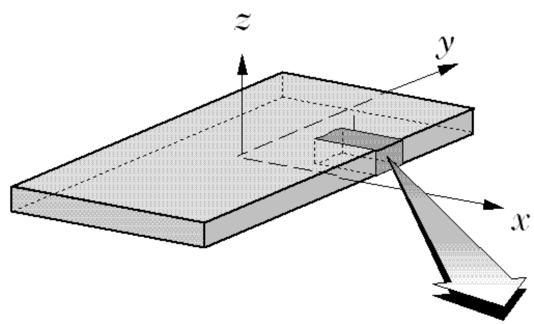
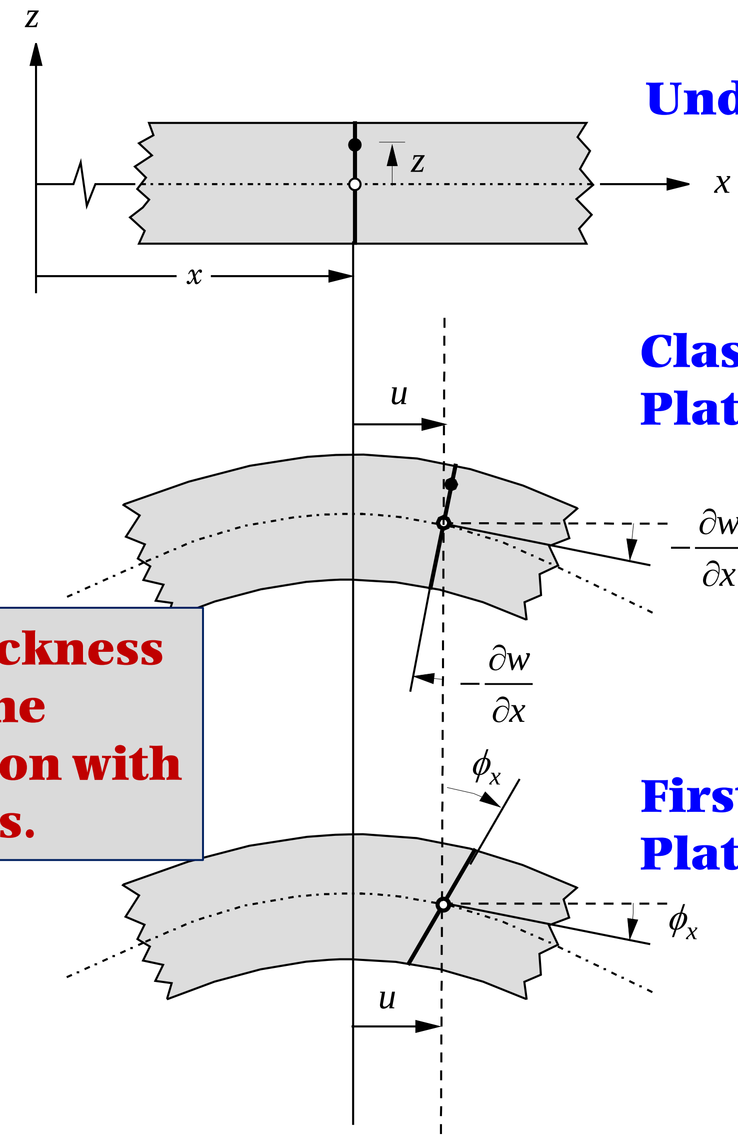
LAMINATE PLATE THEORIES AND NAVIER'S SOLUTIONS OF RECTANGULAR PLATES

For Bending and Vibration

- Kinematics of deformation
- Displacement Fields
- Lamination Scheme and Notation
- Classical Laminate Plate Theory
- First-Order Shear Deformation Theory
- Navier's Solutions

Kinematics of the Classical and Shear Deformation Plate Theories





Undeformed Edge

Classical (Kirchhoff) Plate Theory (CPT)

First-Order (Mindlin) Plate Theory (FST)

Order refers to the thickness coordinate power in the displacement expansion with independent functions.

Displacement Fields of Various Theories

Classical Laminate Plate Theory (CLPT)

$$u_1(x,y,z,t) = u(x,y,t) + z\theta_x(x,y,t) \quad , \quad \theta_x = -(\partial w / \partial x)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\theta_y(x,y,t) \quad , \quad \theta_y = -(\partial w / \partial y)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

First-order Shear Deformation Theory (FSDT)

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

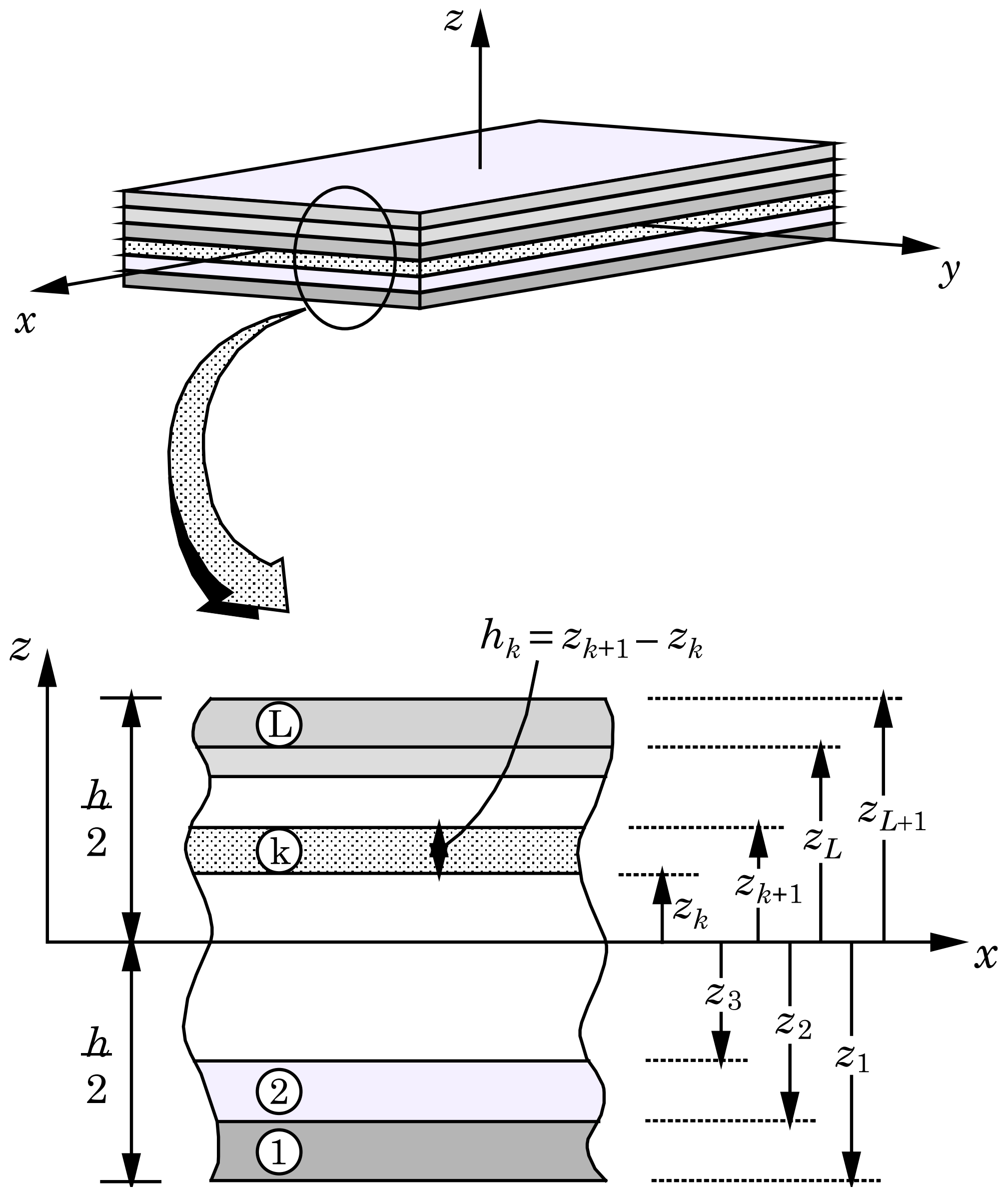
Higher-order Shear Deformation Theories (HSDT)

