

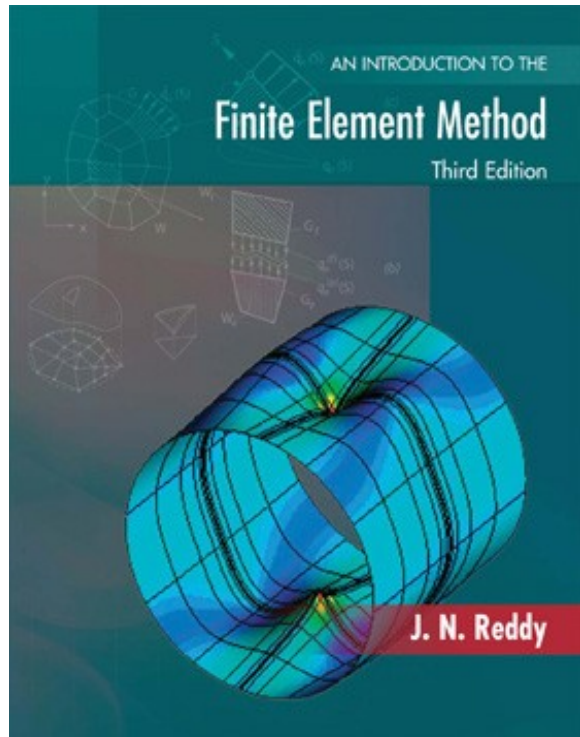
**MEEN 673**

# **Nonlinear Finite Element Analysis**

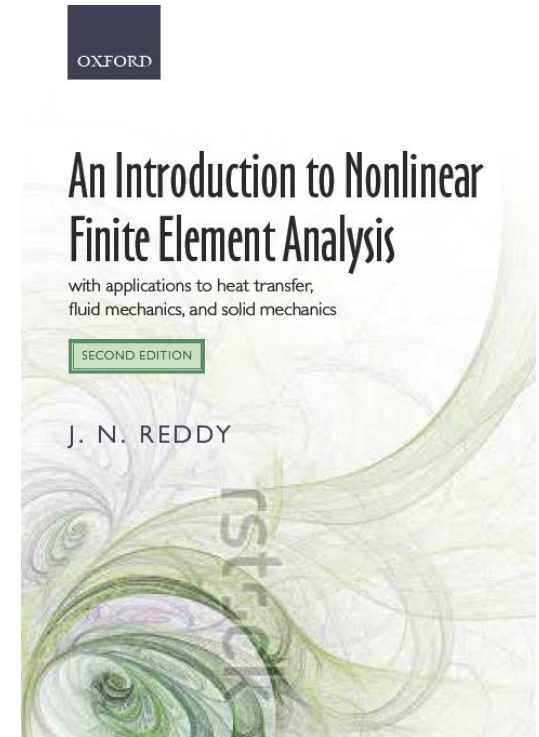
**Fall 2016**

## **Chapter 3**

### **INTRODUCTION AND OVERVIEW OF LINEAR FEM using a 2D model problem**



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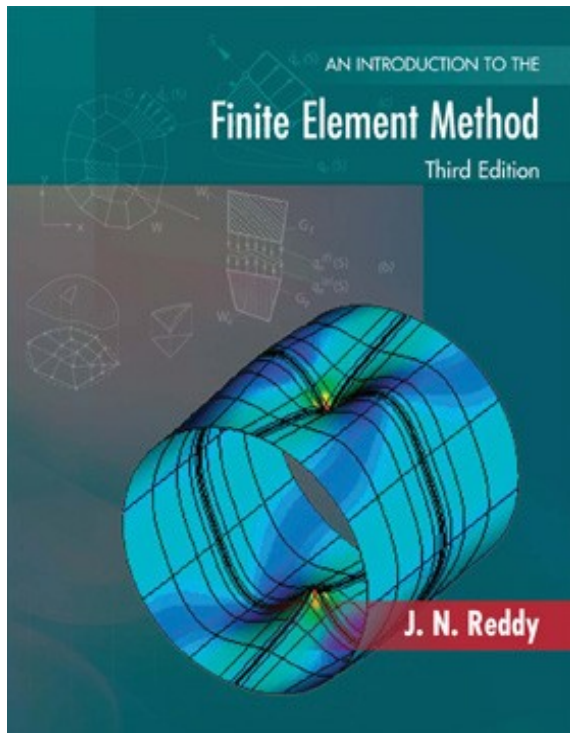


# An Overview of The Finite Element Method

## 2D Problems involving a single unknown

### CONTENTS

- Model equation Discretization
- Weak form development
- Finite element model
- Approximation functions
- Interpolation functions of higher-order elements
- Post-computation of variables
- Numerical examples
- Transient analysis of 2-D problems





# MODEL EQUATION

## Model Differential Equation

$$-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u}{\partial y} \right) = f \quad \text{in } \Omega$$

$a_{ij}$  – coefficients that describe material behavior

$f$  – source term

## Type of boundary conditions

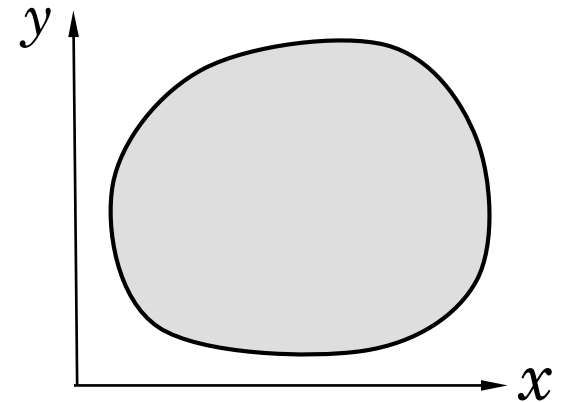
$$u = \hat{u}, \quad \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) n_x - \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) n_y - \hat{q}_n = 0$$

# EXAMPLES OF MODEL EQUATION

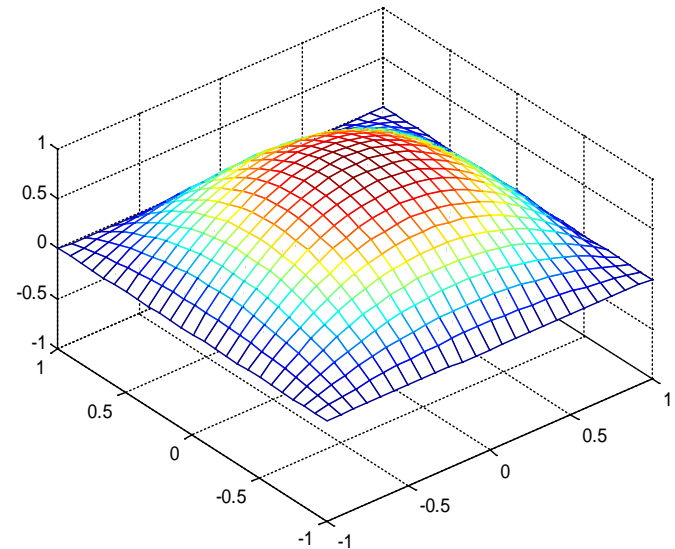
## (1)– Deflection of a membrane

$$-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u}{\partial y} \right) - f = 0$$

$u(x, y)$  = transverse deflection  
of a point in the membrane  
 $f(x, y)$  = applied pressure  
 $a_{11}(x, y), a_{22}(x, y)$  = tensions in the  $x$   
and  $y$  directions, respectively



Transverse deflection of a square membrane fixed on all its sides and subjected to uniform pressure.



# EXAMPLES OF MODEL EQUATION

## (2)– Torsion of a cylindrical member

$$-G\theta \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \text{ in } \Omega$$

$$u = -\theta zy, \quad v = \theta zx, \quad w = \theta \phi(x, y)$$

$$\left( \frac{\partial \phi}{\partial x} - y \right) n_x + \left( \frac{\partial \phi}{\partial y} - x \right) n_y = 0 \text{ on } \Gamma$$

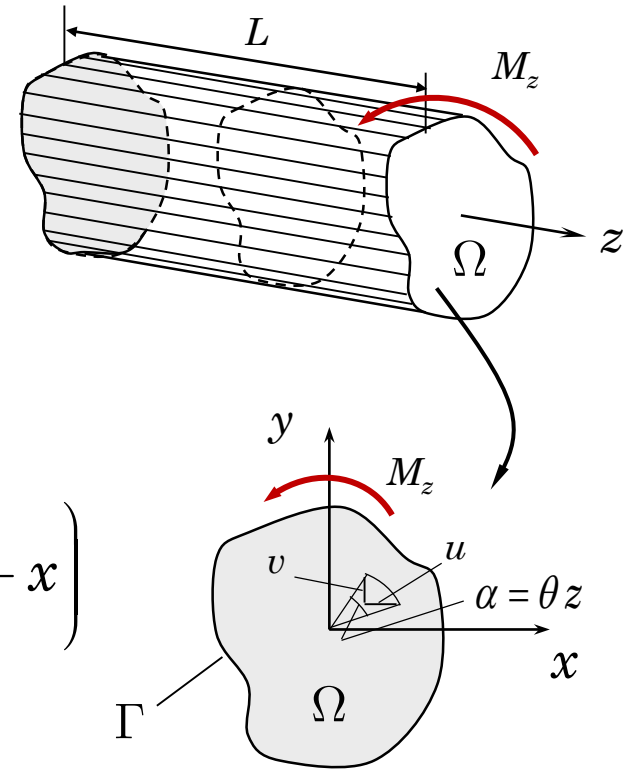
$$\sigma_{xz}(x, y) = G\theta \left( \frac{\partial \phi}{\partial x} - y \right), \quad \sigma_{yz}(x, y) = G\theta \left( \frac{\partial \phi}{\partial y} + x \right)$$

$\phi(x, y) =$  **warping function**

$$a_{11} = a_{22} = G\theta$$

$G =$  shear modulus

$\theta =$  angle of twist



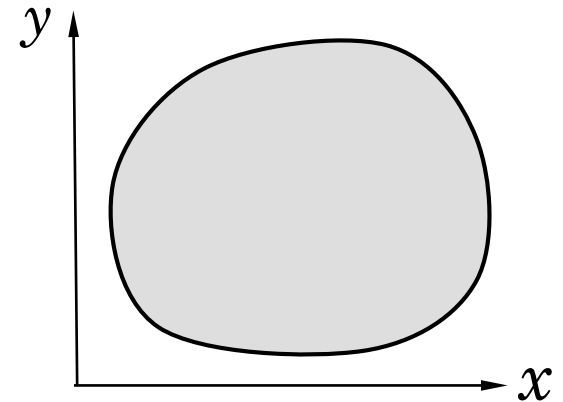
# EXAMPLES OF MODEL EQUATION

## (3)– 2D Heat transfer

$$-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u}{\partial y} \right) - f = 0$$

$$q_x(x, y) = -a_{11} \frac{\partial u}{\partial x}, \quad q_y(x, y) = -a_{22} \frac{\partial u}{\partial y}$$

$$-\left( a_{11} \frac{\partial u}{\partial x} n_x + a_{22} \frac{\partial u}{\partial y} n_y \right) = \hat{q}_n \quad \text{on } \Gamma$$



$u = T$ , temperature;  $a_{11} = k_x$ ,  $a_{22} = k_y$

$k_x$ ,  $k_y$  = thermal conductivities

in the  $x$  and  $y$  directions,  
respectively

$f$  = internal heat generation

$q_n$  = heat flux normal to the boundary

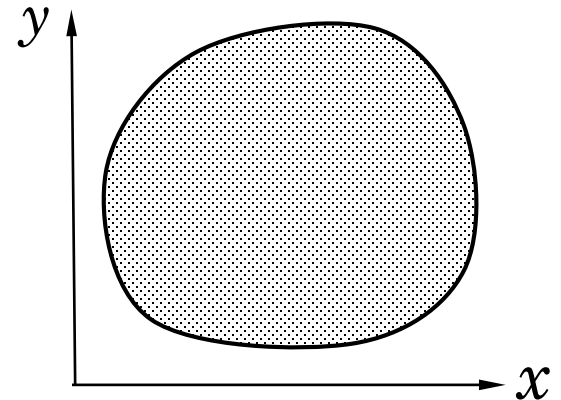
# EXAMPLES OF MODEL EQUATION

## (4) – 2D Inviscid flow

$$-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial \phi}{\partial y} \right) - f = 0$$

$$v_x(x, y) = -a_{11} \frac{\partial \phi}{\partial x}, \quad v_y(x, y) = -a_{22} \frac{\partial \phi}{\partial y}$$

$$-\left( a_{11} \frac{\partial \phi}{\partial x} n_x + a_{22} \frac{\partial \phi}{\partial y} n_y \right) = \hat{q}_n \quad \text{on } \Gamma$$



$u = \phi$ , water head (velocity potential)

$a_{11}$ ,  $a_{22}$  = permeabilities in the  $x$  and  $y$  directions,  
respectively

$f$  = internal infiltration

$q_n$  = flow normal to the boundary

# WEAK FORM DEVELOPMENT

**Approximation**

$$u(x, y) \approx u_h(x, y)$$

**Weak Form Development**

**Step 1**

$$0 = \int_{\Omega^e} w_i \left[ \underbrace{-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right)}_{F_1} - \underbrace{\frac{\partial}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right)}_{F_2} - f \right] dx dy$$

