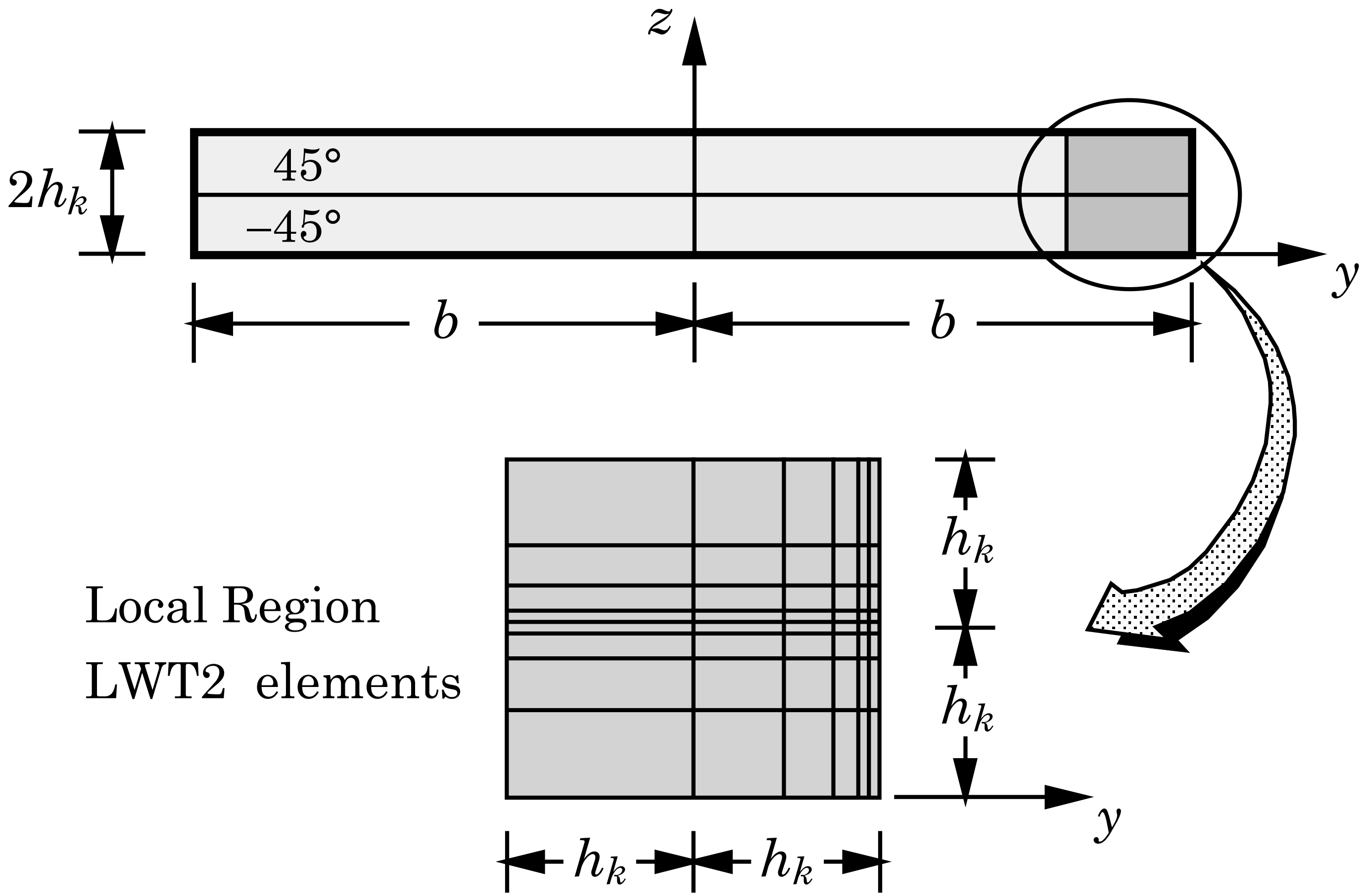
(b) Enforcing **relaxed** subregion compatibility

Free-Edge Problem

$E_1 = 20 \times 10^6$ psi.
$E_2 = 2.1 \times 10^6$ psi.
$E_3 = 2.1 \times 10^6$ psi.
$G_{12} = 0.85 \times 10^6$ psi.
$G_{13} = G_{23} = G_{12}$
$\nu_{13} = \nu_{23} = \nu_{12} = 0.21$



Free-Edge Problem (continued)



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Free-Edge Problem (continued)

Table: Description of global–local meshes for the $(45/{-45})_s$ laminate under axial extension.

