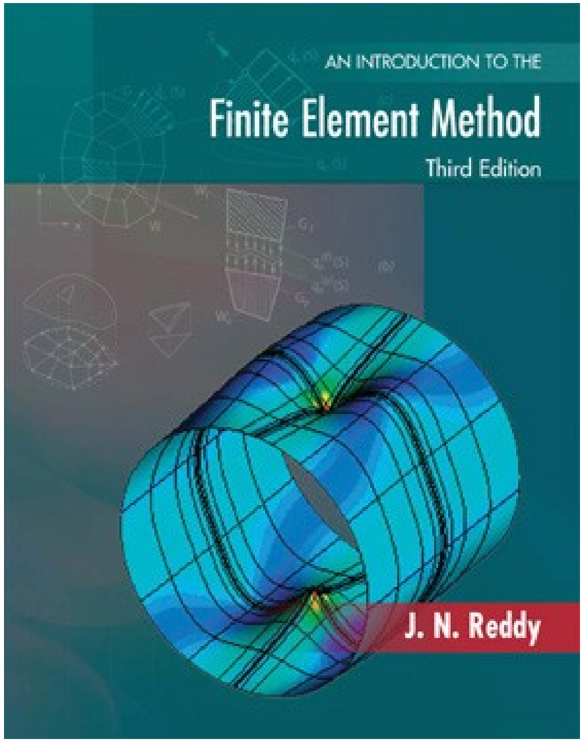
MEEN 673
Nonlinear Finite Element Analysis
Fall 2016
Chapter 3
INTRODUCTION AND OVERVIEW OF LINEAR FEM using a 2D model problem



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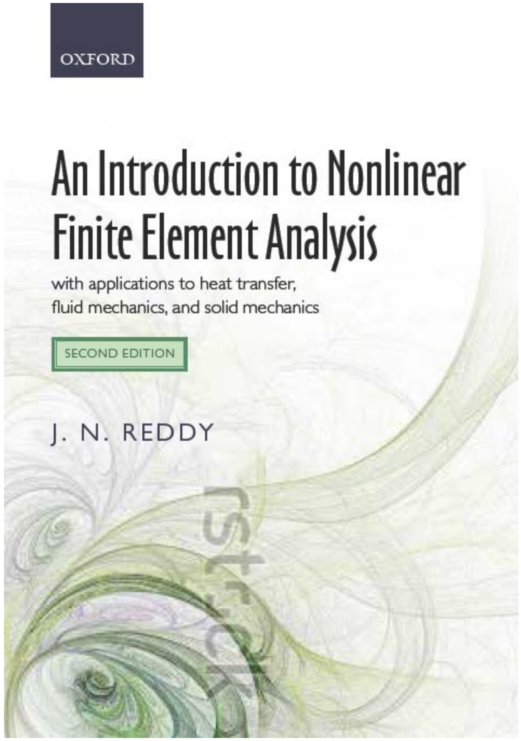
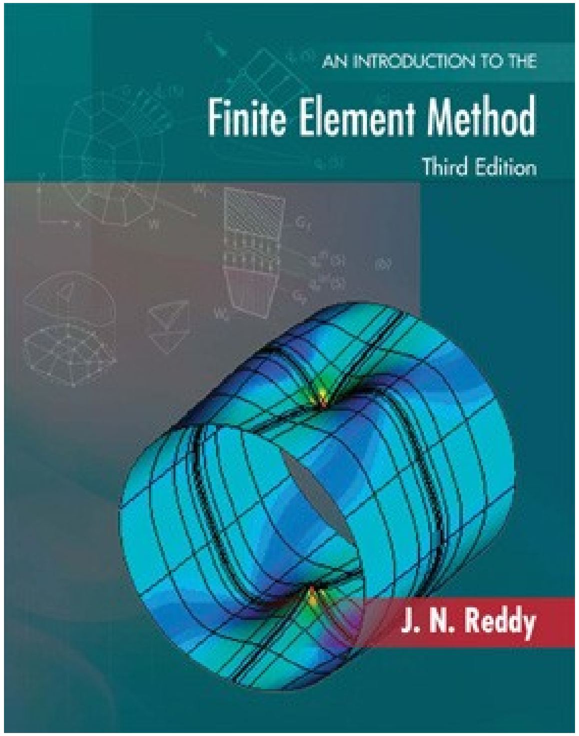


An Overview of The Finite Element Method

2D Problems involving a single unknown



CONTENTS

- Model equation Discretization
- Weak form development
- Finite element model
- Approximation functions
- Interpolation functions of higher-order elements
- Post-computation of variables
- Numerical examples
- Transient analysis of 2-D problems