**Step 2: Trade differentiations between  $w$  and  $u_h$**

Consider the identity:

$$\frac{\partial}{\partial x} (w_i \cdot F_1) = \frac{\partial w_i}{\partial x} F_1 + w_i \frac{\partial F_1}{\partial x} \quad \text{or} \quad -w_i \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x} (w_i \cdot F_1) + \frac{\partial w_i}{\partial x} F_1$$

Green-Gauss Theorem

$$\int_{\Omega^e} \frac{\partial F}{\partial x} dx dy = \oint_{\Gamma^e} n_x F ds, \quad \int_{\Omega^e} \frac{\partial F}{\partial y} dx dy = \oint_{\Gamma^e} n_y F ds$$



# WEAK FORM DEVELOPMENT **continued**

## Step 2:

Consider the identity: 
$$-w_i \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x} \underbrace{(w_i \cdot F_1)}_F + \frac{\partial w_i}{\partial x} F_1$$

$$\int_{\Omega^e} \frac{\partial F}{\partial x} dx dy = \oint_{\Gamma^e} n_x F ds \Rightarrow \int_{\Omega^e} \frac{\partial F}{\partial x} dx dy = \oint_{\Gamma^e} n_x (w_i \cdot F_1) ds$$

$$\int_{\Omega^e} \frac{\partial F}{\partial x} dx dy = \oint_{\Gamma^e} n_x w_i \left( a_{11} \frac{\partial u_h}{\partial x} \right) ds$$

$$\int_{\Omega^e} \frac{\partial G}{\partial y} dx dy = \oint_{\Gamma^e} n_y w_i \left( a_{22} \frac{\partial u_h}{\partial y} \right) ds$$

# WEAK FORM DEVELOPMENT (continued)

**Step 1**

$$0 = \int_{\Omega^e} w_i \left[ -\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - f \right] dx dy$$
$$= \int_{\Omega^e} \left[ \frac{\partial w_i}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial w_i}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - w_i f \right] dx dy$$

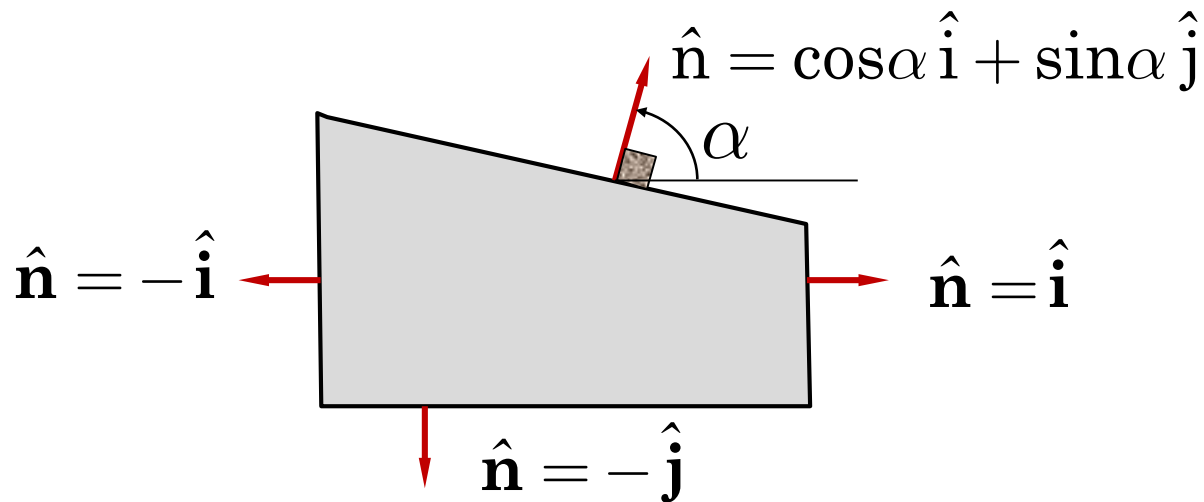
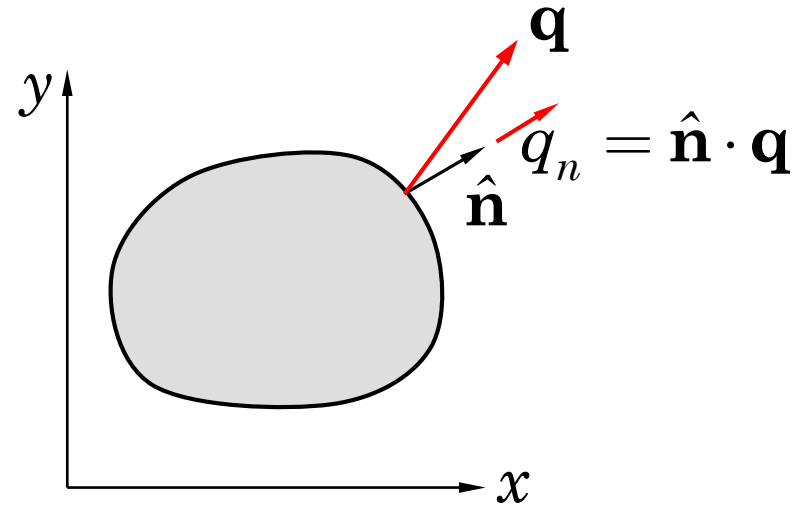
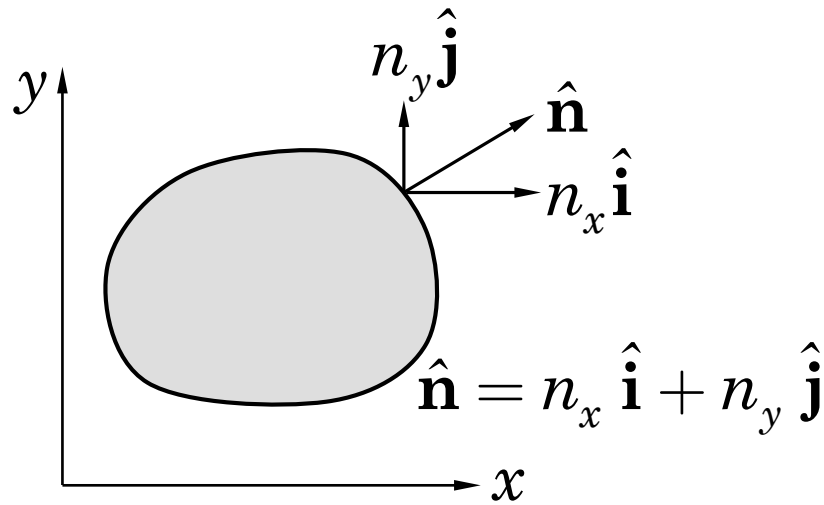
**Step 2**

$$- \oint_{\Gamma^e} w_i \left[ \left( a_{11} \frac{\partial u_h}{\partial x} \right) n_x + \left( a_{22} \frac{\partial u_h}{\partial y} \right) n_y \right] ds$$
$$= \int_{\Omega^e} \left[ \frac{\partial w_i}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial w_i}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - w_i f \right] dx dy$$

**Step 3**

$$- \oint_{\Gamma^e} w_i q_n ds$$
$$q_n = \left( a_{11} \frac{\partial u_h}{\partial x} \right) n_x + \left( a_{22} \frac{\partial u_h}{\partial y} \right) n_y, \text{ flux normal to the boundary}$$

# FINITE ELEMENT APPROXIMATION



# FINITE ELEMENT APPROXIMATION

## Weak form

$$0 = \int_{\Omega^e} \left[ \frac{\partial w_i}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial w_i}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - w_i f \right] dx dy$$
$$- \oint_{\Gamma^e} w_i q_n ds$$

## Approximation

$$u(x, y) \approx u_h(x, y) = c_1 + c_2 x + c_3 y + c_4 xy + \dots \quad (n \text{ terms})$$
$$= \sum_{j=1}^n u_j \psi_j(x, y)$$

**Finite element model** [the  $i$ th equation is obtained by replacing the weight function  $w$  by  $\psi_i$  ( $i = 1, 2, \dots, n$ ) ]

$$0 = \int_{\Omega^e} \left[ \frac{\partial \psi_i}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - \psi_i f \right] dx dy$$
$$- \oint_{\Gamma^e} \psi_i q_n ds$$

# FINITE ELEMENT MODEL DEVELOPMENT

## (continued)

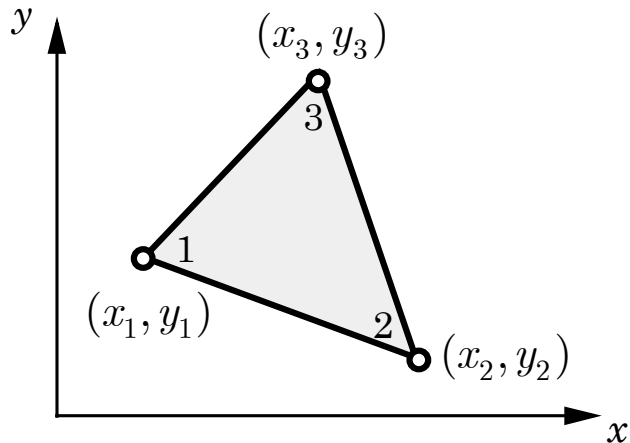
$$\begin{aligned}
 0 &= \int_{\Omega^e} \left[ \frac{\partial \psi_i}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) - \psi_i f \right] dx dy \\
 &\quad - \oint_{\Gamma^e} \psi_i q_n ds \\
 0 &= \int_{\Omega^e} \left[ \frac{\partial \psi_i}{\partial x} \sum_{j=1}^n u_j \left( a_{11} \frac{\partial \psi_j}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \sum_{j=1}^n u_j \left( a_{22} \frac{\partial \psi_j}{\partial y} \right) \right] dx dy \\
 &\quad - \int_{\Omega^e} \psi_i f dx dy - \oint_{\Gamma^e} \psi_i q_n ds \\
 &= \sum_{j=1}^n u_j \int_{\Omega^e} \left( a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy \\
 &\quad - \int_{\Omega^e} \psi_i f dx dy - \oint_{\Gamma^e} \psi_i q_n ds
 \end{aligned}$$

# FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$\begin{aligned} 0 &= \sum_{j=1}^n u_j \int_{\Omega^e} \left( a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy \\ &\quad - \int_{\Omega^e} \psi_i f dx dy - \oint_{\Gamma^e} \psi_i q_n ds \\ &= \sum_{j=1}^n K_{ij}^e u_j^e - f_i^e - Q_i^e = \sum_{j=1}^n K_{ij}^e u_j^e - F_i^e \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e \\ K_{ij}^e &= \int_{\Omega^e} \left( a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy, \\ F_i^e &= \int_{\Omega^e} \psi_i f dx dy + \oint_{\Gamma^e} \psi_i q_n ds \end{aligned}$$

# APPROXIMATION FUNCTIONS

## Linear Triangular Element



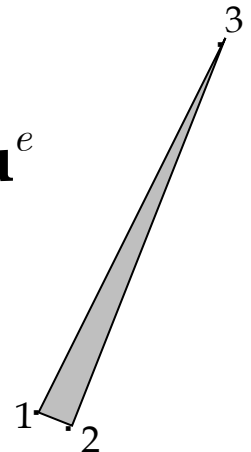
$$u_h^e(x, y) = c_1 + c_2x + c_3y$$

$$u_h^e(x_1, y_1) = c_1 + c_2x_1 + c_3y_1 \equiv u_1^e$$

$$u_h^e(x_2, y_2) = c_1 + c_2x_2 + c_3y_2 \equiv u_2^e$$

$$u_h^e(x_3, y_3) = c_1 + c_2x_3 + c_3y_3 \equiv u_3^e$$

$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \end{Bmatrix} \Rightarrow \mathbf{u}^e = \mathbf{A}^e \mathbf{c}^e \text{ or } \mathbf{c}^e = \mathbf{A}^{-1} \mathbf{u}^e$$