Global-local meshes: Mesh 1 – Mesh 4; 3D mesh: Mesh 5

Remarks	Mesh 1	Mesh 2	Mesh 3	Mesh 4	Mesh 5
Number of Elements in Local LWT2 Region	3×4	3×5	3×6	3×7	5×11
Width of Local Region	$\frac{1}{2}h_k$	h_k	$2h_k$	$3h_k$	$16h_k$
Length of Local Region	$\frac{6}{5}a$	$\frac{6}{5}a$	$\frac{6}{5}a$	$\frac{6}{5}a$	$2a$
Total Number of Active D.O.F. in VKFE Mesh (Strict Compatibility)	1,986	2,400	2,814	3,228	9,116
Total Number of Active D.O.F. in VKFE Mesh (Relaxed Compatibility)	2,354	2,800	3,246	3,690	9,116

h_k = thickness of a single material ply. All five VKFE meshes have the exact same in-plane discretization (5×11).

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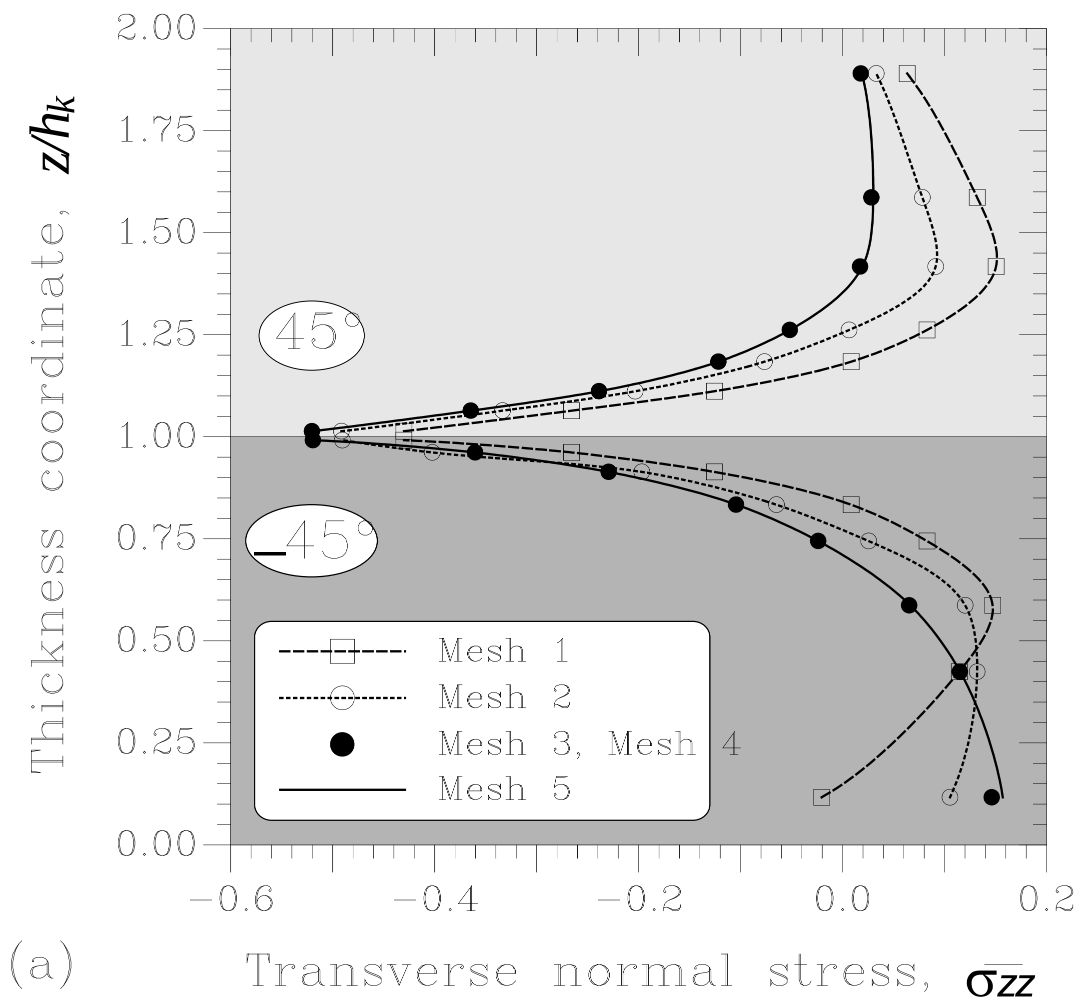
Free-Edge Problem (continued)