$$u_1(x,y,z,t) = \sum_{i=0}^{M} z^i \, \phi_1^{(i)}(x,y,t)$$

$$u_2(x,y,z,t) = \sum_{i=0}^{M} z^i \, \phi_2^{(i)}(x,y,t)$$

$$u_3(x,y,z,t) = \sum_{i=0}^{N} z^i \, \phi_3^{(i)}(x,y,t)$$

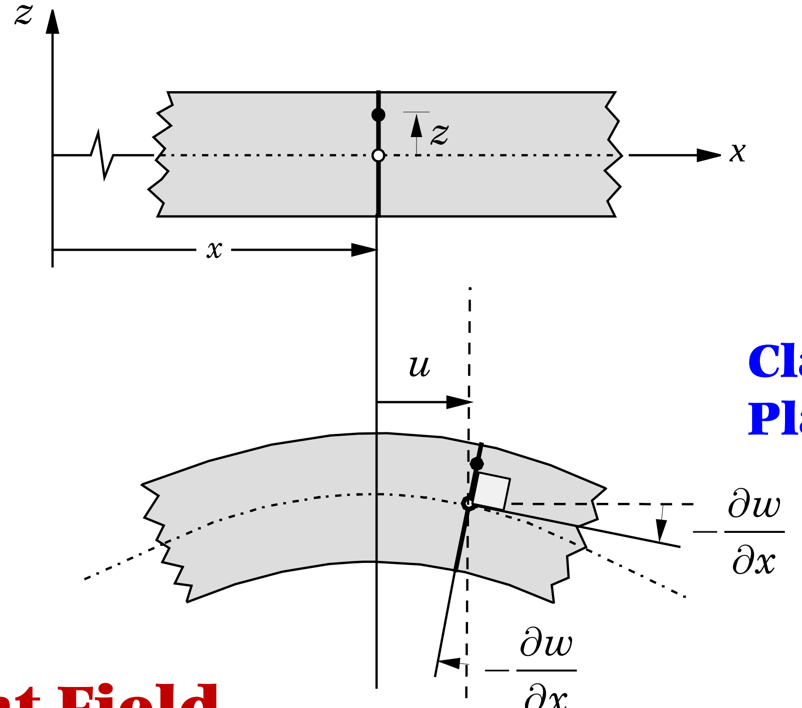
LAMINATION SCHEME AND NOTATION



Notation and Coordinate System Used in a Laminate Analysis

Layers are numbered in the +ve *z* direction

CLASSICAL LAMINATE PLATE THEORY



Undeformed Edge

Classical Laminated Plate Theory (CPT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\theta_x(x,y,t) \quad , \quad \theta_x = -\frac{\partial w}{\partial x}$$

$$u_2(x,y,z,t) = v(x,y,t) + z\theta_y(x,y,t) \quad , \quad \theta_y = -\frac{\partial w}{\partial y}$$

$$u_3(x,y,z,t) = w(x,y,t)$$

CLASSICAL LAMINATE PLATE THEORY

Nonlinear strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial u_m}{\partial x_1}\frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

Strain Field

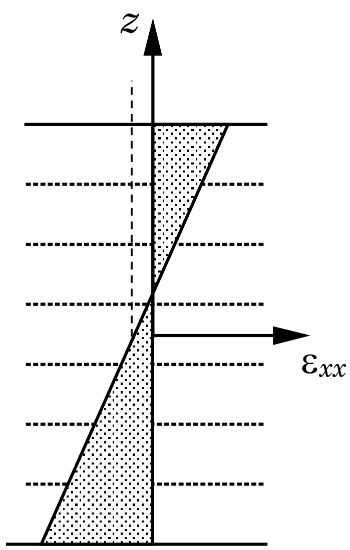
$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \theta_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \theta_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)$$

$$\gamma_{xz} = 0\,, \quad \gamma_{yz} = 0, \quad \theta_x = -\frac{\partial w}{\partial x}\,, \quad \theta_y = -\frac{\partial w}{\partial y}$$

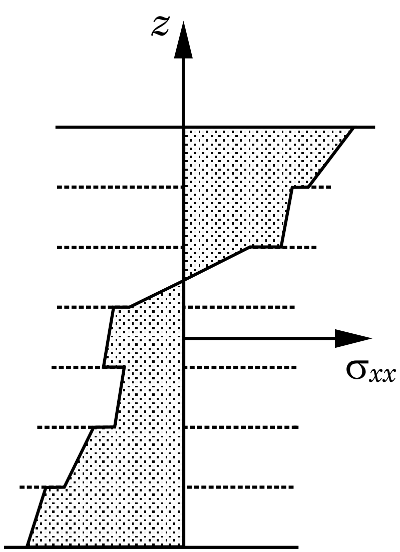
Strain and Stress Distributions Through the Thickness of the Laminate



Strain distribution is *continuous* through laminate thickness

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\}$$

Stress distribution is *discontinuous* through laminate thickness



$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}^{(k)} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\}$$

$$\left\{\begin{array}{c}\sigma_{yz}\\ \sigma_{xz}\end{array}\right\}^{(k)} = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\}$$

Principle of Virtual Displacements

$$0 = \int_{V^e} \sigma_{ij}\delta\varepsilon_{ij}\, dV - \int_{\Omega^e} q\,\delta w\, dxdy - \int_{\Gamma^e} \left(V_n \delta w + M_n\, \delta\theta_n\right) ds$$

$$= \int_{\Omega^e} \int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\,\delta\varepsilon_{yy} + \sigma_{xy}\, 2\delta\varepsilon_{xy}\right) dzdxdy$$

$$- \int_{\Omega^e} q\,\delta w\, dxdy - \int_{\Gamma^e} \left(N_n \delta u_n + N_{ns}\delta u_{ns} + V_n \delta w + M_n\, \delta\theta_n\right) ds$$

$$= \int_{\Omega^e} \left\{ \int_{-h/2}^{h/2} \left[\sigma_{xx}\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} - z\frac{\partial^2 \delta w}{\partial x^2} \right) + \sigma_{yy}\left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} - z\frac{\partial^2 \delta w}{\partial y^2} \right) \right. \right.$$

$$\left. \left. + \sigma_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x} - 2z\frac{\partial^2 \delta w}{\partial x \partial y} \right) \right] dz \right\} dxdy$$

$$- \int_{\Omega^e} q\,\delta w\, dxdy - \int_{\Gamma^e} \left(N_n \delta u_n + N_{ns}\delta u_{ns} + V_n \delta w + M_n\, \delta\theta_n\right) ds$$

Principle of Virtual Displacements (continued)

$$0 = \int_{\Omega^e} \left\{ N_{xx}\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + N_{yy}\left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) \right.$$

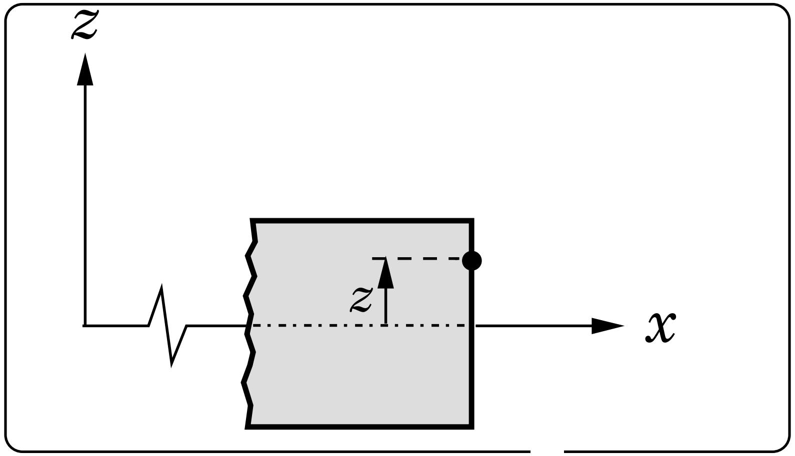
$$+ N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right)$$

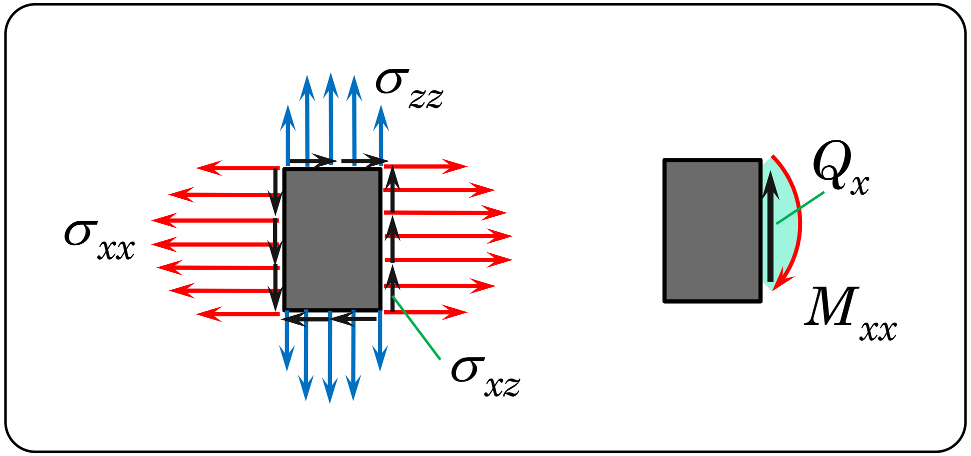
$$\left. - M_{xx}\frac{\partial^2 \delta w}{\partial x^2} - M_{yy}\frac{\partial^2 \delta w}{\partial y^2} - 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} \right\} dxdy$$