MODEL EQUATION

Model Differential Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)=f \quad \text{in} \quad \Omega$$

a_{ij} – coefficients that describe material behavior

f – source term

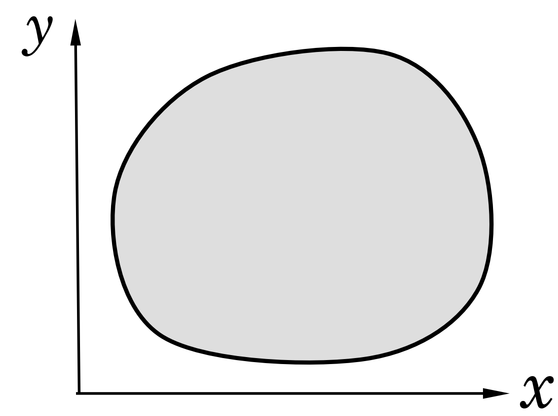
Type of boundary conditions

$$u=\hat{u}, \qquad \left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)n_x-\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)n_y-\hat{q}_n=0$$

EXAMPLES OF MODEL EQUATION

(1)– Deflection of a membrane

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0$$

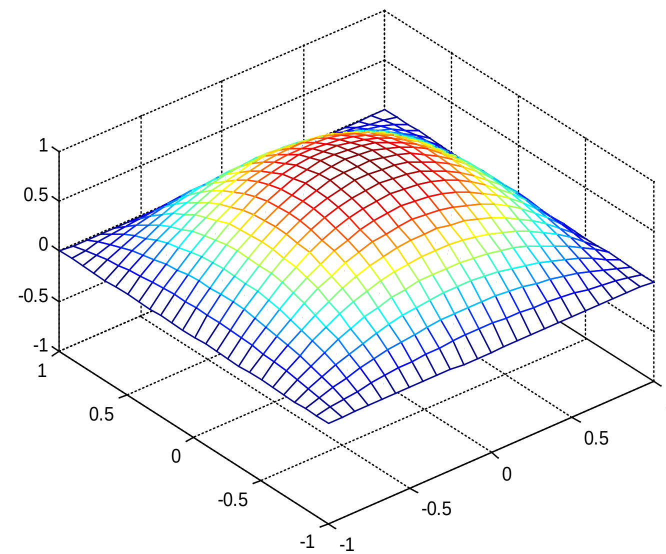


$u(x,y)$ = transverse deflection of a point in the membrane

$f(x,y)$ = applied pressure

$a_{11}(x,y)$, $a_{22}(x,y)$ = tensions in the x and y directions, respecti

Transverse deflection of a square membrane fixed on all its sides and subjected to uniform pressure.



EXAMPLES OF MODEL EQUATION

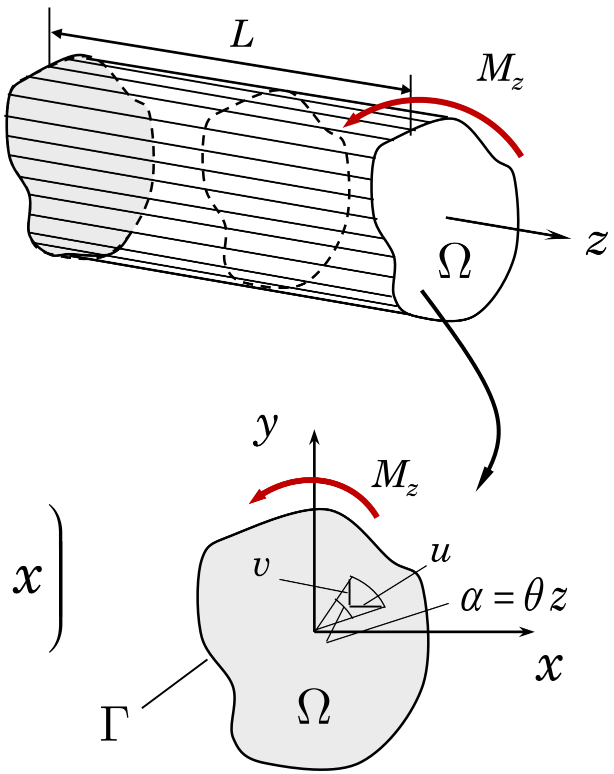
(2)– Torsion of a cylindrical member

$$-G\theta\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0 \text{ in } \Omega$$

$$u=-\theta zy,\quad v=\theta zx,\quad w=\theta\phi(x,y)$$

$$\left(\frac{\partial\phi}{\partial x}-y\right)n_x+\left(\frac{\partial\phi}{\partial y}-x\right)n_y=0 \text{ on } \Gamma$$

$$\sigma_{xz}(x,y)=G\theta\left(\frac{\partial\phi}{\partial x}-y\right),\quad \sigma_{yz}(x,y)=G\theta\left(\frac{\partial\phi}{\partial y}+x\right)$$



$\phi(x,y)=$ warping function

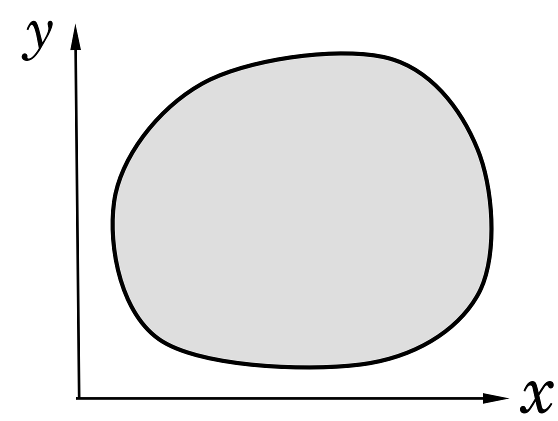
$a_{11}=a_{22}=G\theta$

$G=$ shear modulus

$\theta=$ angle of twist

EXAMPLES OF MODEL EQUATION
(3)– 2D Heat transfer

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0$$



$$q_x(x,y)=-a_{11}\frac{\partial u}{\partial x},\quad q_y(x,y)=-a_{22}\frac{\partial u}{\partial y}$$

$$-\left(a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y\right)=\hat{q}_n \text{ on } \Gamma$$