# APPROXIMATION FUNCTIONS

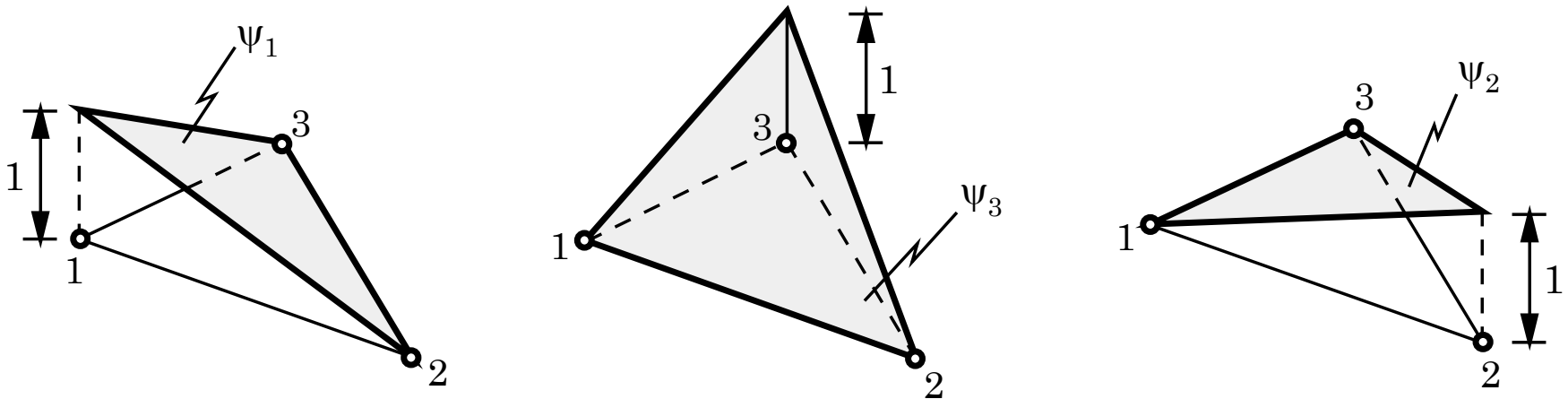
## Linear Triangular Element (continued)

$$\begin{aligned}
 u_h^e(x, y) &= c_1 + c_2x + c_3y = \begin{Bmatrix} 1 & x & y \end{Bmatrix} \begin{Bmatrix} c^e \end{Bmatrix} = \begin{Bmatrix} 1 & x & y \end{Bmatrix} [A]^{-1} \begin{Bmatrix} u^e \end{Bmatrix} \\
 &= \psi_1^e(x, y)u_1^e + \psi_2^e(x, y)u_2^e + \psi_3^e(x, y)u_3^e = \sum_{j=1}^3 u_j^e \psi_j^e(x, y)
 \end{aligned}$$

$$\psi_i^e(x, y) = \frac{1}{2\Delta^e} (\alpha_i + \beta_i x + \gamma_i y)$$

$$\alpha_i = x_j y_k - y_j x_k, \quad \beta_i = y_j - y_k, \quad \gamma_i = -(x_j - x_k)$$

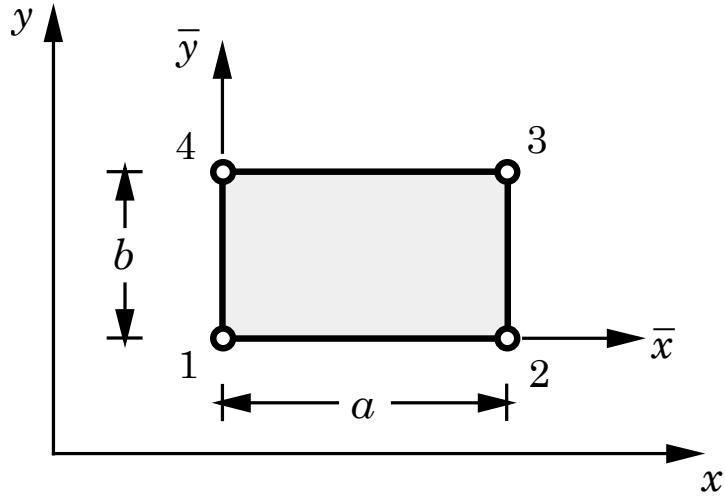
$\Delta^e$  = Area of the triangle,     $2\Delta^e$  = Determinant of  $[A]$





# APPROXIMATION FUNCTIONS

## Linear Rectangular Element



$$u_h^e(\bar{x}, \bar{y}) = c_1 + c_2 \bar{x} + c_3 \bar{y} + c_4 \bar{x} \bar{y}$$

$$u_h^e(\bar{x}_1, \bar{y}_1) = c_1 = u_1^e$$

$$u_h^e(\bar{x}_2, \bar{y}_2) = c_1 + c_2 a \equiv u_2^e$$

$$u_h^e(\bar{x}_3, \bar{y}_3) = c_1 + c_2 a + c_3 b + c_4 ab \equiv u_3^e$$

$$u_h^e(\bar{x}_4, \bar{y}_4) = c_1 + c_3 b \equiv u_4^e$$

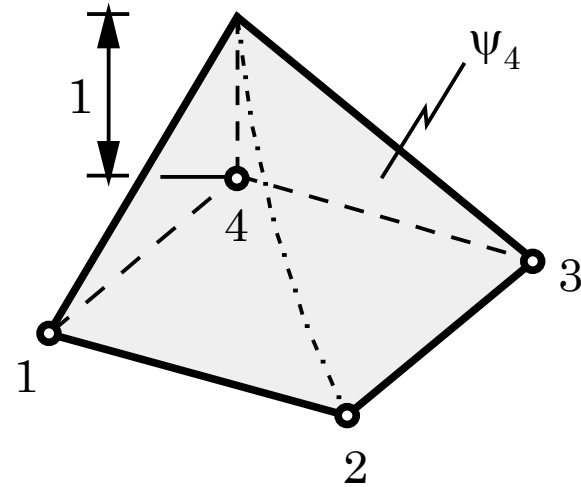
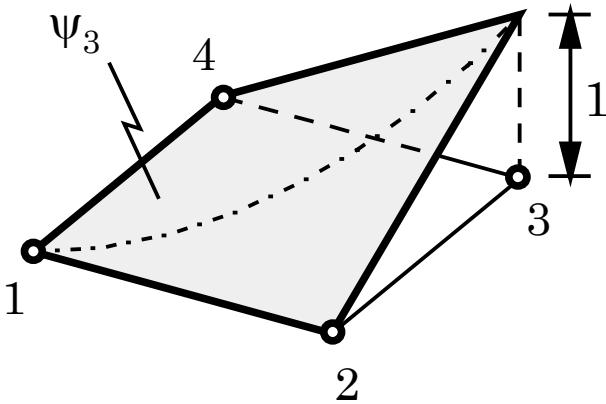
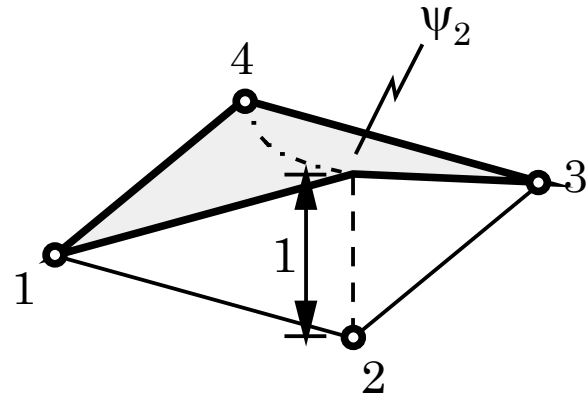
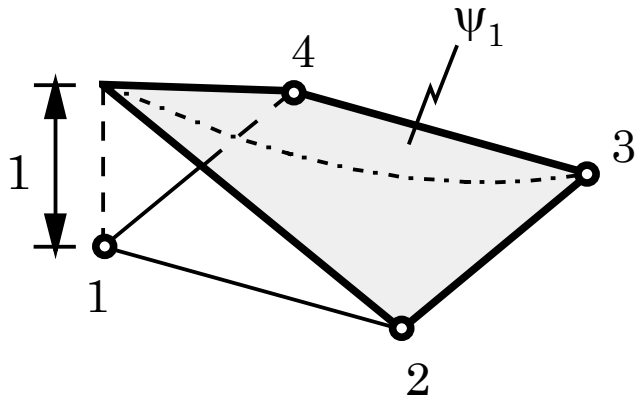
$$\psi_1^e(\bar{x}, \bar{y}) = \left(1 - \frac{\bar{x}}{a}\right) \left(1 - \frac{\bar{y}}{b}\right), \quad \psi_2^e(\bar{x}, \bar{y}) = \frac{\bar{x}}{a} \left(1 - \frac{\bar{y}}{b}\right),$$

$$\psi_3^e(\bar{x}, \bar{y}) = \frac{\bar{x}}{a} \frac{\bar{y}}{b}, \quad \psi_4^e(\bar{x}, \bar{y}) = \left(1 - \frac{\bar{x}}{a}\right) \frac{\bar{y}}{b},$$

$$\psi_j^e(x_i, y_i) = \delta_{ij}$$

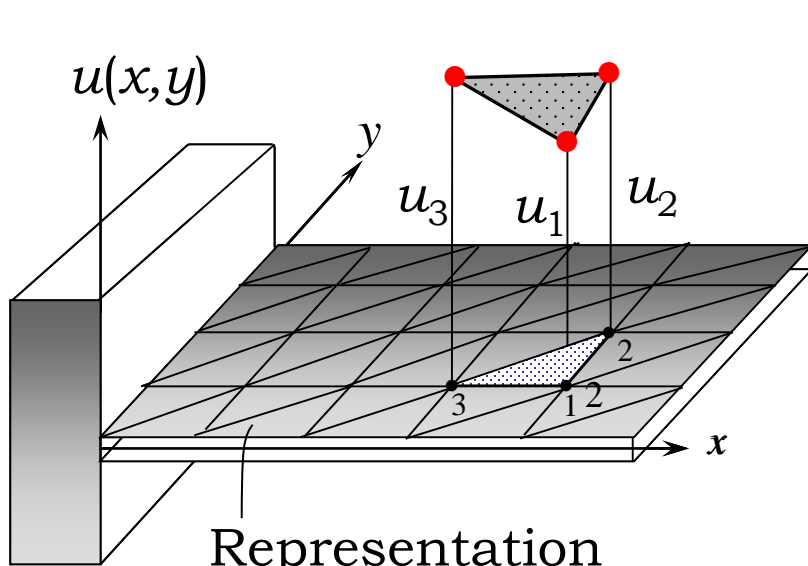
# APPROXIMATION FUNCTIONS

## Linear Rectangular Element (continued)

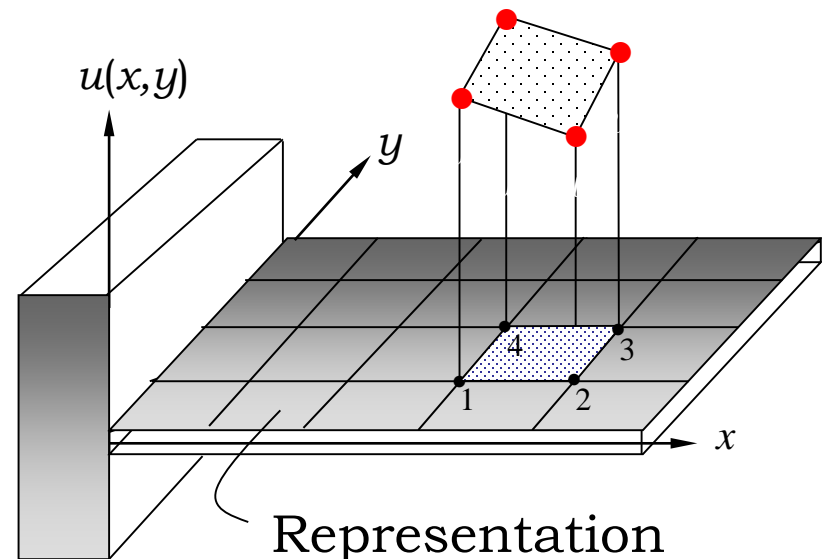


# Interpolation Property of the Approximation Functions

$$\psi_j^e(x_i, y_i) = \begin{cases} 1, & i = j, \text{ a fixed value} \\ 0, & i \neq j, \text{ a fixed value} \end{cases} = \delta_{ij}; \quad \sum_{j=1}^3 \psi_j^e(x, y) = 1$$



Representation of the *domain* by 3-node triangles



Representation of the *domain* by 4-node rectangles

# NUMERICAL EVALUATION OF COEFFICIENT MATRICES

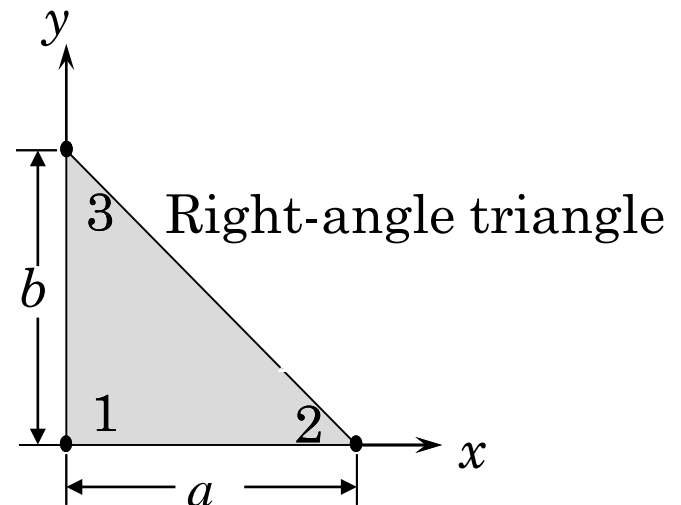
## Linear Right-Angled Triangular Element

$$[K^e] = a_{11}^e \frac{b}{2a} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22}^e \frac{a}{2b} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