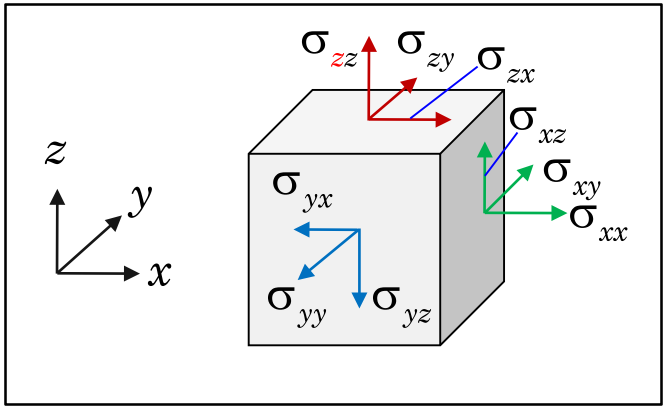
$$- \int_{\Omega^e} q\,\delta w\, dxdy - \int_{\Gamma^e} \left(N_n \delta u_n + N_{ns}\delta u_{ns} + V_n \delta w + M_n\, \delta\theta_n \right) ds$$

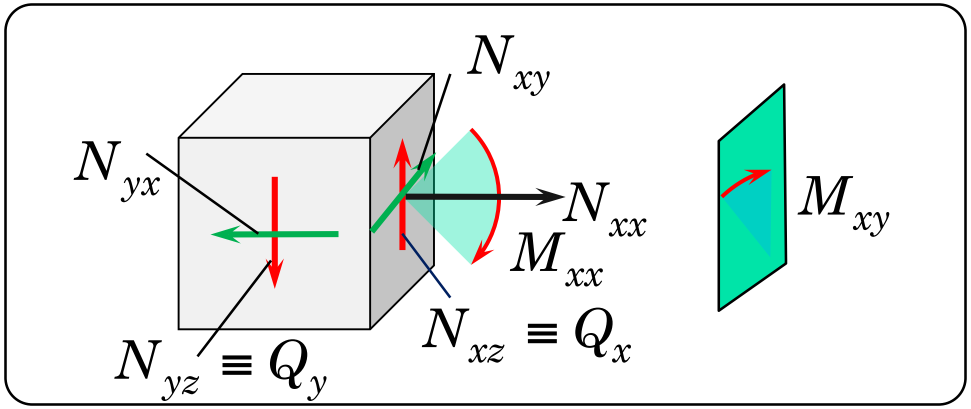
$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\ dz$$

SIGN CONVENTION FOR STRESS RESULTANTS

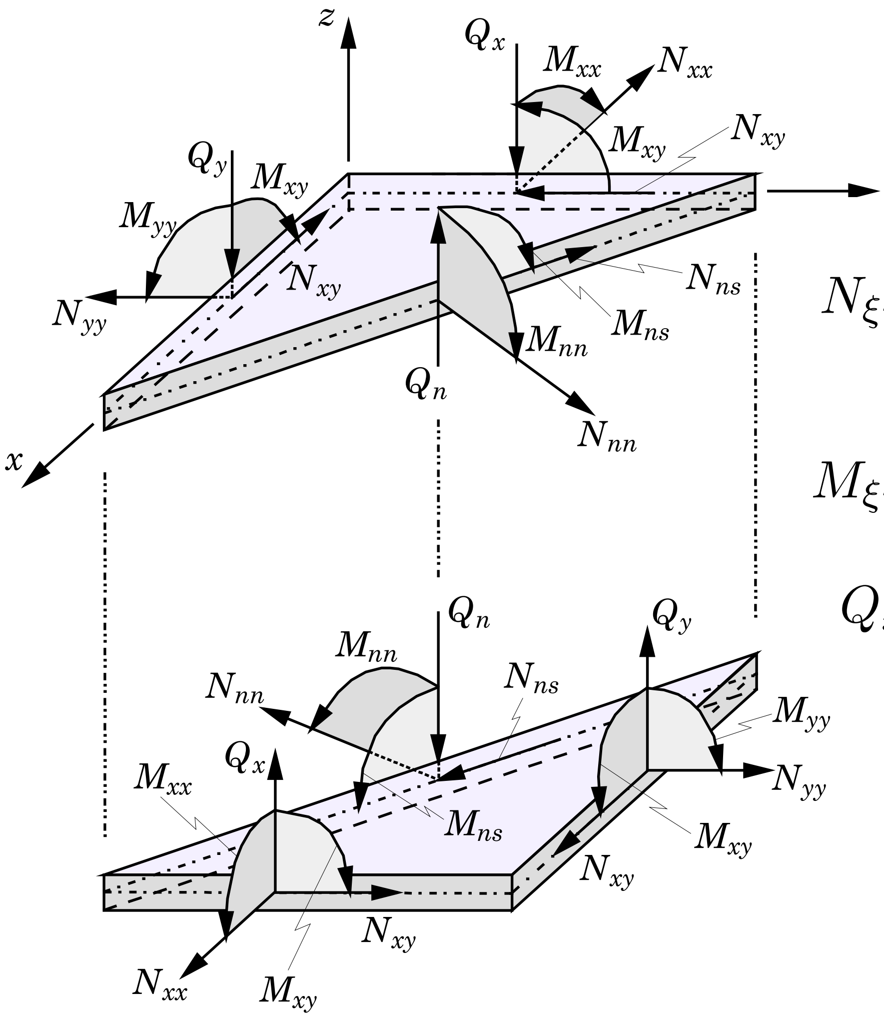








CONVENTIONAL STRESS RESULTANTS



$$N_{\xi\eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi\eta}\, dz \quad (\xi, \eta = x, y, z)$$

$$M_{\xi\eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi\eta} z\, dz \quad (\xi, \eta = x, y)$$

$$Q_x = N_{xz}, \quad Q_y = N_{yz}$$

STRESS RESULTANTS IN TERMS OF THE MEMBRANE AND BENDING STRAINS

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} dz$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z \; dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} z \; dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

DEFINITION OF THE LAMINATE STIFFNESSES

Standard A, B, and D coefficients

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}(1, z, z^2)dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2)dz$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k) \ , \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3)$$

Higher-order coefficients

$$H_{ij}^{(n)} = \sum_{k=1}^{NL} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} z^n \, dz = \frac{1}{n+1} \sum_{k=1}^{NL} \bar{Q}_{ij}^{(k)} \left(z_{k+1}^{n+1} - z_k^{n+1} \right)$$

$$H_{ij}^{(0)} = A_{ij}, \ \ H_{ij}^{(1)} = B_{ij}, \ \ H_{ij}^{(2)} = D_{ij}, \ \text{ and so on}$$

Equations of Motion of the CLPT

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \qquad (1)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \qquad (2)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) + q = I_0 \frac{\partial^2 w}{\partial t^2} \qquad (3)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2} \qquad (4)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2} \qquad (5)$$

GOVERNING EQUATIONS OF CLASSICAL PLATE THEORY (Nonlinear)

Solve Eqs. (4) and (5) for shear forces in terms of the moments and substitute into Eq. (3) to obtain

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} \quad (1)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} \quad (2)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right) +$$

$$\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) + q$$

$$= I_0 \frac{\partial^2 w}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad (3)$$

Stress Resultant-Displacement Relations

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz = \int_{-h/2}^{h/2} \left\{ \bar{Q}_{11} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right\} dz$$

$$+ \int_{-h/2}^{h/2} \bar{Q}_{16} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz$$

$$= A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + A_{16}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right]$$

$$- B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = \int_{-h/2}^{h/2} z \left[\bar{Q}_{11} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right] dz$$

$$+ \int_{-h/2}^{h/2} z\bar{Q}_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz$$

$$= B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right)$$

$$- D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y}$$

Laminated Plate Constitutive Equations

$$N_{xx} = A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$- B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y} - N_{xx}^T$$

$$N_{yy} = A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$- B_{12}\frac{\partial^2 w}{\partial x^2} - B_{22}\frac{\partial^2 w}{\partial y^2} - 2B_{26}\frac{\partial^2 w}{\partial x \partial y} - N_{yy}^T$$

$$N_{xy} = A_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$- B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} - 2B_{66}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xx} = B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T$$

