LAMINATE PLATE THEORIES AND NAVIER’S SOLUTIONS OF RECTANGULAR PLATES
For Bending and Vibration

- Kinematics of deformation
- Displacement Fields
- Lamination Scheme and Notation
- Classical Laminate Plate Theory
- First-Order Shear Deformation Theory
- Navier’s Solutions
Kinematics of the Classical and Shear Deformation Plate Theories

- **Undeformed Edge**
- **Classical (Kirchhoff) Plate Theory (CPT)**
- **First-Order (Mindlin) Plate Theory (FST)**

Order refers to the thickness coordinate power in the displacement expansion with independent functions.

JN Reddy
Displacement Fields of Various Theories

Classical Laminate Plate Theory (CLPT)

\[ u_1(x, y, z, t) = u(x, y, t) + z\theta_x(x, y, t) \quad \theta_x = -(\partial w / \partial x) \]
\[ u_2(x, y, z, t) = v(x, y, t) + z\theta_y(x, y, t) \quad \theta_y = -(\partial w / \partial y) \]
\[ u_3(x, y, z, t) = w(x, y, t) \]

First-order Shear Deformation Theory (FSDT)

\[ u_1(x, y, z, t) = u(x, y, t) + z\phi_x(x, y, t) \]
\[ u_2(x, y, z, t) = v(x, y, t) + z\phi_y(x, y, t) \]
\[ u_3(x, y, z, t) = w(x, y, t) \]

Higher-order Shear Deformation Theories (HSDT)

\[ u_1(x, y, z, t) = \sum_{i=0}^{M} z^i \phi_1^{(i)}(x, y, t) \]
\[ u_2(x, y, z, t) = \sum_{i=0}^{M} z^i \phi_2^{(i)}(x, y, t) \]
\[ u_3(x, y, z, t) = \sum_{i=0}^{N} z^i \phi_3^{(i)}(x, y, t) \]
Notation and Coordinate System Used in a Laminate Analysis

Layers are numbered in the +ve $z$ direction
CLASSICAL LAMINATE PLATE THEORY

**Displacement Field**

\[ u_1(x, y, z, t) = u(x, y, t) + z\theta_x(x, y, t) \quad , \quad \theta_x = -\frac{\partial w}{\partial x} \]

\[ u_2(x, y, z, t) = v(x, y, t) + z\theta_y(x, y, t) \quad , \quad \theta_y = -\frac{\partial w}{\partial y} \]

\[ u_3(x, y, z, t) = w(x, y, t) \]
Nonlinear strains

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \]

\[ \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \frac{\partial u_m}{\partial x_1} \frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} \right)^2 \]

Strain Field

\[ \varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x} \right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + z \frac{\partial \theta_x}{\partial x} \]

\[ \varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2} \left( \frac{\partial u_3}{\partial y} \right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + z \frac{\partial \theta_y}{\partial y} \]

\[ \gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + z \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \]

\[ \gamma_{xz} = 0, \quad \gamma_{yz} = 0, \quad \theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y} \]
Strain and Stress Distributions Through the Thickness of the Laminate

Strain distribution is continuous through laminate thickness

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\varepsilon^{(0)}_{xx} \\
\varepsilon^{(0)}_{yy} \\
\gamma^{(0)}_{xy}
\end{bmatrix} + z \begin{bmatrix}
\varepsilon^{(1)}_{xx} \\
\varepsilon^{(1)}_{yy} \\
\gamma^{(1)}_{xy}
\end{bmatrix}
\]

Stress distribution is discontinuous through laminate thickness

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{xz}
\end{bmatrix}^{(k)} = 
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}^{(k)}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{yz} \\
\sigma_{xz}
\end{bmatrix}^{(k)} =
\begin{bmatrix}
\bar{Q}_{44} & \bar{Q}_{45} \\
\bar{Q}_{45} & \bar{Q}_{55}
\end{bmatrix}^{(k)}
\begin{bmatrix}
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix}
\]