When  $a_{11}^e = a_{22}^e = k_e$ , we have

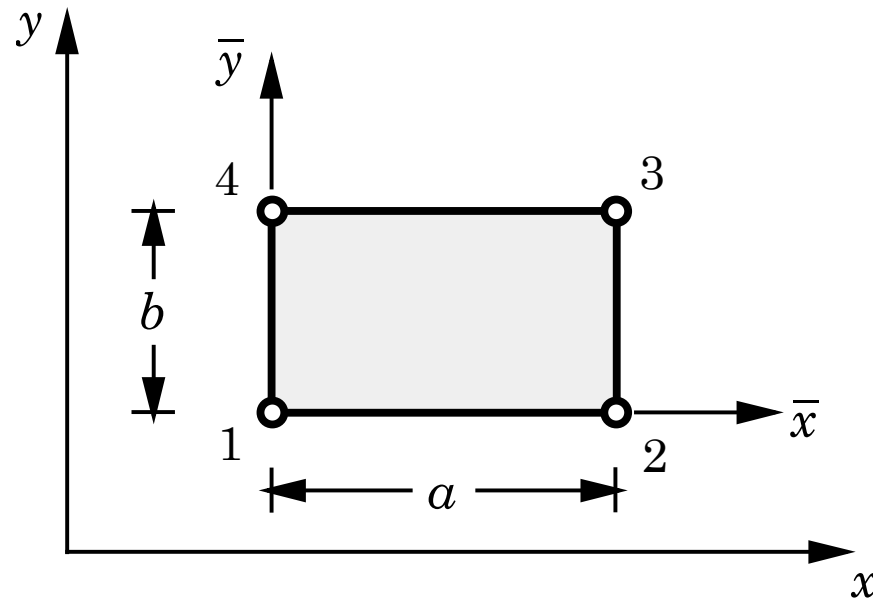
$$[K^e] = \frac{k_e}{2ab} \begin{bmatrix} a^2 + b^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix}$$

$$\{f^e\} = \frac{f_e \Delta^e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



# NUMERICAL EVALUATION OF COEFFICIENT MATRICES

## Linear Rectangular Element



$$[K^e] = a_{11}^e \frac{b}{6a} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + a_{22}^e \frac{a}{6b} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}, \quad \{f^e\} = \frac{f_e ab}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

# PARAMETRIC FORMULATIONS (2-D)

**Geometry:** 
$$x = \sum_{j=1}^m x_j^e \hat{\psi}_j^e(\xi, \eta), \quad y = \sum_{j=1}^m y_j^e \hat{\psi}_j^e(\xi, \eta)$$

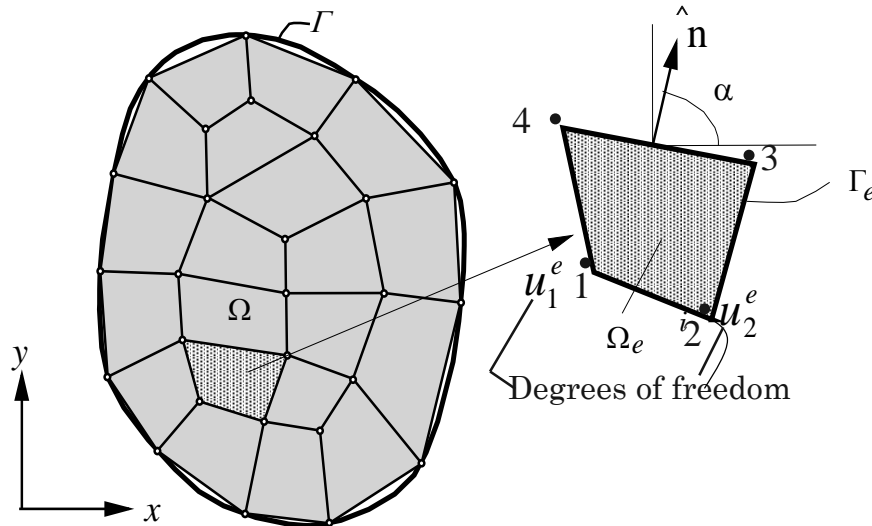
**Solution:** 
$$u(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta))$$

Thus, there are two meshes in finite element analysis.

1. **Superparametric ( $m > n$ ):** The polynomial degree of approximation used for the geometry is of higher order than that used for the dependent variable.
2. **Isoparametric ( $m = n$ ):** Equal degree of approximation is used for both geometry and dependent variables.
3. **Subparametric ( $m < n$ ):** Higher-order approximation of the dependent variable is used.

# NUMERICAL EVALUATION OF INTEGRAL COEFFICIENTS

- Transformation of the integrals posed on arbitrary-shaped element to the master element domain so that evaluation of the integrals is made easy.
- The Gauss integration rule that evaluates an integral expression as a linear sum of the integrand evaluated at certain points (Gauss points) and weights (Gauss weights) is used.

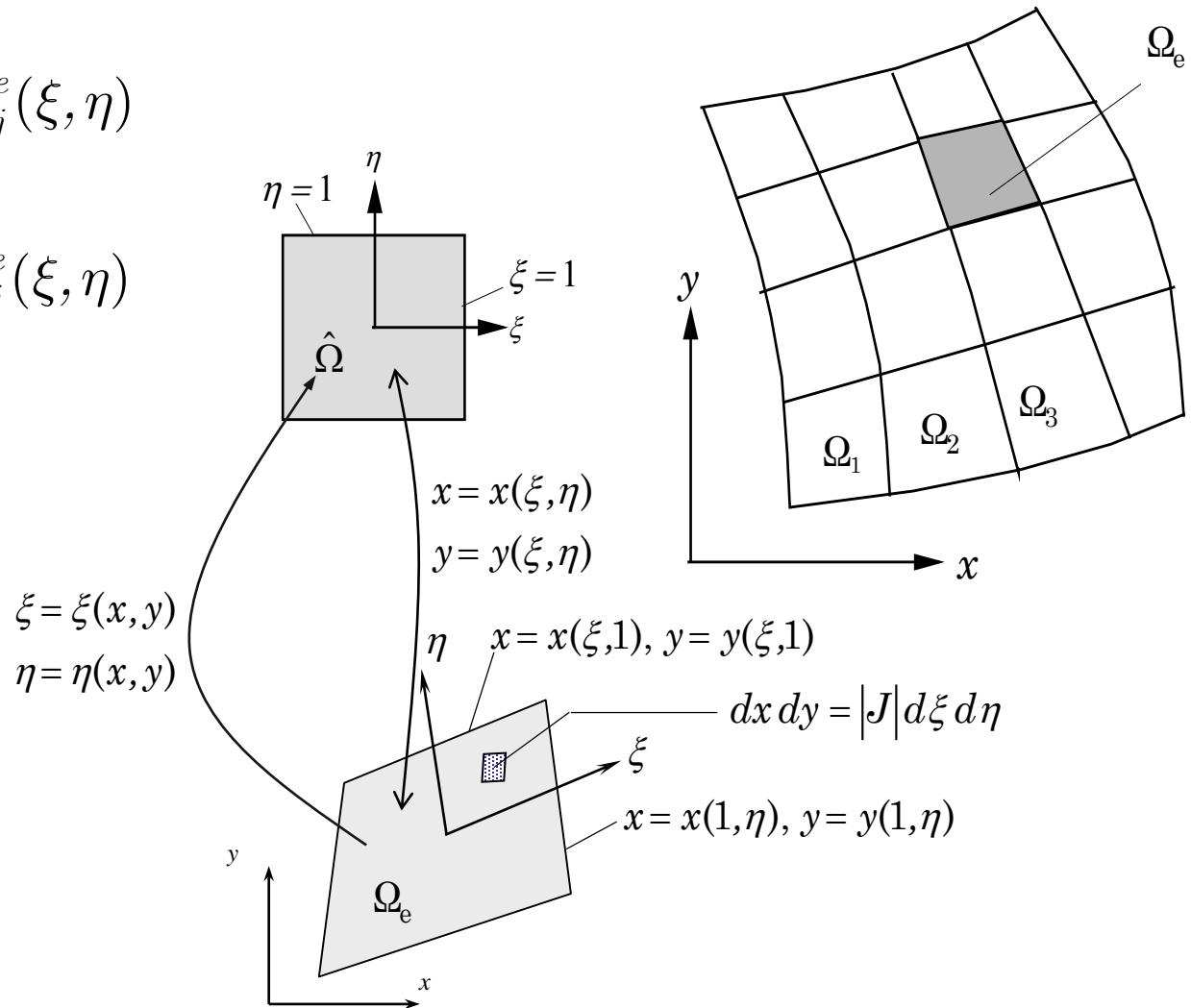


$$\begin{aligned}
 u(x, y) &\approx u_h^e(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x, y) \\
 &\approx u_h^e(x(\xi, \eta), y(\xi, \eta)) \\
 &= \sum_{j=1}^n u_j^e \psi_j^e(\xi, \eta)
 \end{aligned}$$

# NUMERICAL INTEGRATION

$$x(\xi, \eta) = \sum_{j=1}^m x_j^e \hat{\psi}_j^e(\xi, \eta)$$

$$y(\xi, \eta) = \sum_{j=1}^m y_j^e \hat{\psi}_j^e(\xi, \eta)$$





# NUMERICAL INTEGRATION

$$G_{ij} = \int_{x_a}^{x_b} F_{ij}(x) dx \approx \sum_{I=1}^{NPT} F_{ij}(x_I) W_I - \text{General}$$

$$K_{ij} = \int_{-1}^{+1} F_{ij}(\xi) d\xi \approx \sum_{I=1}^{NGP} F_{ij}(\xi_I) W_I - \text{Gauss rule}$$

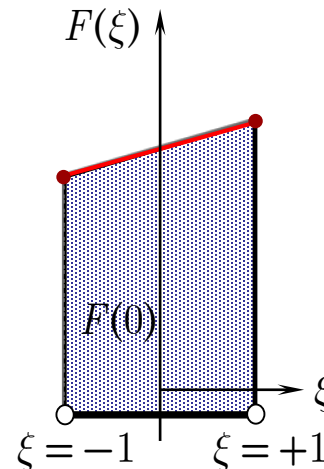
$$NGP = \left\lceil \frac{p+1}{2} \right\rceil, \quad \text{Nearest larger integer equal to } (p+1)/2$$

$$p = 1 \quad \Rightarrow \text{NGP} = 1$$

$$p = 2 \text{ or } 3 \Rightarrow \text{NGP} = 2$$

$$p = 4 \text{ or } 5 \Rightarrow \text{NGP} = 3$$

$$\begin{aligned} I_i &= \int_{-1}^{+1} F(\xi) d\xi = \frac{h}{2} [F_1 + F_2] = 2 \frac{F_1 + F_2}{2} \\ &= 2 \times F(0) = F(\xi_1) W_1 \end{aligned}$$



# TRANSFORMATION OF THE INTEGRAL

$$\begin{aligned}
 K_{ij}^e &= \int_{\Omega_e} \left[ a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dx dy \\
 &= \int_{\Omega_e} F_{ij}(x, y) dx dy = \int_{\hat{\Omega}} F_{ij}(x(\xi, \eta), y(\xi, \eta)) J d\xi d\eta \\
 &= \int_{\hat{\Omega}} F_{ij}(\xi, \eta) J d\xi d\eta \approx \sum_{I=1}^{NGP} \sum_{J=1}^{NGP} W_I W_J \hat{F}_{ij}(\xi_I, \eta_J) \quad \begin{array}{l} W_I \quad - \text{Weights} \\ (x_I, y_I) - \text{Integration points} \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \psi_i}{\partial \xi} &= \frac{\partial \psi_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \psi_i}{\partial y} \frac{\partial y}{\partial \xi} \\
 \frac{\partial \psi_i}{\partial \eta} &= \frac{\partial \psi_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \psi_i}{\partial y} \frac{\partial y}{\partial \eta}
 \end{aligned}
 \Rightarrow
 \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}
 =
 \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}
 \begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix}
 =
 \mathbf{J}
 \begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix}$$

# NUMERICAL INTEGRATION

Jacobian  
matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m x_i \frac{\partial \hat{\psi}_i}{\partial \xi} & \sum_{i=1}^m y_i \frac{\partial \hat{\psi}_i}{\partial \xi} \\ \sum_{i=1}^m x_i \frac{\partial \hat{\psi}_i}{\partial \eta} & \sum_{i=1}^m y_i \frac{\partial \hat{\psi}_i}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \dots & \frac{\partial \hat{\psi}_m}{\partial \xi} \\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \dots & \frac{\partial \hat{\psi}_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