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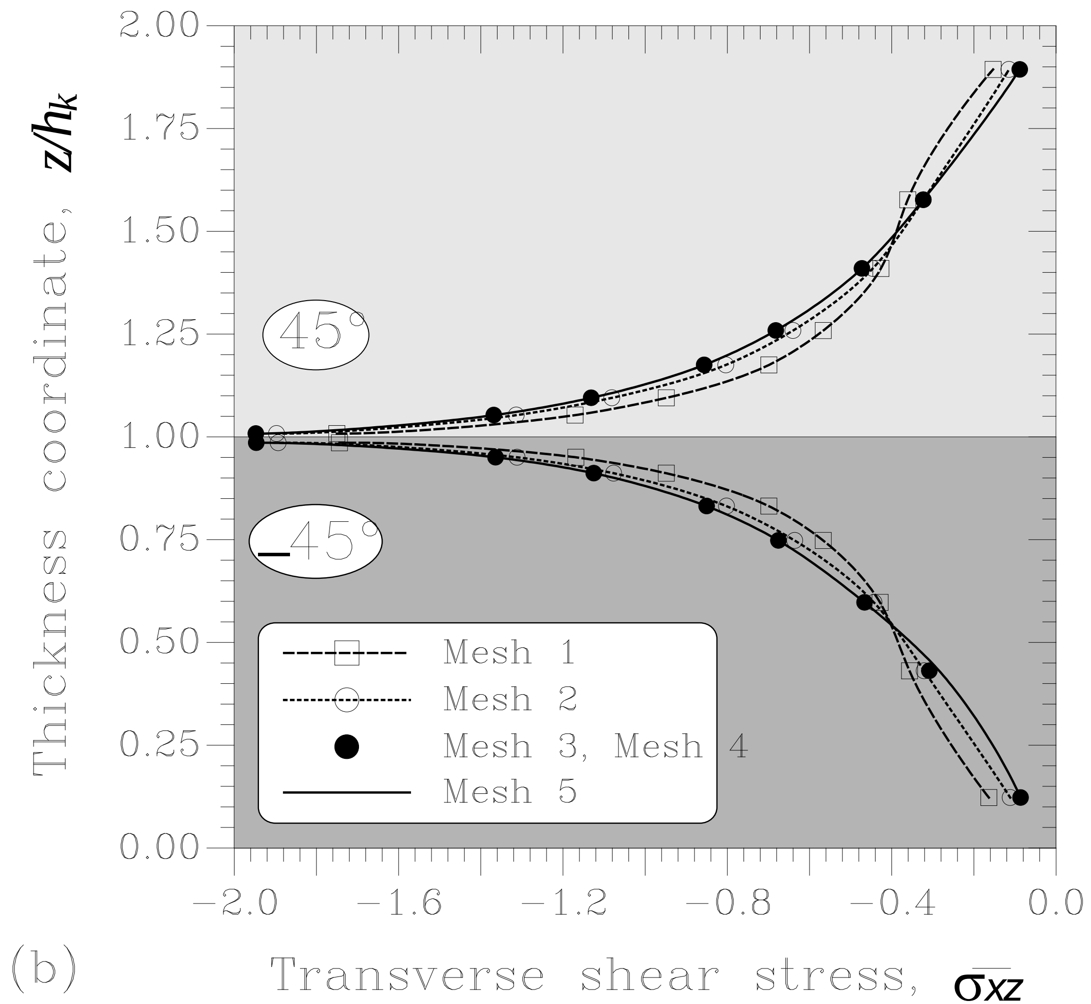
Free-Edge Problem (continued)