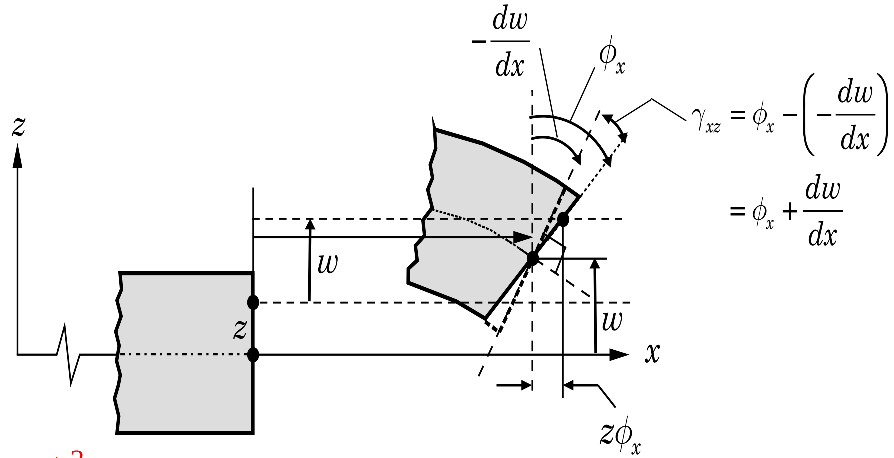
$$M_{yy} = B_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T$$

$$M_{xy} = B_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T$$

FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$



Nonlinear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x},$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y}$$

STRAIN-DISPLACEMENT EQUATIONS (FSDT)

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{yz}^{(1)}\\ \gamma_{xz}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \frac{\partial w}{\partial y} + \phi_y\\ \frac{\partial w}{\partial x} + \phi_x\\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{array}\right\} + z\left\{\begin{array}{c}\frac{\partial \phi_x}{\partial x}\\ \frac{\partial \phi_y}{\partial y}\\ 0\\ 0\\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \frac{\partial w}{\partial y} + \phi_y\\ \frac{\partial w}{\partial x} + \phi_x\\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{array}\right\} + z\left\{\begin{array}{c}\frac{\partial \phi_x}{\partial x}\\ \frac{\partial \phi_y}{\partial y}\\ 0\\ 0\\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\end{array}\right\}$$

EQUATIONS OF MOTION OF FSDT

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y}\right)$$

$$+ \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y}\right) + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

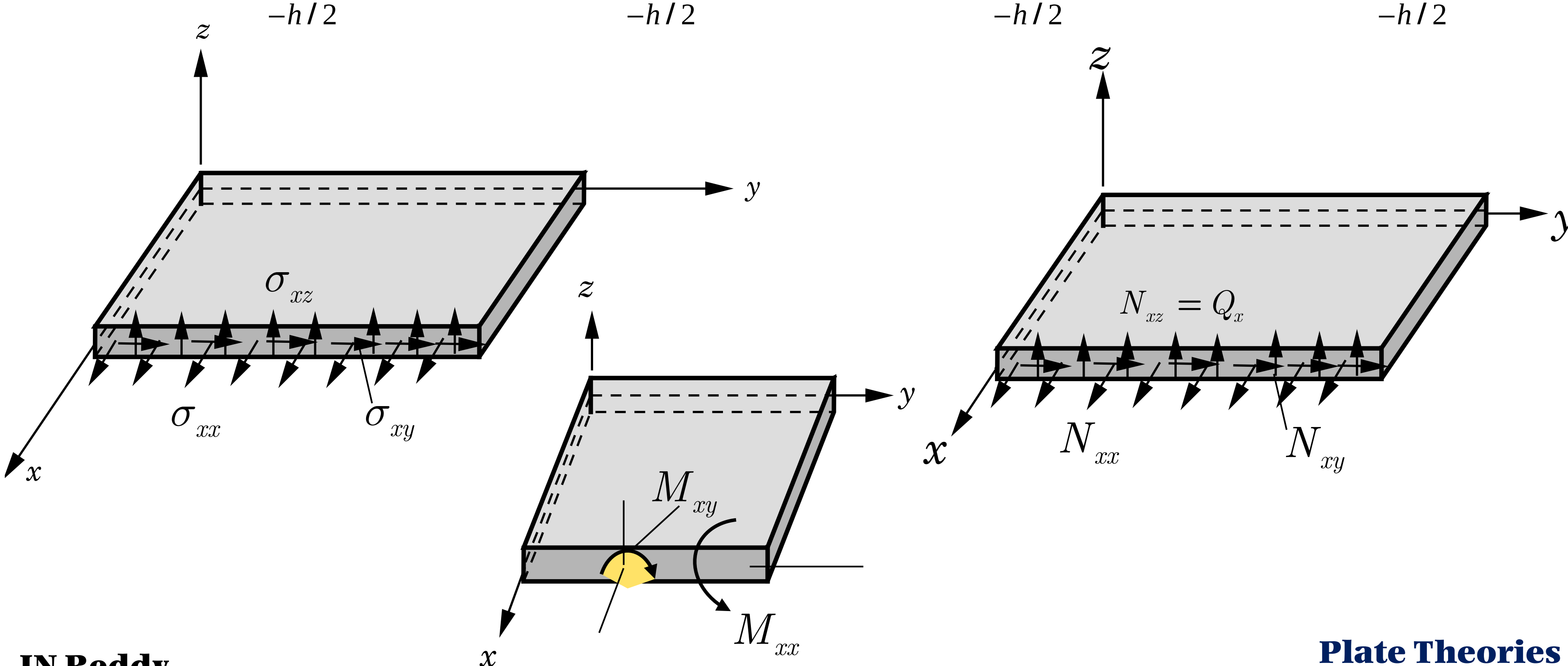
$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2}$$

Stresses and Stress Resultants on an edge of a Plate

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\,dz,\quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\,dz,\quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\,dz,\quad Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz}\,dz$$

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\,dz,\quad M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\,dz,\quad M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\,dz,\quad Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz}\,dz$$



THE FIRST-ORDER SHEAR DEFORMATION THEORY

Stress Resultants (Nonlinear)

$$N_{xx} = A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$
$$+ B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{xx}^T$$

$$N_{yy} = A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$
$$+ B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{yy}^T$$

$$M_{xx} = B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T$$

$$M_{yy} = B_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T$$

$$M_{xy} = B_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \quad Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

Linear Equations for Pure Bending of orthotropic plates

Equation of equilibrium

$$\frac{\partial}{\partial x}\left(\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}\right)+q=0$$

Moment-deflection relations

$$M_{xx}=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}-2D_{16}\frac{\partial^2 w}{\partial x\partial y}$$

$$M_{yy}=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}-2D_{26}\frac{\partial^2 w}{\partial x\partial y}$$

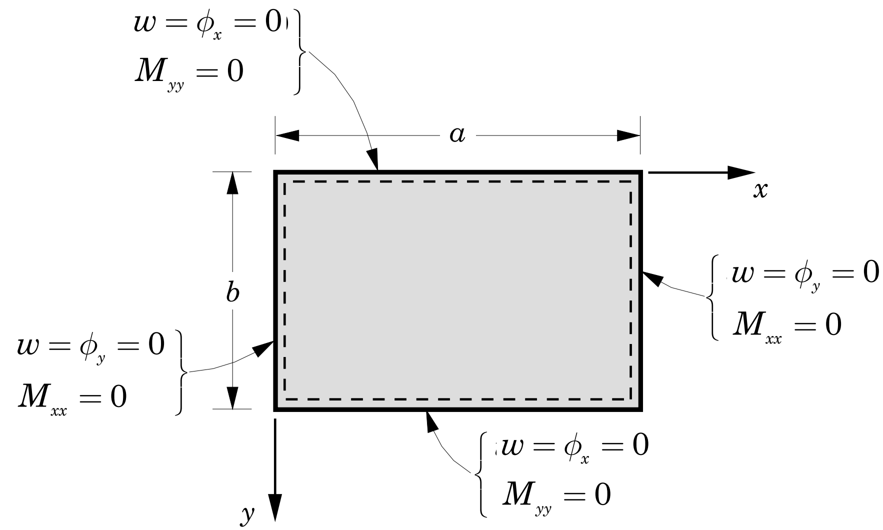
$$M_{xy}=-D_{16}\frac{\partial^2 w}{\partial x^2}-D_{26}\frac{\partial^2 w}{\partial y^2}-2D_{66}\frac{\partial^2 w}{\partial x\partial y}$$