Global derivatives in terms of the local derivatives

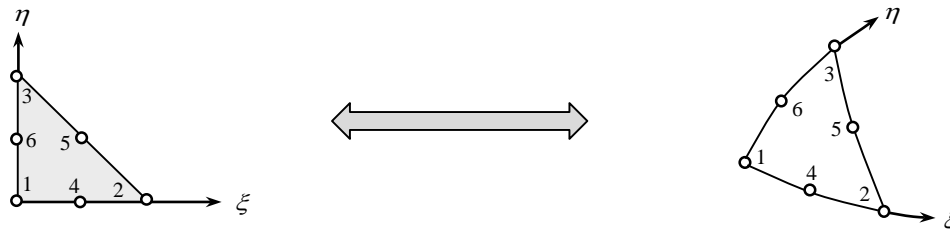
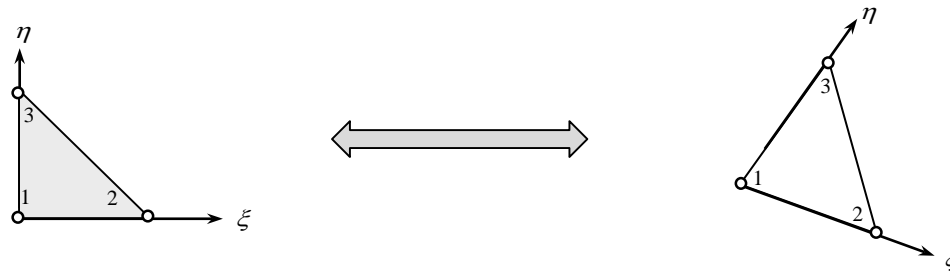
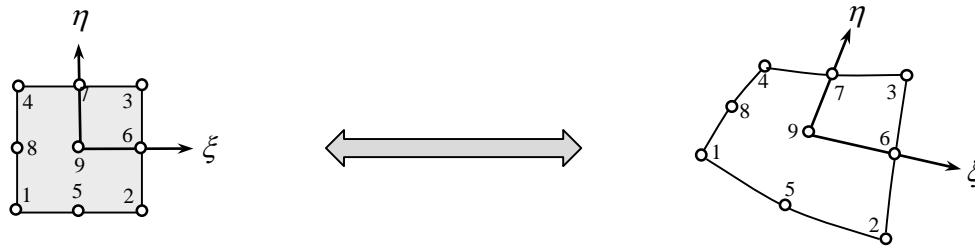
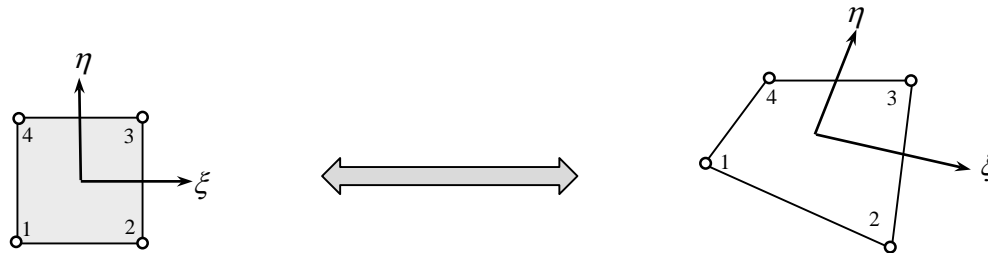
$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = \mathbf{J}^* \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}$$

# ELEMENT CALCULATIONS

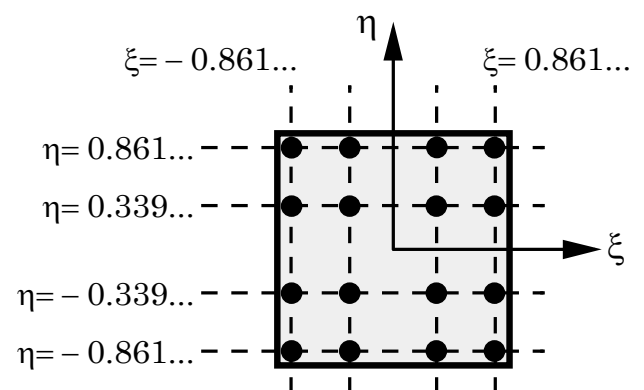
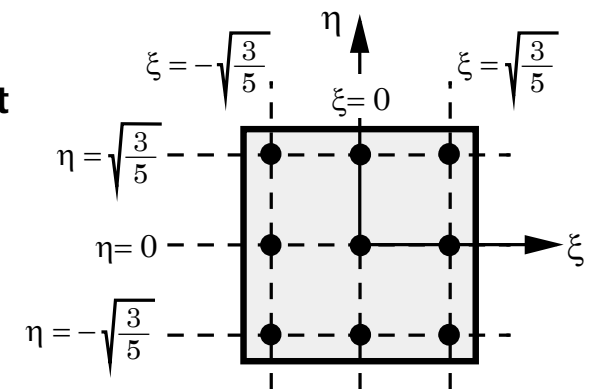
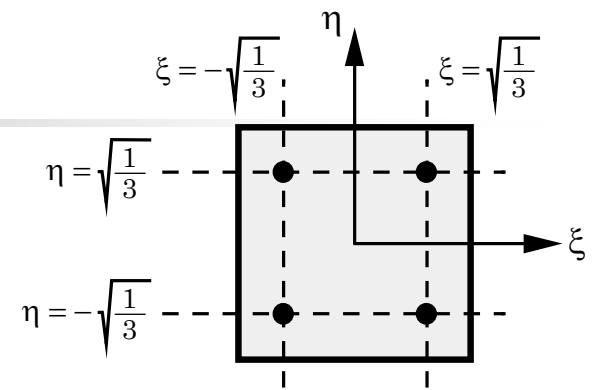
$$\begin{aligned}
 K_{ij}^e &= \int_{\Omega_e} \left[ a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dx dy && \boxed{dx dy = J d\xi d\eta, \quad J = |\mathbf{J}|} \\
 &= \int_{\hat{\Omega}} \left\{ a_{11}(\xi, \eta) \left( J_{11}^* \frac{\partial \psi_i}{\partial \xi} + J_{12}^* \frac{\partial \psi_i}{\partial \eta} \right) \left( J_{11}^* \frac{\partial \psi_j}{\partial \xi} + J_{12}^* \frac{\partial \psi_j}{\partial \eta} \right) \right. \\
 &\quad \left. + a_{22}(\xi, \eta) \left( J_{21}^* \frac{\partial \psi_i}{\partial \xi} + J_{22}^* \frac{\partial \psi_i}{\partial \eta} \right) \left( J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta} \right) \right\} J d\xi d\eta \\
 &= \int_{\hat{\Omega}} \hat{F}_{ij}^e(\xi, \eta) d\xi d\eta = \int_{-1}^1 \int_{-1}^1 \hat{F}_{ij}^e(\xi, \eta) d\xi d\eta \\
 &\approx \sum_{I=1}^{NGP_\xi} \sum_{J=1}^{NGP_\eta} \hat{F}_{ij}^e(\xi_I, \eta_J) W_I W_J
 \end{aligned}$$

# Master element

# Actual element



# GAUSS QUADRATURE



$$\int_{\Omega_e} F_{ij}(x, y) dx dy = \int_{\hat{\Omega}_e} \hat{F}_{ij}(\xi, \eta) d\xi d\eta$$

Domain of the master element

$$\approx \sum_{I, J=1}^N W_I W_J \hat{F}_{ij}(\xi_I, \eta_J)$$

Gauss points

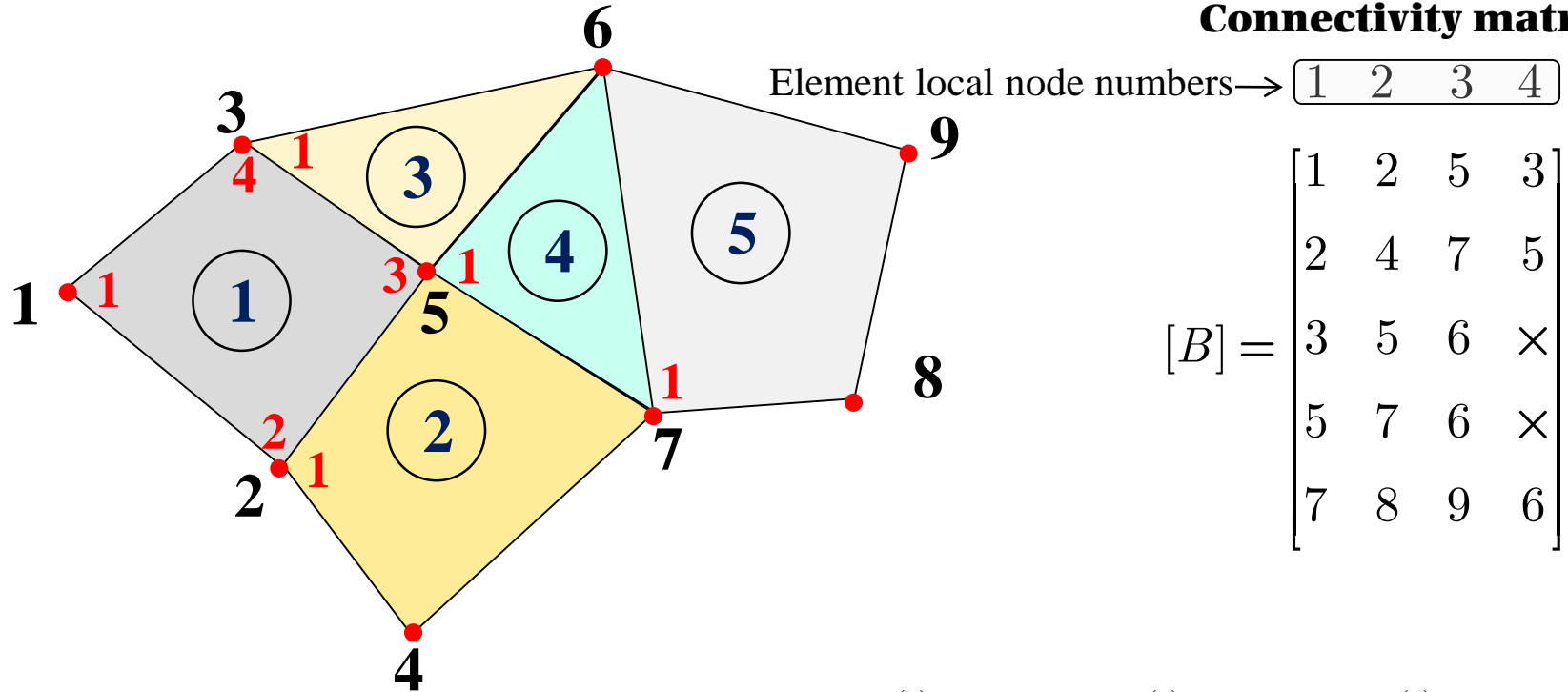
Gauss weights

Domain of the physical element

# ASSEMBLY OF ELEMENTS/EQUATIONS

(1 DoF per node)

Connectivity matrix

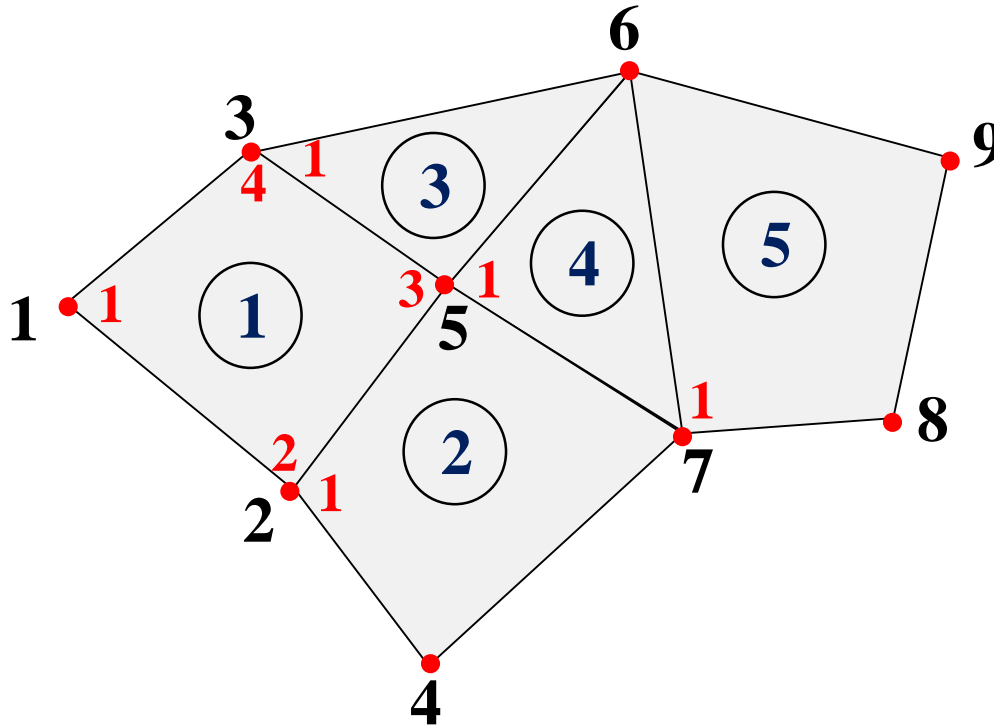


$$[B] = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 2 & 4 & 7 & 5 \\ 3 & 5 & 6 & \times \\ 5 & 7 & 6 & \times \\ 7 & 8 & 9 & 6 \end{bmatrix}$$

$K_{IJ}$  = a relationship between certain property of global node  $I$  and global node  $J$ .  
 = 0, if node  $I$  and  $J$  do not belong to the same element

$$\begin{aligned} K_{11} &= K_{11}^{(1)}, & K_{13} &= K_{14}^{(1)}, & K_{15} &= K_{13}^{(1)}, & K_{14} &= 0, \\ K_{22} &= K_{22}^{(1)} + K_{11}^{(2)}, & K_{25} &= K_{23}^{(1)} + K_{14}^{(2)}, & K_{26} &= 0, \\ K_{55} &= K_{33}^{(1)} + K_{44}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)}, \\ K_{56} &= K_{23}^{(3)} + K_{13}^{(4)}, & F_1 &= F_1^{(1)}, & F_2 &= F_2^{(1)} + F_1^{(2)}, \\ F_5 &= F_3^{(1)} + F_4^{(2)} + F_2^{(3)} + F_1^{(4)} \end{aligned}$$

# POST-COMPUTATION OF VARIABLES



$$u_h^e(x, y) = \sum_{j=1}^n u_j^{(e)} \psi_j^{(e)}(x, y), \quad (x, y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x}, \quad \frac{\partial u_h^e}{\partial y} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y}, \quad (x, y) \in \Omega^e$$



# Derivatives of the Solution

## Linear triangular element

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^n u_j^e \beta_j^e, \quad (x, y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^n u_j^e \gamma_j^e, \quad (x, y) \in \Omega^e$$