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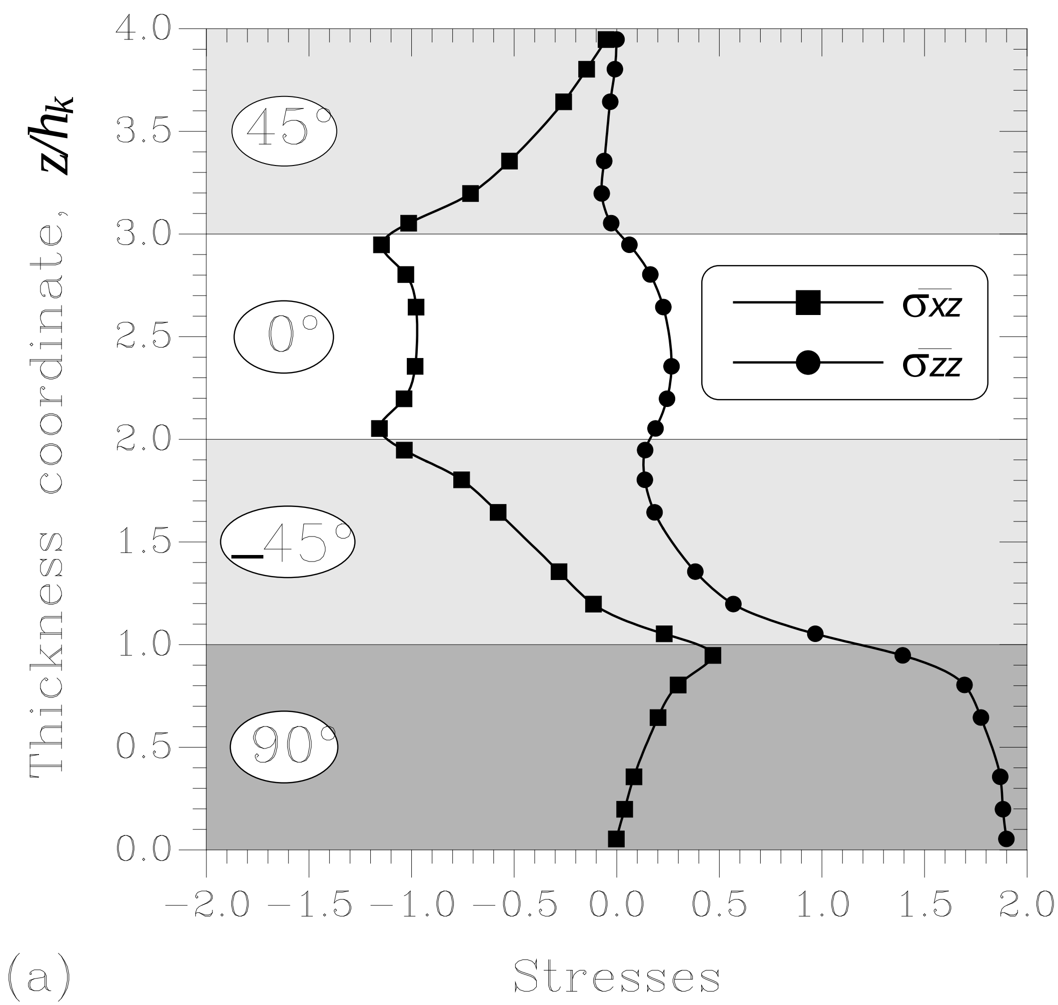
Free-Edge Problem (continued)



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Summary

In this lecture, we have discussed the following topics:

Third-order Shear Deformation Plate Theory

- Development of governing equations
- Numerical results

Layerwise Laminate Theory

- Development of governing equations
- Global-local analysis
- Numerical results

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