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### Principle of Virtual Displacements

\[
0 = \int_{V^e} \sigma_{ij} \delta \varepsilon_{ij} \, dV - \int_{\Omega^e} q \, \delta w \, dxdy - \int_{\Gamma^e} (V_n \delta w + M_n \delta \theta_n) \, ds
\]

\[
= \int_{\Omega^e} \int_{-h/2}^{h/2} (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \sigma_{xy} 2\delta \varepsilon_{xy}) \, dz \, dxdy
\]

\[
- \int_{\Omega^e} q \, \delta w \, dxdy - \int_{\Gamma^e} (N_n \delta u_n + N_{ns} \delta u_{ns} + V_n \delta w + M_n \delta \theta_n) \, ds
\]

\[
= \int_{\Omega^e} \left\{ \int_{-h/2}^{h/2} \left[ \sigma_{xx} \left( \frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} - z \frac{\partial^2 \delta w}{\partial x^2} \right) + \sigma_{yy} \left( \frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} - z \frac{\partial^2 \delta w}{\partial y^2} \right) + \sigma_{xy} \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} - 2z \frac{\partial^2 \delta w}{\partial x \partial y} \right) \right] \, dz \right\} \, dxdy
\]

\[
- \int_{\Omega^e} q \, \delta w \, dxdy - \int_{\Gamma^e} (N_n \delta u_n + N_{ns} \delta u_{ns} + V_n \delta w + M_n \delta \theta_n) \, ds
\]
Principle of Virtual Displacements (continued)

\[ 0 = \int_{\Omega_e} \left\{ N_{xx} \left( \frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + N_{yy} \left( \frac{\partial^2 \delta v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) \right. \]

\[ + N_{xy} \left( \frac{\partial \delta u}{\partial y} \frac{\partial \delta w}{\partial x} + \frac{\partial \delta v}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) \]

\[ - M_{xx} \frac{\partial^2 \delta w}{\partial x^2} - M_{yy} \frac{\partial^2 \delta w}{\partial y^2} - 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} \right\} \, dx \, dy \]

\[ - \int_{\Omega_e} q \delta w \, dx \, dy - \int_{\Gamma_e} \left( N_n \delta u_n + N_{ns} \delta u_{ns} + V_n \delta w + M_n \delta \theta_n \right) \, ds \]

\[
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \, dz, \quad \begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} z \, dz
\]
SIGN CONVENTION FOR STRESS RESULTANTS

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CONVENTIONAL STRESS RESULTANTS

\[ N_{\xi \eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi \eta} \, dz \quad (\xi, \eta = x, y, z) \]

\[ M_{\xi \eta} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{\xi \eta z} \, dz \quad (\xi, \eta = x, y) \]

\[ Q_x = N_{xz}, \quad Q_y = N_{yz} \]
STRESS RESULTANTS IN TERMS OF
THE MEMBRANE AND BENDING STRAINS

\[
\begin{align*}
\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \, dz \\
&= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{bmatrix} \, dz \\
\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} \\
\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} \, z \, dz \\
&= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{bmatrix} \, z \, dz \\
\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix}
\end{align*}
\]
DEFINITION OF THE LAMINATE STIFFNESSES

Standard A, B, and D coefficients

\[
(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}(1, z, z^2) dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2) dz
\]

\[
A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k), \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2)
\]

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3)
\]

Higher-order coefficients

\[
H_{ij}^{(n)} = \sum_{k=1}^{NL} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} z^n dz = \frac{1}{n+1} \sum_{k=1}^{NL} \bar{Q}_{ij}^{(k)} (z_{k+1}^{n+1} - z_k^{n+1})
\]

\[
H_{ij}^{(0)} = A_{ij}, \quad H_{ij}^{(1)} = B_{ij}, \quad H_{ij}^{(2)} = D_{ij}, \quad \text{and so on}
\]
Equations of Motion of the CLPT

\[ \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \]  \hspace{1cm} (1)

\[ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \]  \hspace{1cm} (2)

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) + q = I_0 \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (3)

\[ \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2} \]  \hspace{1cm} (4)

\[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2} \]  \hspace{1cm} (5)
GOVERNING EQUATIONS OF
CLASSICAL PLATE THEORY (Nonlinear)

Solve Eqs. (4) and (5) for shear forces in terms of the moments and substitute into Eq. (3) to obtain

\[
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} \tag{1}
\]