## Linear Rectangular element

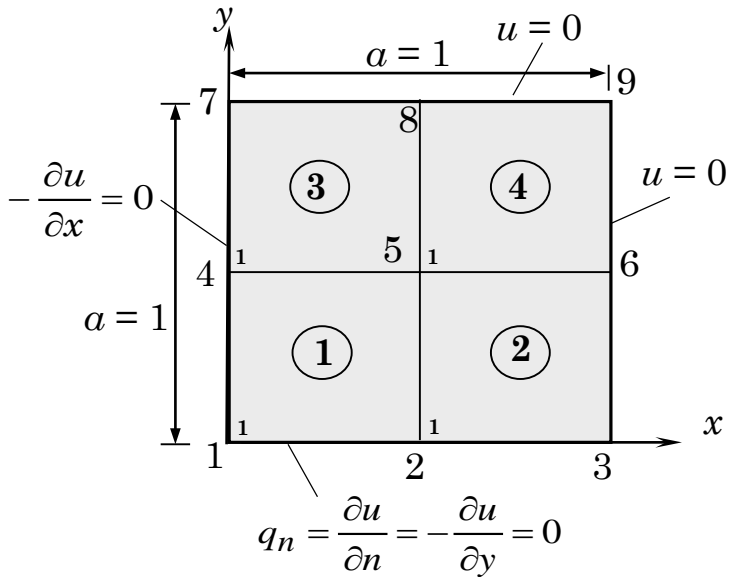
$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^n u_j^e (\beta_j^e + \mu_j^e y), \quad (x, y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^n u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^n u_j^e (\gamma_j^e + \mu_j^e x), \quad (x, y) \in \Omega^e$$

# Example 8.3.1 from the book

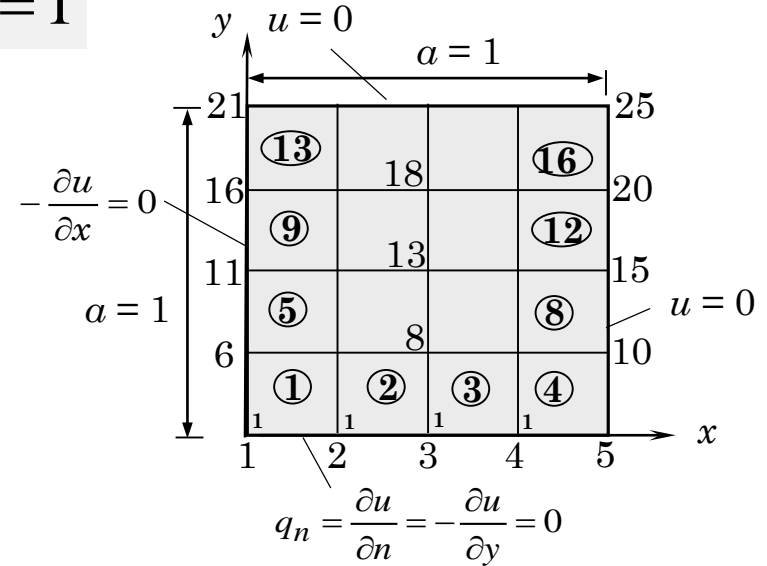
$$-\nabla^2 u = 1$$

2x2 mesh



(a)

4x4 mesh



(b)

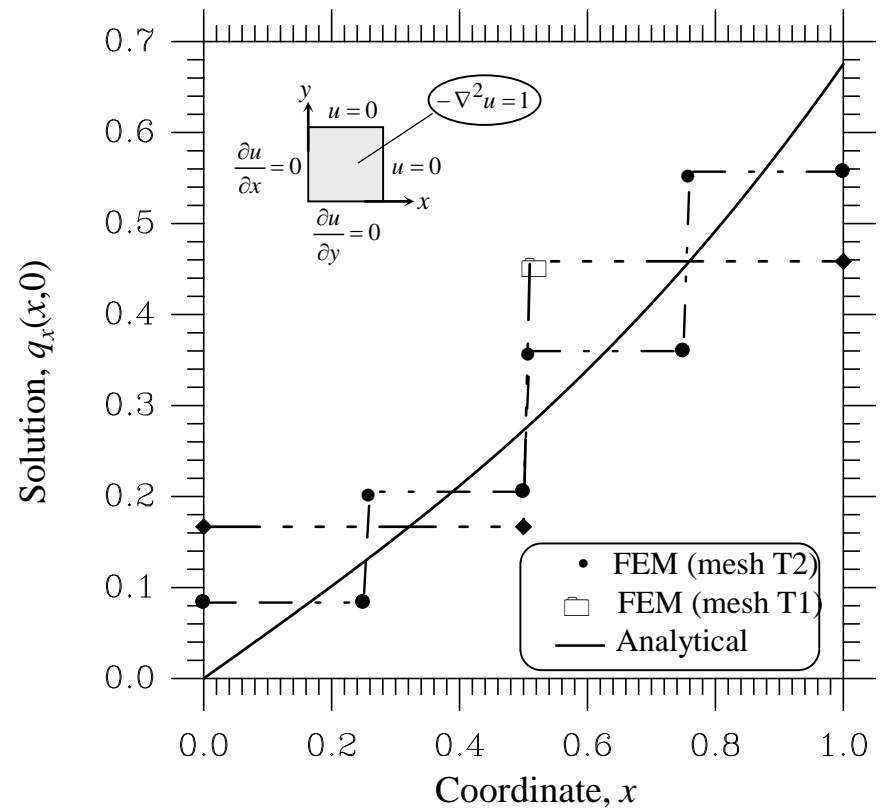
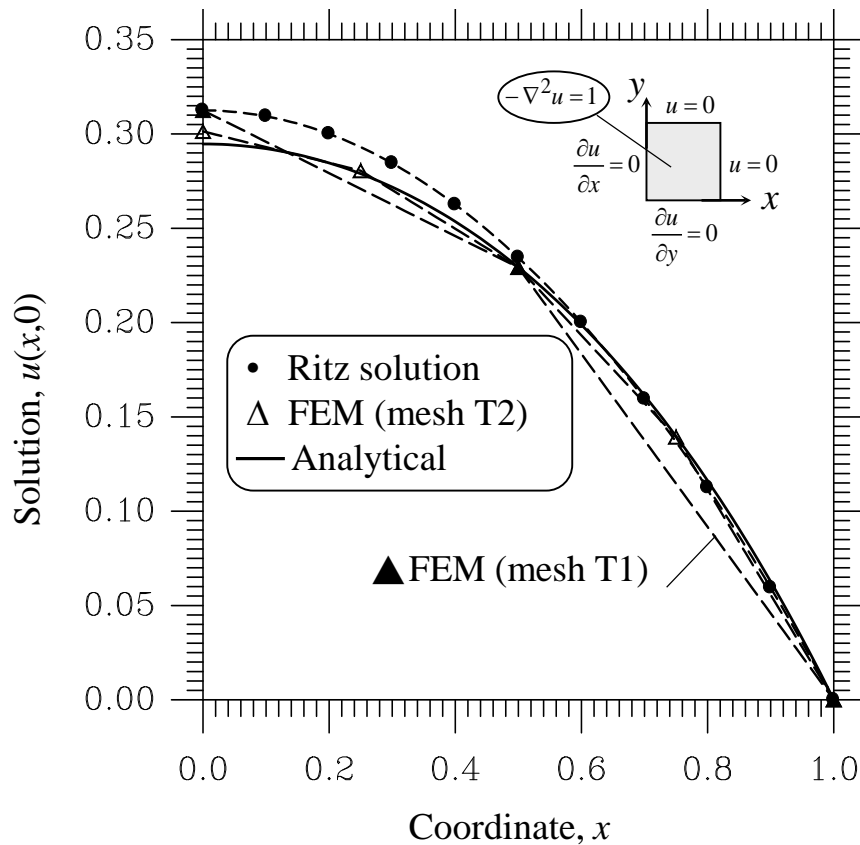
For example, equation for node 1 is:

$$K_{11}U_1 + K_{12}U_2 + K_{14}U_4 + K_{15}U_5 = F_1$$

or

$$K_{11}^{(1)}U_1 + K_{12}^{(1)}U_2 + K_{14}^{(1)}U_4 + K_{13}^{(1)}U_5 = f_1^{(1)} + Q_1^{(1)} = f_1^{(1)}$$

# Finite Element, Ritz, and Exact Solutions



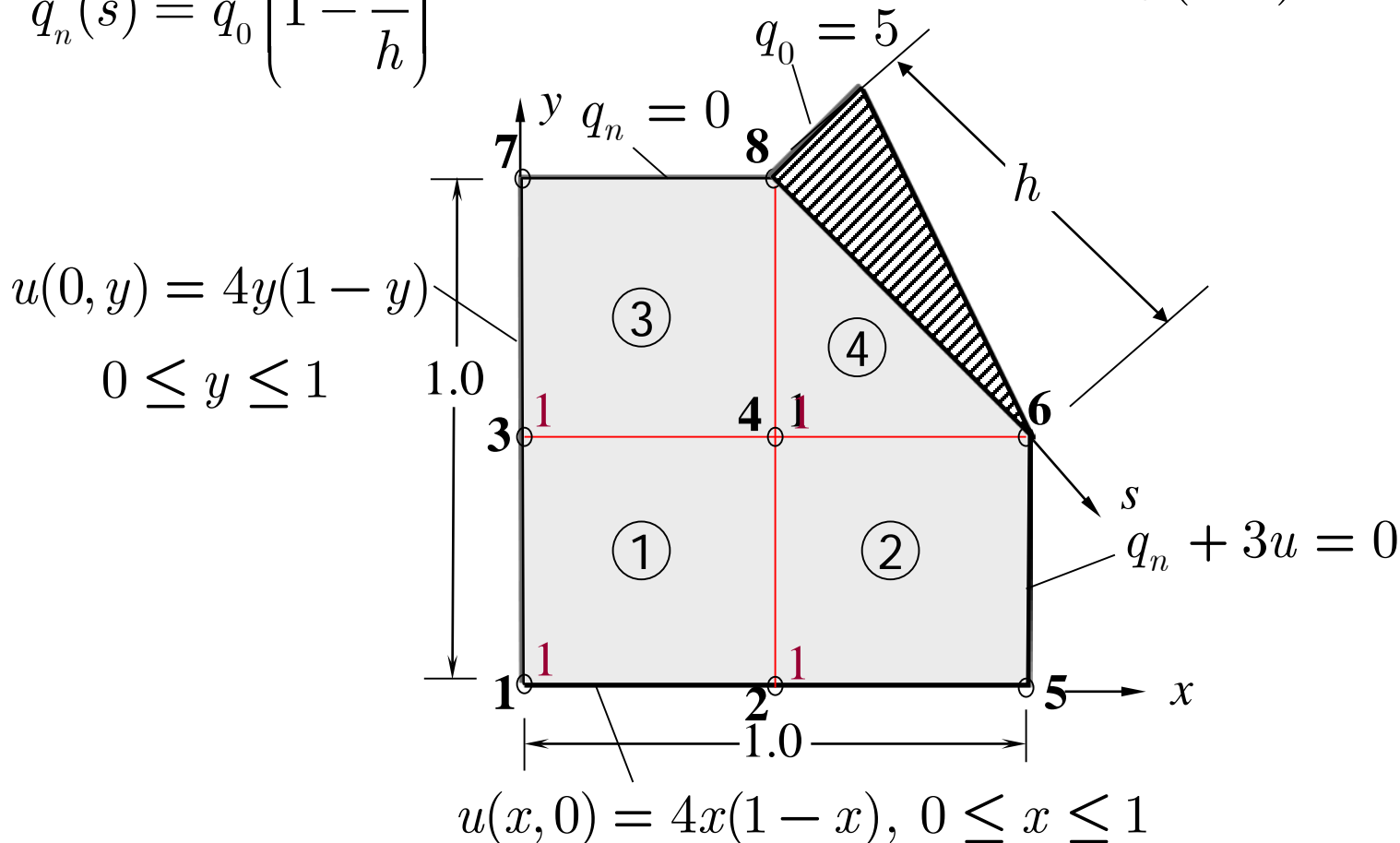
# DISCUSSION (distributed boundary source and convection type boundary condition)

For linear elements

$$Q_i = \int_0^h q_n(s) \psi_i(s) ds$$

$$q_n(s) = q_0 \left(1 - \frac{s}{h}\right)$$

$$Q_8 = \frac{2}{3} \left(\frac{q_n h}{2}\right), \quad Q_6 = \frac{1}{3} \left(\frac{q_n h}{2}\right)$$



$$\begin{aligned}
 Q_i &= \int_0^h q_n(s) \psi_i(s) ds = \int_0^h -\beta(u - u_0) \psi_i(s) ds \\
 &= \int_0^h -\beta(\sum_{j=1}^n u_j \psi_j - u_0) \psi_i(s) ds \\
 &= -\beta \sum_{j=1}^n u_j \int_0^h \psi_i \psi_j ds + \beta \int_0^h u_0 \psi_i ds
 \end{aligned}$$