Equilibrium equation in terms of the deflection

$$-\left[D_{11}\frac{\partial^4 w}{\partial x^4}+2\left(D_{66}+2D_{12}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}\right]$$

$$-4\frac{\partial^2}{\partial x\partial y}\left(D_{16}\frac{\partial^2 w}{\partial x^2}+D_{26}\frac{\partial^2 w}{\partial y^2}\right)+q=0$$

Navier Solution of Simply Supported ORTHOTROPIC PLATES



$$w_0(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y$$

$$q(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y$$

$$Q_{mn} = \frac{4}{ab} \int_0^b \int_0^a q(x,y) \sin \alpha x \sin \beta y \, dx dy$$

Navier Solution of Simply Supported ORTHOTROPIC PLATES

$$-\left[D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4}\right] + q = 0$$

$$w_0(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn} \ \sin\alpha x \ \sin\beta y$$

$$q(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{mn} \sin\alpha x \ \sin\beta y$$

$$Q_{mn} = \frac{4}{ab}\int_0^b\int_0^a \ q(x,y) \ \sin\alpha x \ \sin\beta y \ dxdy$$

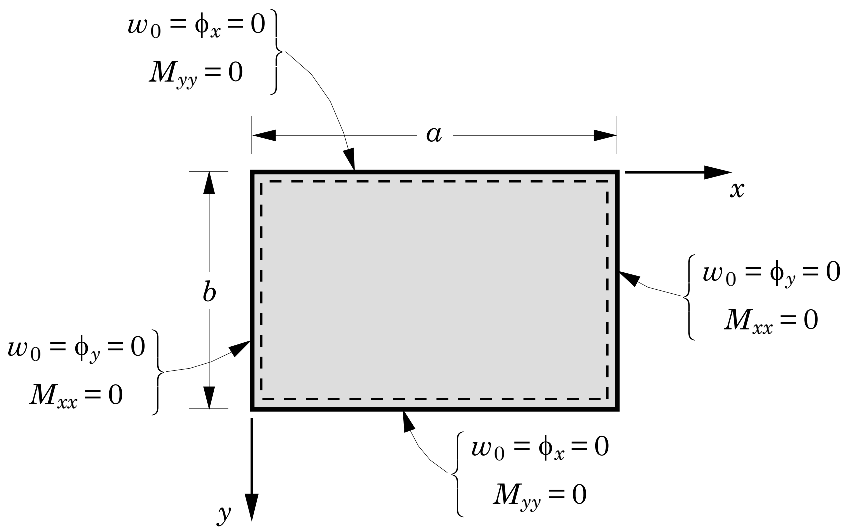
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big\{-W_{mn}\left[D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right] + Q_{mn}\Big\}\sin\alpha x \ \sin\beta y = 0$$

$$W_{mn} = \frac{Q_{mn}}{d_{mn}}, \ d_{mn} = \frac{\pi^4}{b^4}\left[D_{11}m^4 s^4 + 2(D_{12} + 2D_{66})m^2 n^2 s^2 + D_{22}n^4\right]$$

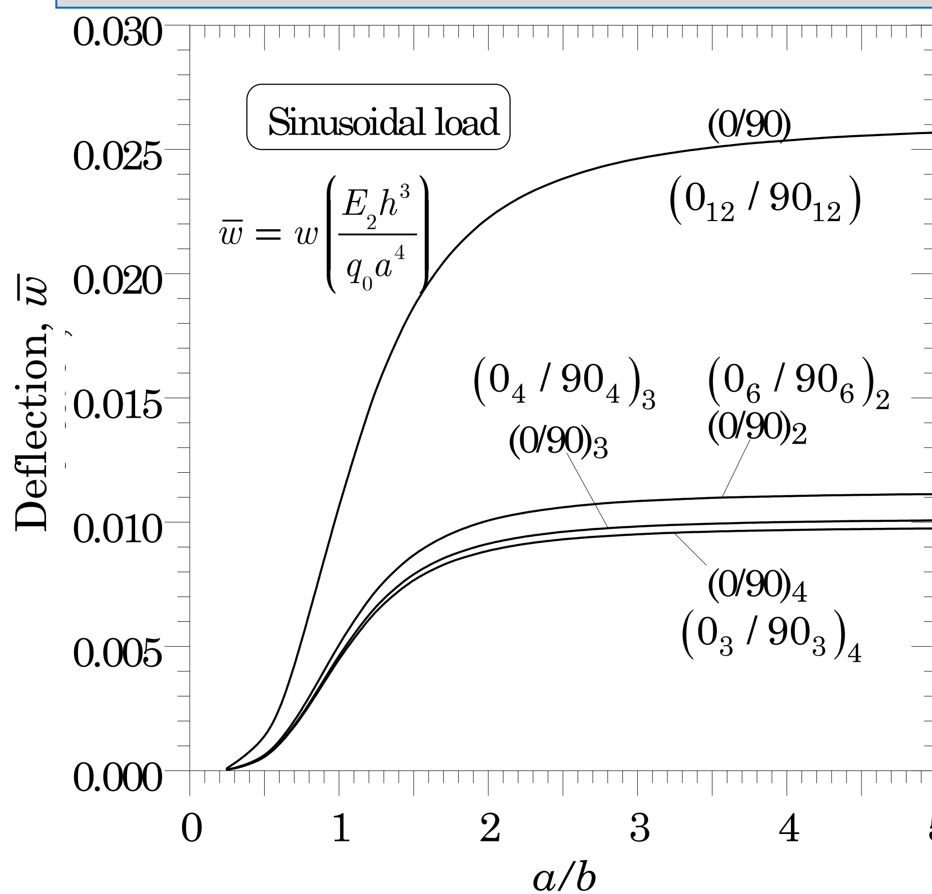
ANALYTICAL SOLUTIONS (continued)

BENDING OF CROSS-PLY PLATES -CLPT

The *total thickness* of all laminates is the same (i.e., all contain 24 layers of the same thickness)



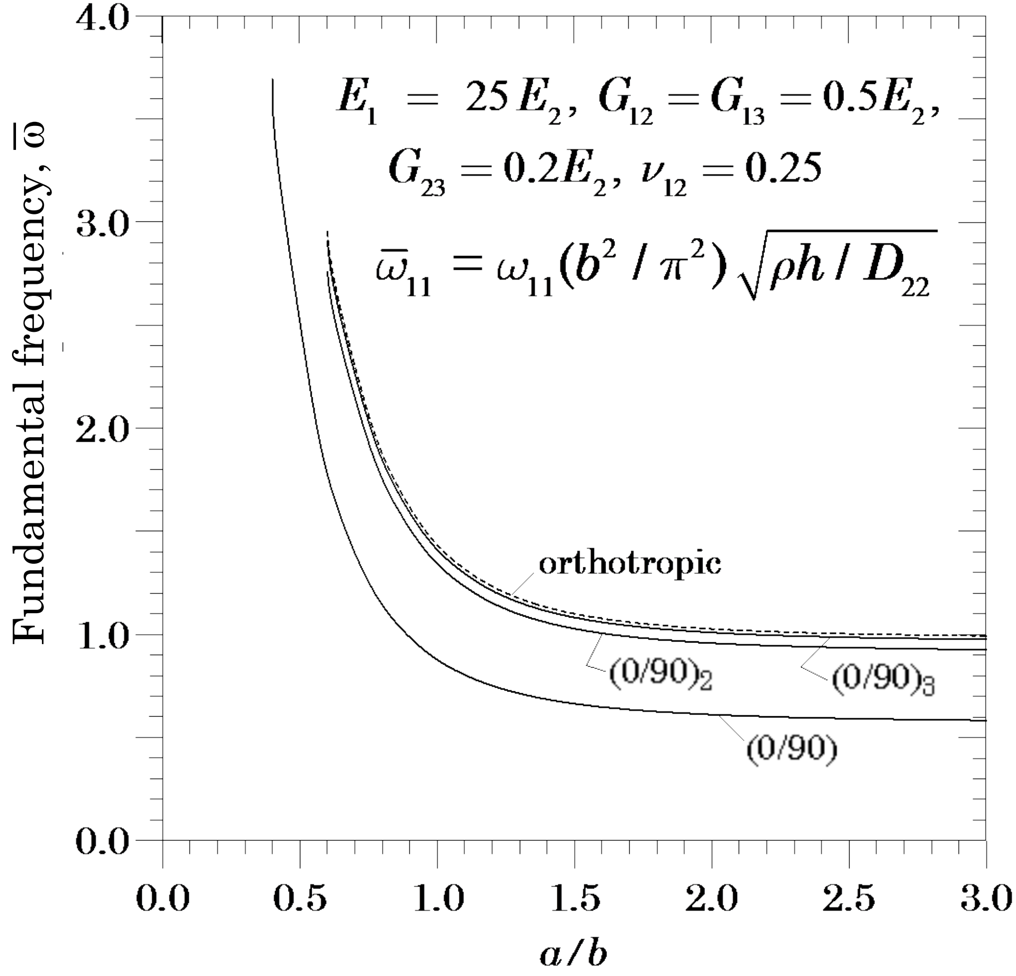
SS-1 Boundary Conditions



$$E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,\ \nu_{12} = 0.25$$

ANALYTICAL SOLUTIONS (continued)