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} \tag{2}
\]

\[
\frac{\partial}{\partial x} \left( \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) + \\
\frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) + q + \\
I_0 \frac{\partial^2 w}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{3}
\]
Stress Resultant-Displacement Relations

\[
N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} \, dz = \int_{-h/2}^{h/2} \left[ \bar{Q}_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \right) \right] \, dz \\
+ \int_{-h/2}^{h/2} \bar{Q}_{16} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2z \frac{\partial^2 w}{\partial x \partial y} \right) \, dz
\]

\[
= A_{11} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{12} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{16} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \\
- B_{11} \frac{\partial^2 w}{\partial x^2} - B_{12} \frac{\partial^2 w}{\partial y^2} - 2B_{16} \frac{\partial^2 w}{\partial x \partial y}
\]

\[
M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z \, dz = \int_{-h/2}^{h/2} z \left[ \bar{Q}_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \right) \right] \, dz \\
+ \int_{-h/2}^{h/2} z \bar{Q}_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2z \frac{\partial^2 w}{\partial x \partial y} \right) \, dz
\]

\[
= B_{11} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{12} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + B_{16} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \\
- D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y}
\]
Laminated Plate Constitutive Equations

\[ N_{xx} = A_{11} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{12} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \]

\[ -B_{11} \frac{\partial^2 w}{\partial x^2} - B_{12} \frac{\partial^2 w}{\partial y^2} - 2B_{16} \frac{\partial^2 w}{\partial x \partial y} - N_{xx}^T \]

\[ N_{yy} = A_{12} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{22} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \]

\[ -B_{12} \frac{\partial^2 w}{\partial x^2} - B_{22} \frac{\partial^2 w}{\partial y^2} - 2B_{26} \frac{\partial^2 w}{\partial x \partial y} - N_{yy}^T \]

\[ N_{xy} = A_{16} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{26} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \]

\[ -B_{16} \frac{\partial^2 w}{\partial x^2} - B_{26} \frac{\partial^2 w}{\partial y^2} - 2B_{66} \frac{\partial^2 w}{\partial x \partial y} \]

\[ M_{xx} = B_{11} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{12} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T \]

\[ M_{yy} = B_{12} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{22} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + B_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{12} \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^2 w}{\partial y^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T \]

\[ M_{xy} = B_{16} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{26} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + B_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{16} \frac{\partial^2 w}{\partial x^2} - D_{26} \frac{\partial^2 w}{\partial y^2} - 2D_{66} \frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T \]
**FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)**

**Displacement Field**

\[ u_1(x, y, z, t) = u(x, y, t) + z\phi_x(x, y, t) \]
\[ u_2(x, y, z, t) = v(x, y, t) + z\phi_y(x, y, t) \]
\[ u_3(x, y, z, t) = w(x, y, t) \]

**Nonlinear strains**

\[ \varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x} \right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + z \frac{\partial \phi_x}{\partial x} , \]
\[ \varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2} \left( \frac{\partial u_3}{\partial y} \right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + z \frac{\partial \phi_y}{\partial y} \]
\[ \gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial w}{\partial y} + z \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \]
\[ \gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x} , \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y} \]
STRAIN-DISPLACEMENT EQUATIONS
(FSDT)

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon^{(0)}_{xx} \\
\varepsilon^{(0)}_{yy} \\
\gamma^{(0)}_{yz} \\
\gamma^{(0)}_{xz} \\
\gamma^{(0)}_{xy}
\end{bmatrix} + z
\begin{bmatrix}
\varepsilon^{(1)}_{xx} \\
\varepsilon^{(1)}_{yy} \\
\gamma^{(1)}_{yz} \\
\gamma^{(1)}_{xz} \\
\gamma^{(1)}_{xy}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
\frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\
\frac{\partial w}{\partial y} + \phi_y \\
\frac{\partial w}{\partial x} + \phi_x \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
\end{bmatrix} + z
\begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
0 \\
0 \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{bmatrix}
\]
EQUATIONS OF MOTION OF FSDT

\[\begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x &= I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y &= I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) + q &= I_0 \frac{\partial^2 w}{\partial t^2} \\
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y &= I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2}
\end{align*}\]
Stresses and Stress Resultants on an edge of a Plate

\[ \begin{align*}
N_{xx} &= \int_{-h/2}^{h/2} \sigma_{xx} \, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} \, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz, \quad Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz} \, dz \\
M_{xx} &= \int_{-h/2}^{h/2} z\sigma_{xx} \, dz, \quad M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy} \, dz, \quad M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy} \, dz, \quad Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz} \, dz
\end{align*} \]
THE FIRST-ORDER SHEAR DEFORMATION THEORY