## DISCUSSION

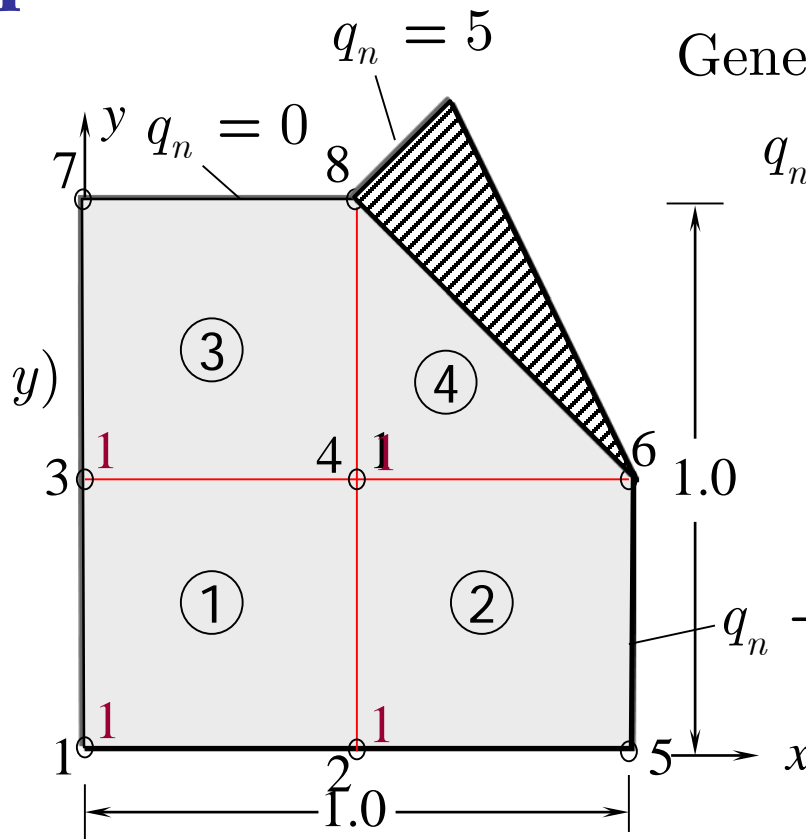
(computation of convection type BC)

General form of the BC:

$$q_n + \beta(u - u_0) = 0$$

$$u(0, y) = 4y(1 - y)$$

$$0 \leq y \leq 1$$



$$u(x, 0) = 4x(1 - x), 0 \leq x \leq 1$$

$$q_n + 3u = 0 \quad (\beta = 3, u_0 = 0)$$

# Transient Analysis of 2-D Problems

## Model Governing Differential Equation

$$c_1 \frac{\partial u}{\partial t} + c_2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u}{\partial y} \right) = f(x, y, t)$$

## Weak formulation

$$\begin{aligned} 0 &= \int_{\Omega_e} w_i \left[ c_1 \frac{\partial u_h}{\partial t} + c_2 \frac{\partial^2 u_h}{\partial t^2} - \frac{\partial}{\partial x} \left( a_{11} \frac{\partial u_h}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_{22} \frac{\partial u_h}{\partial y} \right) \right. \\ &\quad \left. - f(x, y, t) \right] dx dy \\ &= \int_{\Omega_e} \left[ w_i c_1 \frac{\partial u_h}{\partial t} + w_i c_2 \frac{\partial^2 u_h}{\partial t^2} + a_{11} \frac{\partial w_i}{\partial x} \frac{\partial u_h}{\partial x} + a_{22} \frac{\partial w_i}{\partial y} \frac{\partial u_h}{\partial y} \right. \\ &\quad \left. - w_i f \right] dx dy - \oint_{\Gamma_e} q w_i ds \end{aligned}$$



# SPATIAL DISCRETIZATION: Finite Element Model

## Approximation

$$u(x, y, t) \approx u_h^e(x, y, t) = \sum_{j=1}^n u_j^e(t) \psi_j^e(x, y)$$

## Finite element model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

$$C_{ij}^e = \int_{\Omega_e} c_1 \psi_i \psi_j \, dx dy, \quad M_{ij}^e = \int_{\Omega_e} c_2 \psi_i \psi_j \, dx dy$$

$$K_{ij}^e = \int_{\Omega_e} \left( a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy$$

$$F_i^e = \int_{\Omega_e} f \psi_i \, dx dy + \oint_{\Gamma_e} q \psi_i \, ds$$

# TIME APPROXIMATIONS (Parabolic)

## Semidiscrete FE model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}, \quad 0 < t < T$$

**NOTE:** From here, the procedure is the same as in 1-D

## Alfa-family of approximation

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \Delta t_{s+1} [\alpha \dot{\mathbf{u}}_{s+1} + (1 - \alpha) \dot{\mathbf{u}}_s]$$

$$\mathbf{C}\mathbf{u}_{s+1} = \mathbf{C}\mathbf{u}_s + \Delta t_{s+1} [\alpha \mathbf{C}\dot{\mathbf{u}}_{s+1} + (1 - \alpha) \mathbf{C}\dot{\mathbf{u}}_s]$$

$$\mathbf{C}\dot{\mathbf{u}}_{s+1} = \mathbf{F}_{s+1} - \mathbf{K}_{s+1}\mathbf{u}_{s+1} \quad \mathbf{C}\dot{\mathbf{u}}_s = \mathbf{F}_s - \mathbf{K}_s\mathbf{u}_s$$

$$\begin{aligned} (\mathbf{C} + \alpha \Delta t_{s+1} \mathbf{K}_{s+1}) \mathbf{u}_{s+1} &= (\mathbf{C} - (1 - \alpha) \Delta t_s \mathbf{K}_s) \mathbf{u}_s \\ &+ \Delta t_{s+1} [\alpha \mathbf{F}_{s+1} + (1 - \alpha) \mathbf{F}_s] \end{aligned}$$

## Fully discretized model

$$\hat{\mathbf{K}}_{s+1} \mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1}, \quad \hat{\mathbf{K}}_{s+1} = \alpha \Delta t_{s+1} \mathbf{K}_{s+1} + \mathbf{C},$$

$$\hat{\mathbf{F}}_{s+1} = \left[ (1 - \alpha) \Delta t_{s+1} \mathbf{K}_{s+1} + \mathbf{C} \right] \mathbf{u}_s + \Delta t_{s+1} [\alpha \mathbf{F}_{s+1} + (1 - \alpha) \mathbf{F}_s]$$



# TIME APPROXIMATIONS (Hyperbolic)

## Semidiscrete FE model

$$\mathbf{C}^e \dot{\mathbf{u}}^e + \mathbf{M}^e \ddot{\mathbf{u}}^e + \mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e$$

## Newmark scheme (second-order equations)

$$\begin{aligned} \mathbf{u}_{s+1} &= \mathbf{u}_s + \Delta t \dot{\mathbf{u}}_s + \frac{1}{2} (\Delta t)^2 \ddot{\mathbf{u}}_{s,\gamma} \\ \dot{\mathbf{u}}_{s+1} &= \dot{\mathbf{u}}_s + \Delta t \ddot{\mathbf{u}}_{s,\alpha}, \quad \ddot{\mathbf{u}}_{s,\alpha} = (1 - \alpha) \ddot{\mathbf{u}}_s + \alpha \ddot{\mathbf{u}}_{s+1} \end{aligned}$$

## Fully discretized model

$$\hat{\mathbf{K}}_{s+1} \mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1},$$

$$\hat{\mathbf{K}}_{s+1} = \mathbf{K}_{s+1} + a_3 \mathbf{M}_{s+1} + a_5 \mathbf{C}_{s+1}$$

$$\hat{\mathbf{F}}_{s+1} = \mathbf{F}_{s+1} + \mathbf{M}_{s+1} (a_3 \mathbf{u}_s + a_4 \dot{\mathbf{u}}_s + a_5 \ddot{\mathbf{u}}_s) + \mathbf{C}_{s+1} (a_5 \mathbf{u}_s + a_6 \dot{\mathbf{u}}_s + a_7 \ddot{\mathbf{u}}_s)$$

# An Example: Transient Heat Conduction Problem

## Governing equation

$$c_1 \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) = f(x, y, t)$$

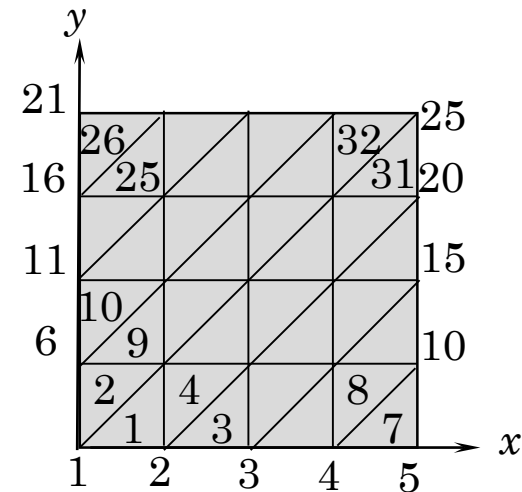
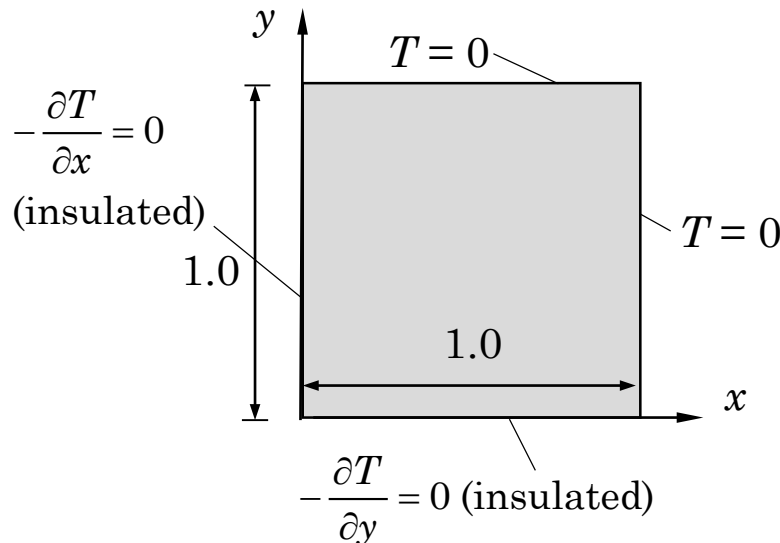
## Boundary conditions

$$k \left( \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) = 0 \text{ on } x = 0 \text{ and } y = 0$$

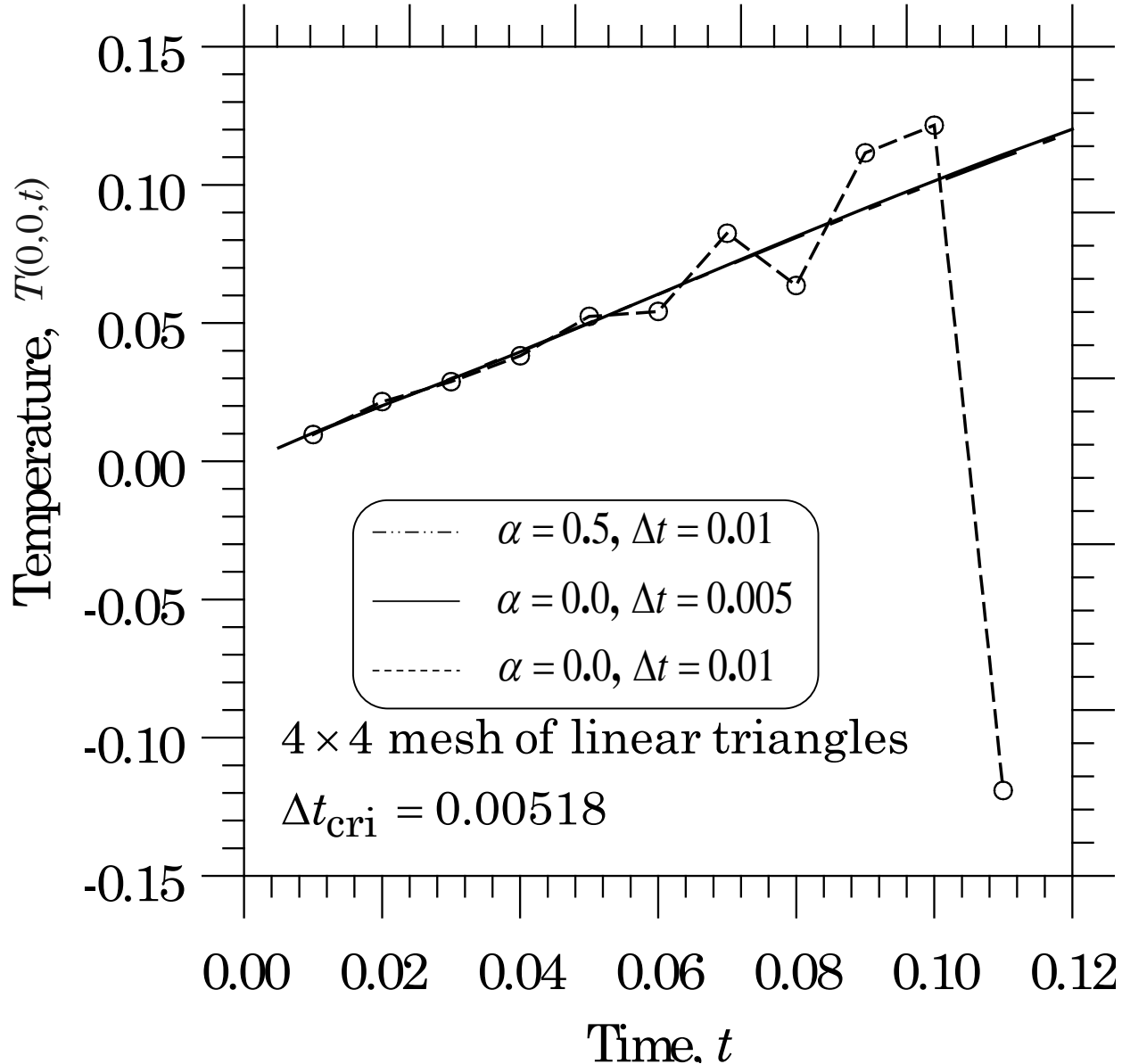
$$T = 0 \text{ on } y = 1 \text{ and } x = 1$$