VIBRATION OF CROSS-PLY PLATES -CLPT



$(0/90) = (0_{12}/90_{12})$

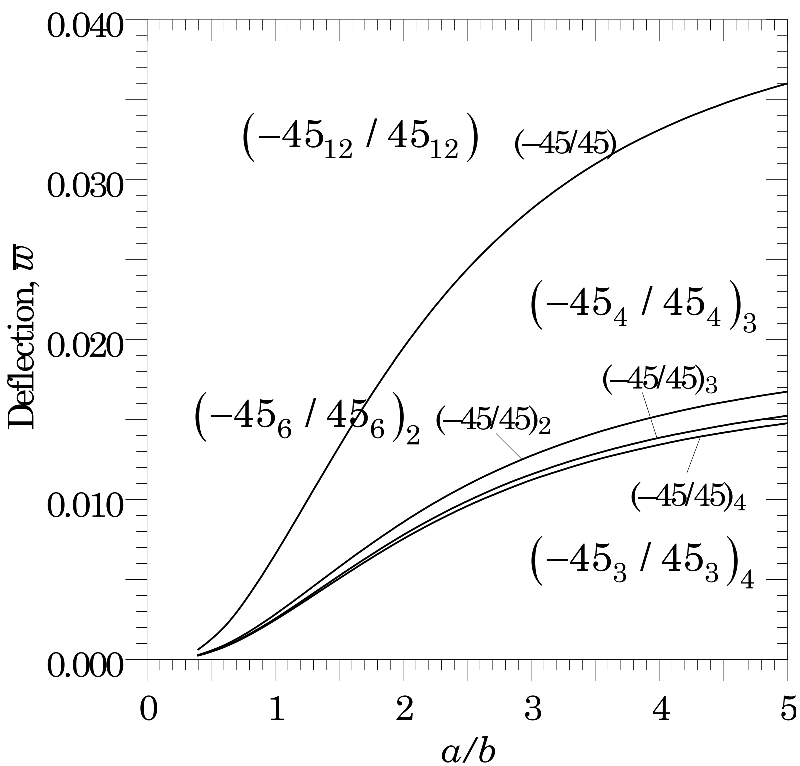
$(0/90)_2 = (0_6/90_6)_2$

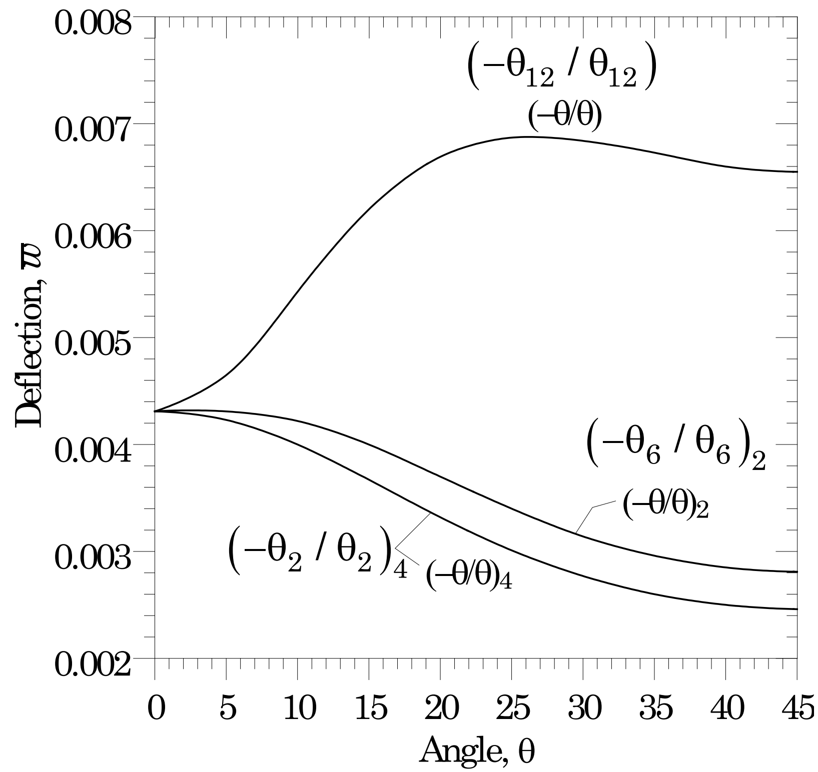
$(0/90)_4 = (0_3/90_3)_4$

$(0/90)_3 = (0_4/90_4)_3$

ANALYTICAL SOLUTIONS (continued)