Stress Resultants (Nonlinear)

\[ N_{xx} = A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \frac{\partial \phi_x}{\partial x} + B_{12} \frac{\partial \phi_y}{\partial y} + B_{16} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) - N_{xx}^T \]

\[ N_{yy} = A_{12} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{22} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{12} \frac{\partial \phi_x}{\partial x} + B_{22} \frac{\partial \phi_y}{\partial y} + B_{26} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) - N_{yy}^T \]

\[ M_{xx} = B_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T \]

\[ M_{yy} = B_{12} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{22} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{12} \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^2 w}{\partial y^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T \]

\[ M_{xy} = B_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - D_{16} \frac{\partial^2 w}{\partial x^2} - D_{26} \frac{\partial^2 w}{\partial y^2} - 2D_{66} \frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T \]

\[ Q_x = K_s A_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) + K_s A_{45} \left( \phi_y + \frac{\partial w}{\partial y} \right); \quad Q_y = K_s A_{45} \left( \phi_x + \frac{\partial w}{\partial x} \right) + K_s A_{44} \left( \phi_y + \frac{\partial w}{\partial y} \right)\]
Linear Equations for Pure Bending of orthotropic plates

Equation of equilibrium
\[
\frac{\partial}{\partial x} \left( \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) + q = 0
\]

Moment-deflection relations
\[
M_{xx} = -D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y}
\]
\[
M_{yy} = -D_{12} \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^2 w}{\partial y^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y}
\]
\[
M_{xy} = -D_{16} \frac{\partial^2 w}{\partial x^2} - D_{26} \frac{\partial^2 w}{\partial y^2} - 2D_{66} \frac{\partial^2 w}{\partial x \partial y}
\]

Equilibrium equation in terms of the deflection
\[
- \left[ D_{11} \frac{\partial^4 w}{\partial x^4} + 2 \left( D_{66} + 2D_{12} \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \right] \\
- 4 \frac{\partial^2}{\partial x \partial y} \left( D_{16} \frac{\partial^2 w}{\partial x^2} + D_{26} \frac{\partial^2 w}{\partial y^2} \right) + q = 0
\]
Navier Solution of Simply Supported ORTHOTROPIC PLATES

\[ w = \phi_x = 0, \quad M_{yy} = 0 \]

\[ w = \phi_y = 0, \quad M_{xx} = 0 \]

\[ w_0(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y \]

\[ q(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \]

\[ Q_{mn} = \frac{4}{ab} \int_{0}^{b} \int_{0}^{a} q(x, y) \sin \alpha x \sin \beta y \, dx \, dy \]
Navier Solution of Simply Supported ORTHOTROPIC PLATES

\[- \left[ D_{11} \frac{\partial^4 w_0}{\partial x^4} + 2 (D_{12} + 2D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_0}{\partial y^4} \right] + q = 0 \]

\[ w_0(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y \]

\[ q(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \]

\[ Q_{mn} = \frac{4}{ab} \int_{0}^{b} \int_{0}^{a} q(x, y) \sin \alpha x \sin \beta y \, dx \, dy \]

\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ -W_{mn} \left[ D_{11} \alpha^4 + 2(D_{12} + 2D_{66})\alpha^2 \beta^2 + D_{22} \beta^4 \right] + Q_{mn} \right\} \sin \alpha x \sin \beta y = 0 \]

\[ W_{mn} = \frac{Q_{mn}}{d_{mn}}, \quad d_{mn} = \frac{\pi^4}{b^4} \left[ D_{11} m^4 s^4 + 2(D_{12} + 2D_{66})m^2 n^2 s^2 + D_{22} n^4 \right] \]
The total thickness of all laminates is the same (i.e., all contain 24 layers of the same thickness)