## Initial condition

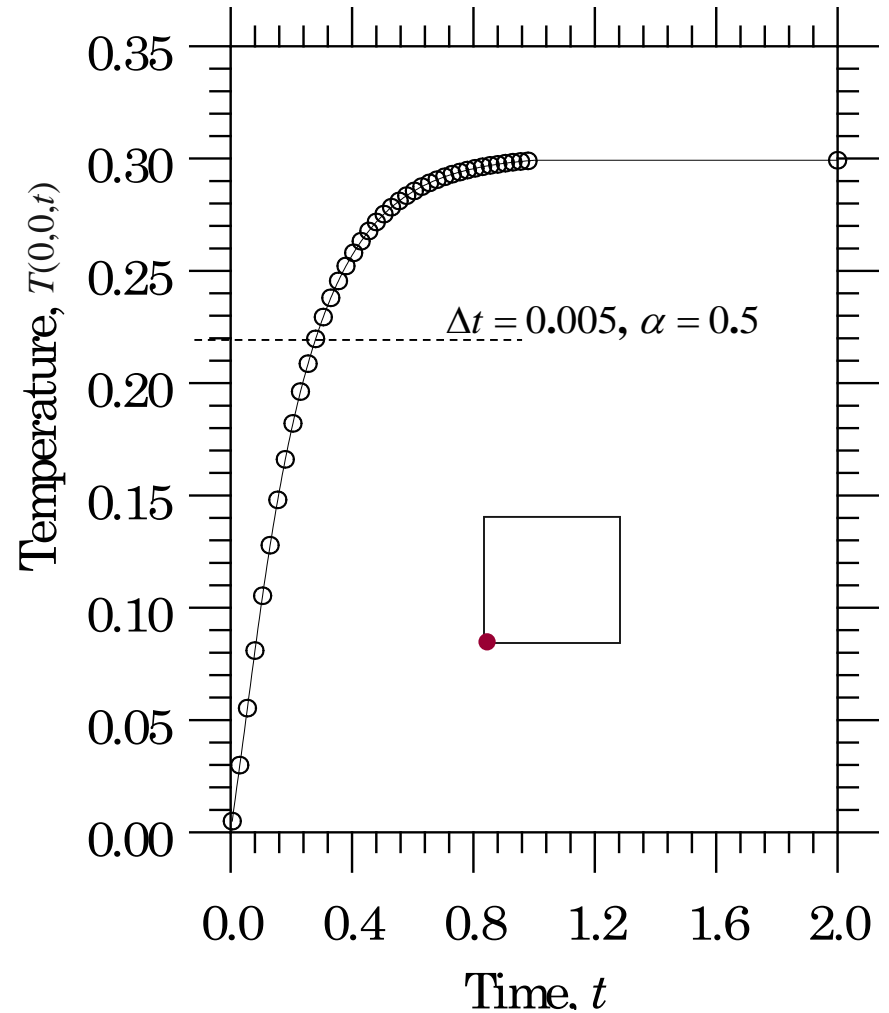
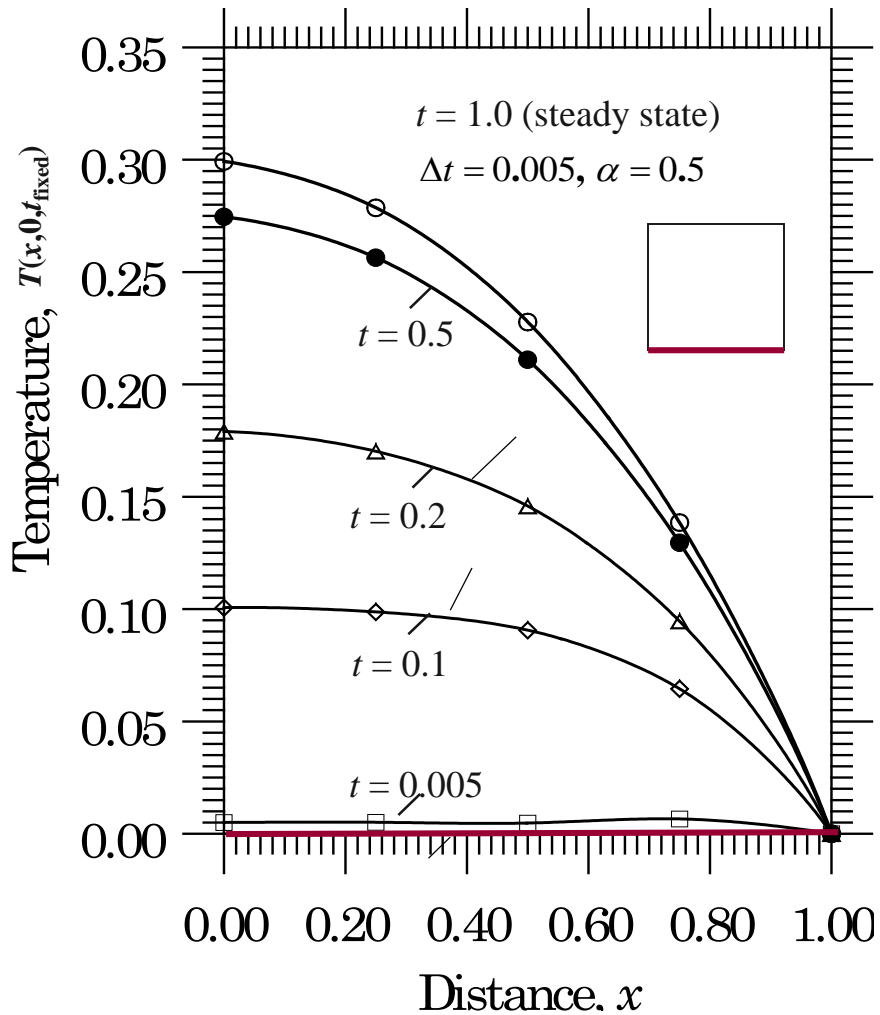
$$T(x, y, 0) = 1$$



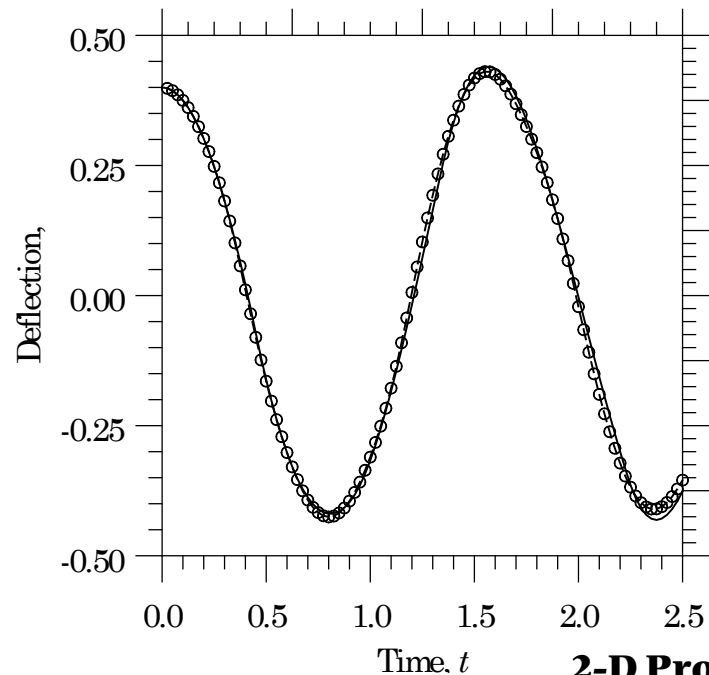
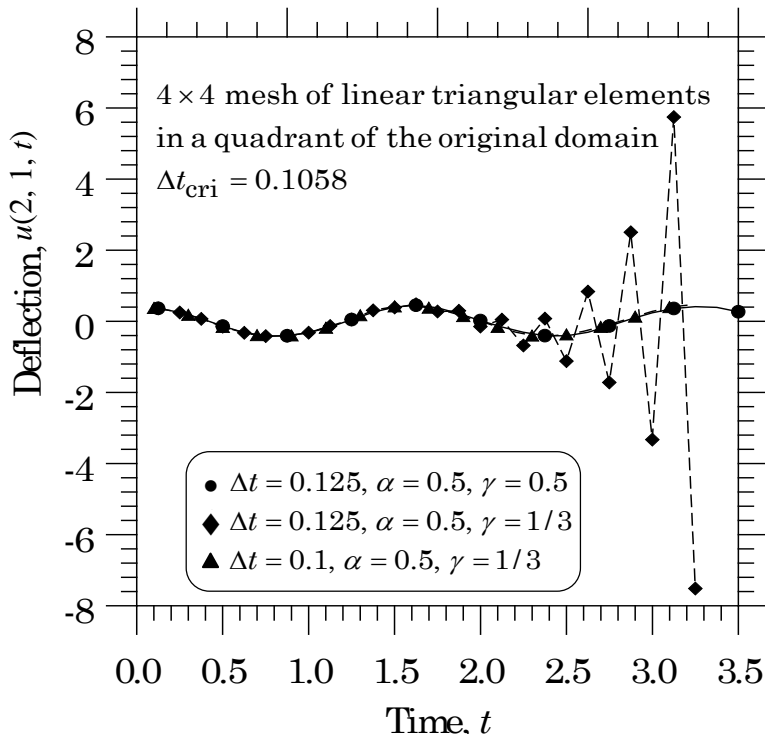
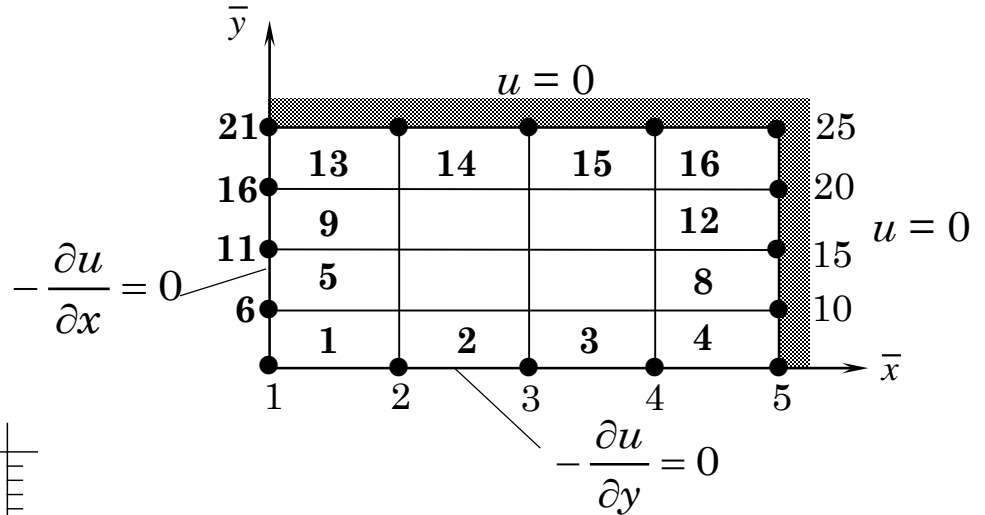
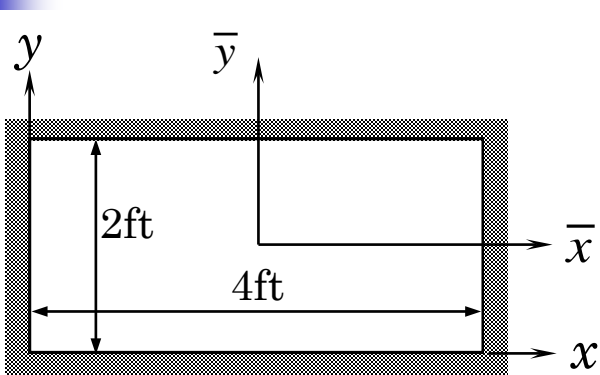
# Stability Characteristics



# A NUMERICAL EXAMPLE (Parabolic)



# TRANSIENT RESPONSE OF A MEMBRANE





# SUMMARY

1. Starting point is a model differential equation in  $u$
2. Construct an integral statement – weak form, which has three steps. Integration by parts that (a) relaxes (“weakens” differentiability on the variable  $u$ , and (b) brings in secondary variable into the integral form.
3. Substitute suitable approximation for  $u$  and obtain the finite element model (i.e., a set of algebraic relations between nodal values of  $u$  and  $Q$ ).
4. Numerical evaluation of coefficients  $K_{ij}$  and  $f_i$
5. Assemble equations, impose boundary conditions, and solve the assembled equations.
6. Post-computation of variables follows same procedure as in 1-D FEM.
7. Reviewed time-dependent problems.