BENDING OF ANGLE-PLY PLATES -CLPT

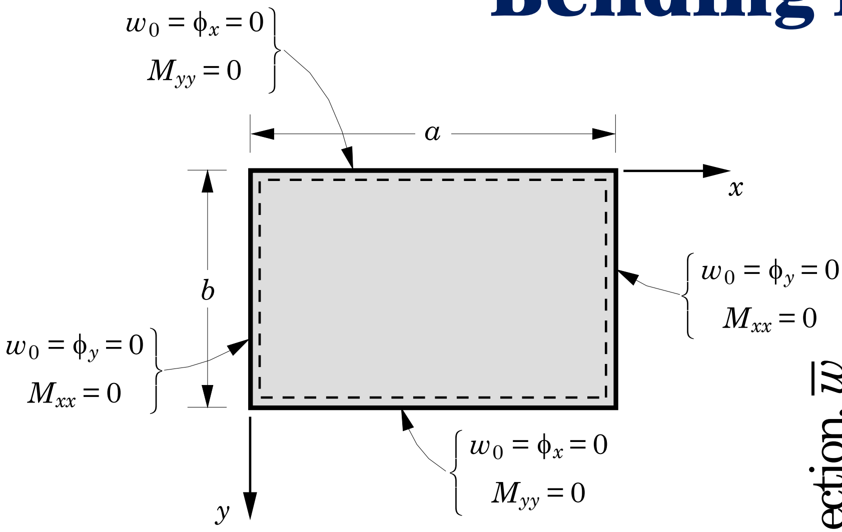




$$E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,\ \nu_{12} = 0.25$$

$$\bar{w} = w\left(\frac{E_2 h^3}{q_0 a^4}\right)$$

Effect of Shear Deformation on Bending Deformation



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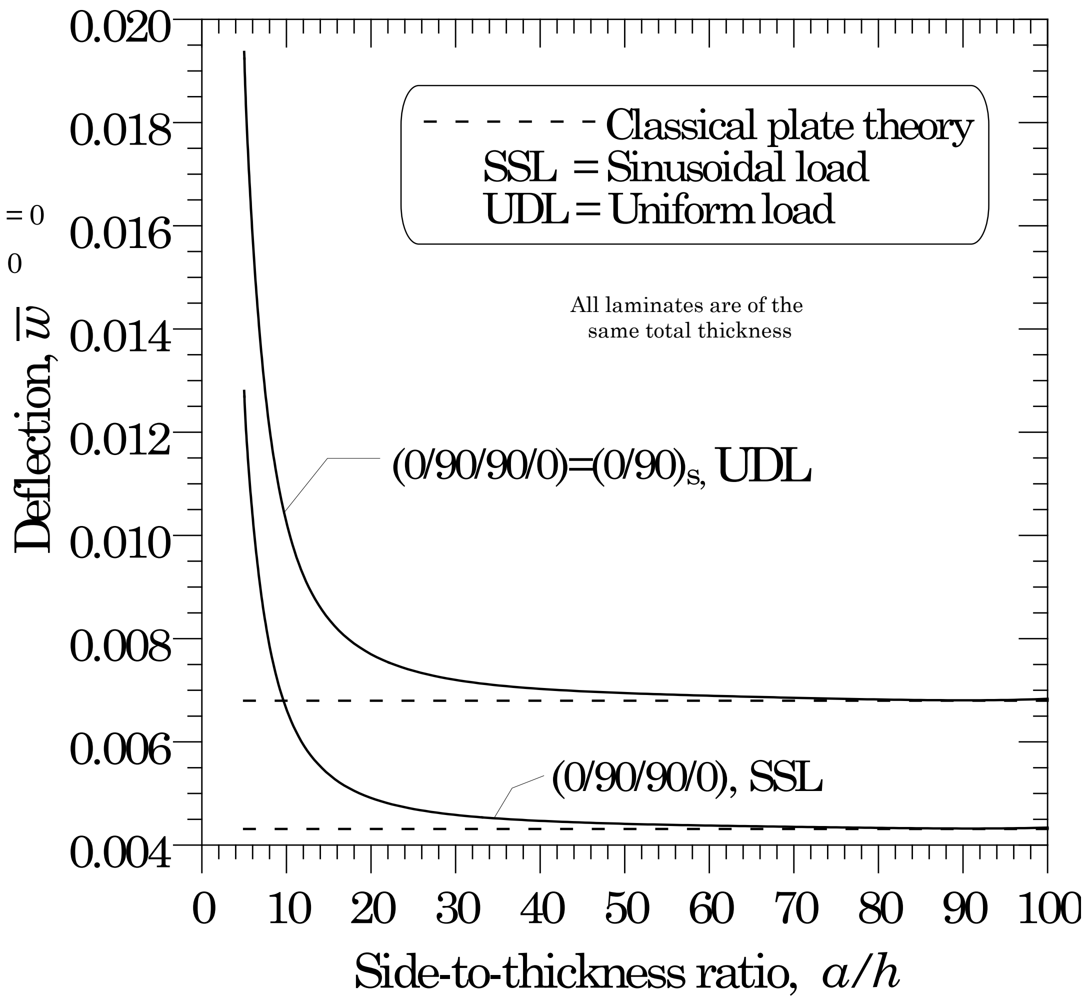

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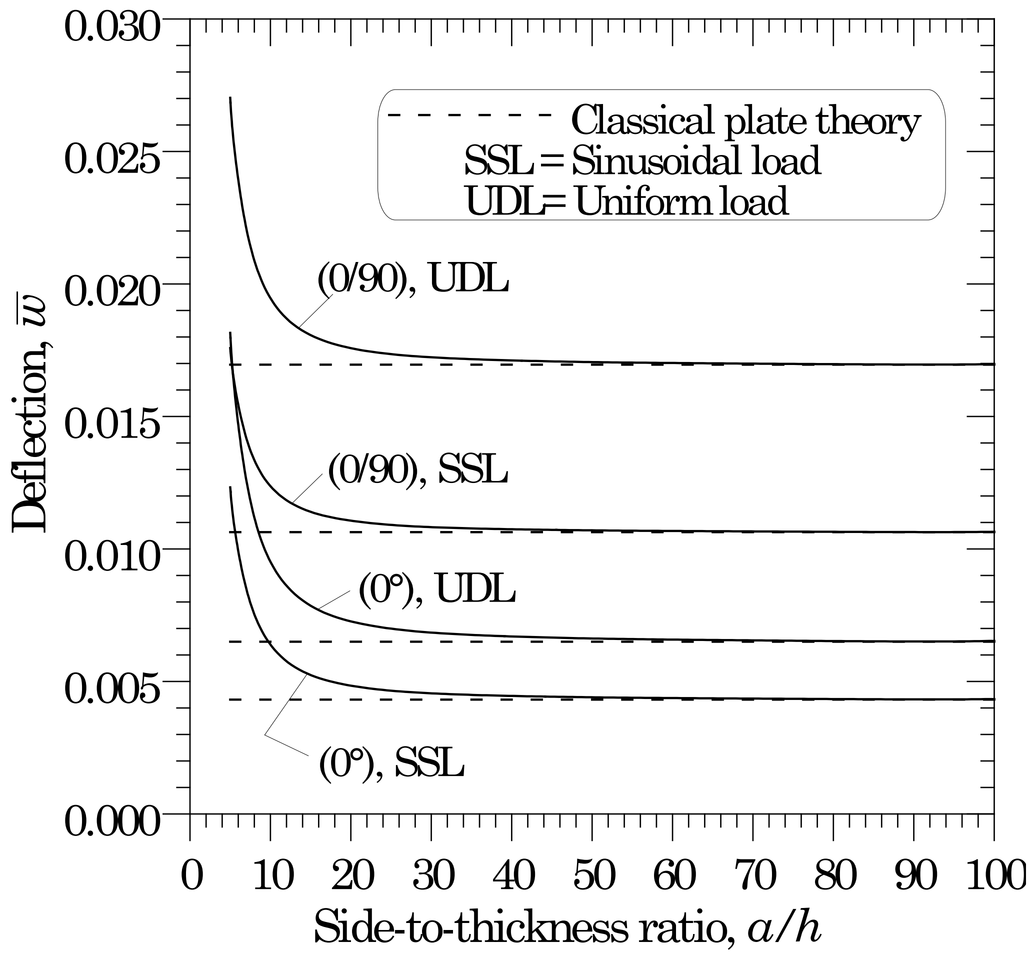
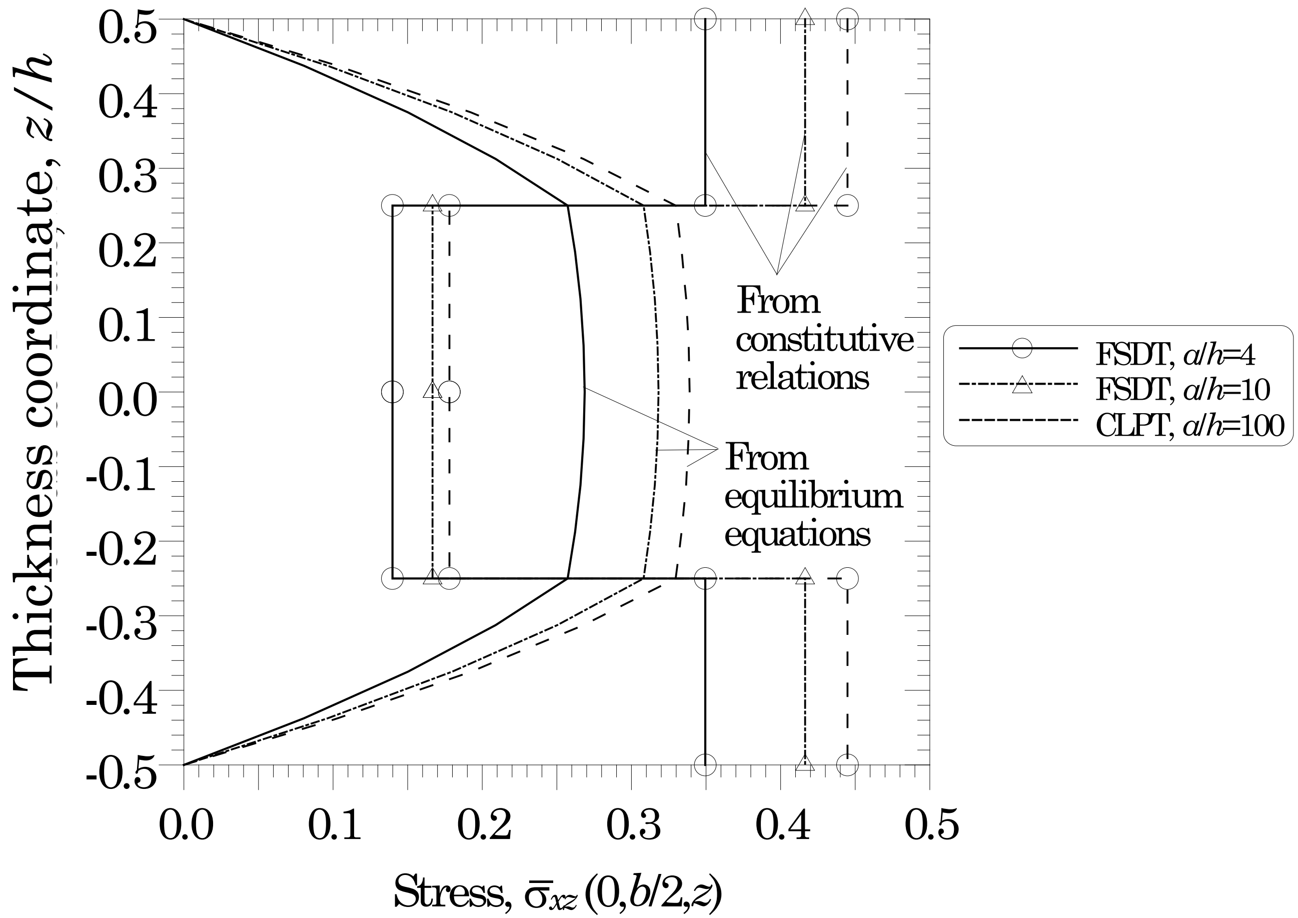
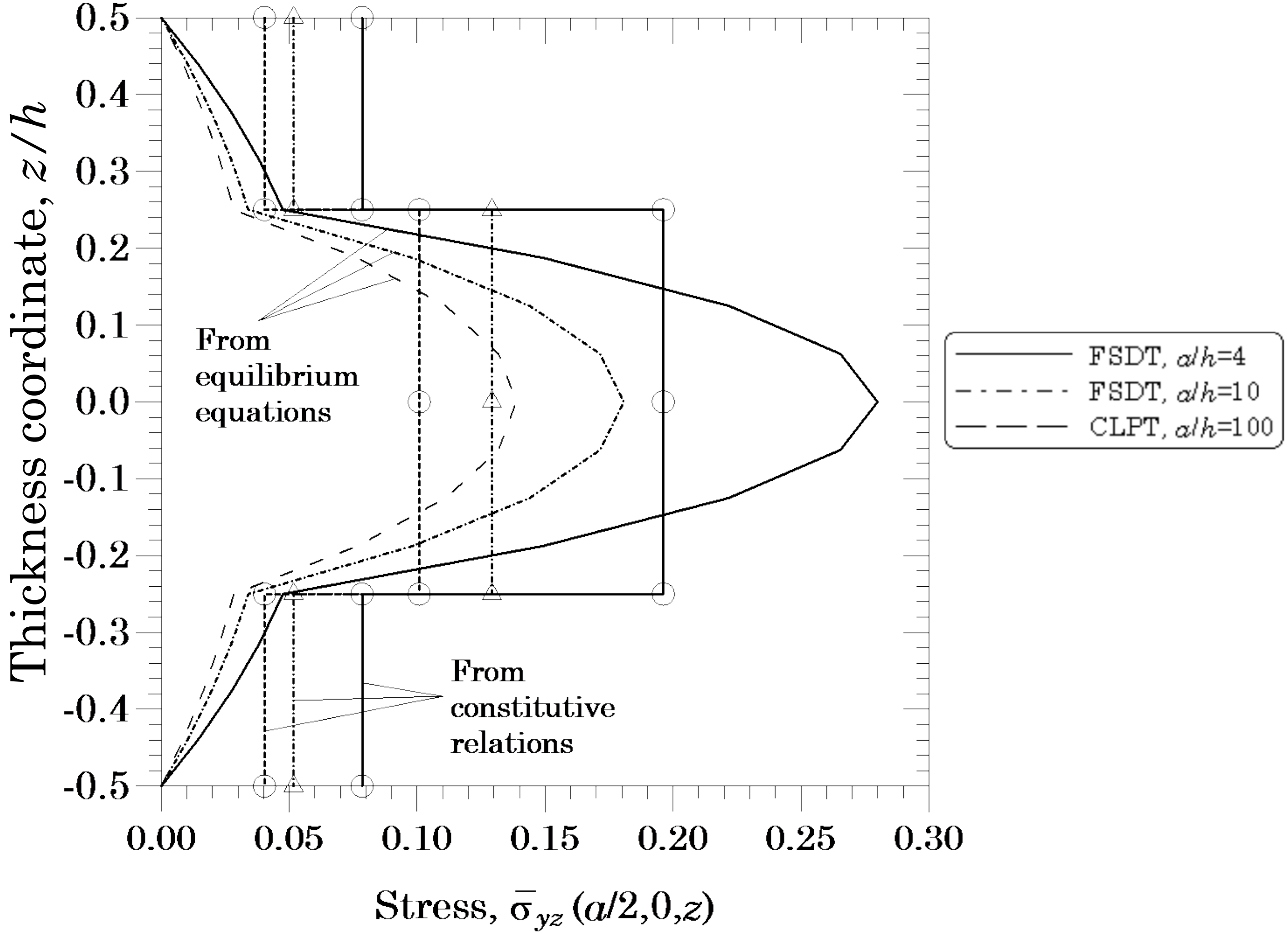
All laminates are of the same total thickness