SS-1 Boundary Conditions

\[
\begin{align*}
w_0 &= \phi_x = 0 \\
M_{xy} &= 0 \\
w_0 &= \phi_y = 0 \\
M_{xx} &= 0 \\
w_0 &= \phi_x = 0 \\
M_{xy} &= 0 \\
w_0 &= \phi_y = 0 \\
M_{xx} &= 0 \\
w_0 &= \phi_x = 0 \\
M_{xy} &= 0
\end{align*}
\]

\[
E_1 = 25 E_2, \quad G_{12} = G_{13} = 0.5 E_2, \quad G_{23} = 0.2 E_2, \quad \nu_{12} = 0.25
\]

Sinusoidal load

\[
\bar{w} = w \left( \frac{E_2 h^3}{q_0 a^4} \right)
\]

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\[ E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2, \]
\[ G_{23} = 0.2E_2, \quad \nu_{12} = 0.25 \]
\[ \bar{\omega}_{11} = \omega_{11} \left( \frac{b^2}{\pi^2} \right) \sqrt{\frac{\rho h}{D_{22}}} \]

\[(0/90) = (0_{12} / 90_{12})\]
\[(0/90)_2 = (0_6 / 90_6)_2\]
\[(0/90)_4 = (0_3 / 90_3)_4\]
\[(0/90)_3 = (0_4 / 90_4)_3\]
ANALYTICAL SOLUTIONS (continued)

BENDING OF ANGLE-PLY PLATES -CLPT

\[ E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2, \]
\[ G_{23} = 0.2E_2, \quad \nu_{12} = 0.25 \]

\[ \bar{w} = w \left( \frac{E_2 h^3}{q_0 a^4} \right) \]
Effect of Shear Deformation on Bending Deformation

\[ w_0 = \phi_x = 0 \]
\[ M_{yy} = 0 \]
\[ w_0 = \phi_y = 0 \]
\[ M_{xx} = 0 \]

\[ E_1 = 25E_2, \]
\[ G_{12} = G_{13} = 0.5E_2, \]
\[ G_{23} = 0.2E_2, \]
\[ \nu_{12} = 0.25 \]

\[ \bar{w} = w \left( \frac{E_2 h^3}{q_0 a^4} \right) \]

-Arrow indicators for boundary conditions:
-\( w_0 = \phi_x = 0 \)
-\( M_{yy} = 0 \)
-\( w_0 = \phi_y = 0 \)
-\( M_{xx} = 0 \)

All laminates are of the same total thickness

SSL = Sinusoidal load
UDL = Uniform load

Classical plate theory

(0/90/90/0) = (0/90)_s, UDL

(0/90/90/0), SSL
Effect of Shear Deformation on Bending Deformation

\[ E_1 = 25 E_2, \]
\[ G_{12} = G_{13} = 0.5 E_2, \]
\[ G_{23} = 0.2 E_2, \]
\[ \nu_{12} = 0.25 \]

\[ \bar{w} = w \left( \frac{E_2 h^3}{q_0 a^4} \right) \]

All laminates are of the same total thickness
Transverse Shear Stresses (0/90/90/0)

From constitutive relations

From equilibrium equations

Thickness coordinate, \( z/h \)

Stress, \( \sigma_{xz}(0, b/2, z) \)

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Transverse Shear Stresses (0/90/90/0)

From equilibrium equations

From constitutive relations

Thickness coordinate, $z/h$

Stress, $\bar{\sigma}_{yz} (a/2, 0, z)$

FSDT, $a/h=4$
FSDT, $a/h=10$
CLPT, $a/h=100$
Effect of Shear Deformation on Bending Deformation

\[ E_1 = 25E_2, \]
\[ G_{12} = G_{13} = 0.5E_2, \]
\[ G_{23} = 0.2E_2, \]
\[ \nu_{12} = 0.25 \]

All laminates are of the same total thickness
SUMMARY

- Kinematics of deformation
- Displacement Fields
- Lamination Scheme and Notation
- Classical Laminate Plate Theory
- First-Order Shear Deformation Theory
- Navier’s Solutions