Transverse Shear Stresses (0/90/90/0)



Transverse Shear Stresses (0/90/90/0)



Effect of Shear Deformation on Bending Deformation

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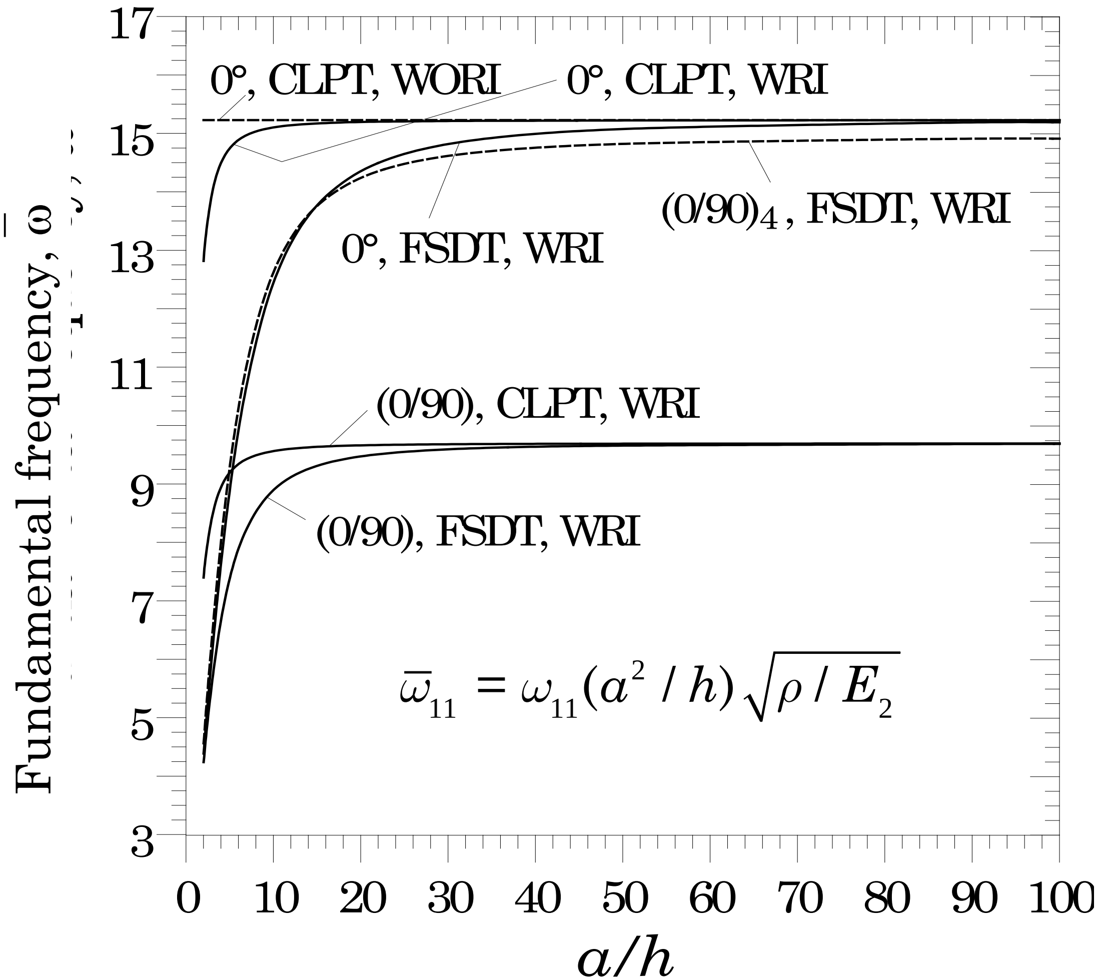

SUMMARY

- Kinematics of deformation
- Displacement Fields
- Lamination Scheme and Notation
- Classical Laminate Plate Theory
- First-Order Shear Deformation Theory
- Navier's Solutions