$u=T$, temperature; $a_{11}=k_x$, $a_{22}=k_y$

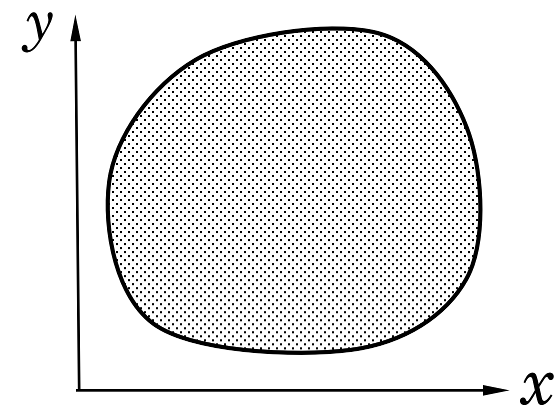
k_x, k_y = thermal conductivities in the x and y directions, respectively

$f =$ internal heat generation

$q_n =$ heat flux normal to the boundary

EXAMPLES OF MODEL EQUATION
(4) – 2D Inviscid flow

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial \phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial \phi}{\partial y}\right)-f=0$$



$$v_x(x,y)=-a_{11}\frac{\partial \phi}{\partial x},\quad v_y(x,y)=-a_{22}\frac{\partial \phi}{\partial y}$$

$$-\left(a_{11}\frac{\partial \phi}{\partial x}n_x+a_{22}\frac{\partial \phi}{\partial y}n_y\right)=\hat{q}_n \quad \text{on } \Gamma$$

$u=\phi$, water head (velocity potential)

$a_{11},\ a_{22}=$ permeabilities in the x and y directions, respectively

$f=$ internal infiltration

$q_n=$ flow normal to the boundary

WEAK FORM DEVELOPMENT

Approximation $u(x,y) \approx u_h(x,y)$

Weak Form Development

Step 1

$$0 = \int_{\Omega^e} w_i \left[-\frac{\partial}{\partial x}\underbrace{\left(a_{11}\frac{\partial u_h}{\partial x}\right)}_{F_1} - \frac{\partial}{\partial y}\underbrace{\left(a_{22}\frac{\partial u_h}{\partial y}\right)}_{F_2} - f \right] dxdy$$

Step 2: Trade differentiations between w and u_h

Consider the identity:

$$\frac{\partial}{\partial x}\left(w_i \cdot F_1\right) = \frac{\partial w_i}{\partial x} F_1 + w_i \frac{\partial F_1}{\partial x} \quad \text{or} \quad -w_i \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x}\underbrace{\left(w_i \cdot F_1\right)}_{F} + \frac{\partial w_i}{\partial x} F_1$$

Green-Gauss Theorem

$$\int_{\Omega^e} \frac{\partial F}{\partial x}\, dxdy = \oint_{\Gamma^e} n_x F\, ds, \qquad \int_{\Omega^e} \frac{\partial F}{\partial y}\, dxdy = \oint_{\Gamma^e} n_y F\, ds$$

WEAK FORM DEVELOPMENT continued

Step 2:

Consider the identity: $$-w_i\frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x}\underbrace{\left(w_i \cdot F_1\right)}_{F} + \frac{\partial w_i}{\partial x}F_1$$

$$\int_{\Omega^e}\frac{\partial F}{\partial x}dxdy = \oint_{\Gamma^e} n_x F\, ds \Rightarrow \int_{\Omega^e}\frac{\partial F}{\partial x}dxdy = \oint_{\Gamma^e} n_x\left(w_i \cdot F_1\right)ds$$

$$\int_{\Omega^e}\frac{\partial F}{\partial x}dxdy = \oint_{\Gamma^e} n_x w_i\left(a_{11}\frac{\partial u_h}{\partial x}\right)ds$$

$$\int_{\Omega^e}\frac{\partial G}{\partial y}dxdy = \oint_{\Gamma^e} n_y w_i\left(a_{22}\frac{\partial u_h}{\partial y}\right)ds$$

WEAK FORM DEVELOPMENT (continued)

Step 1

$$0 = \int_{\Omega^e} w_i \left[-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - f \right] dxdy$$

$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - w_i f \right] dxdy$$

Step 2

$$- \oint_{\Gamma^e} w_i \left[\left(a_{11}\frac{\partial u_h}{\partial x}\right) n_x + \left(a_{22}\frac{\partial u_h}{\partial y}\right) n_y \right] ds$$

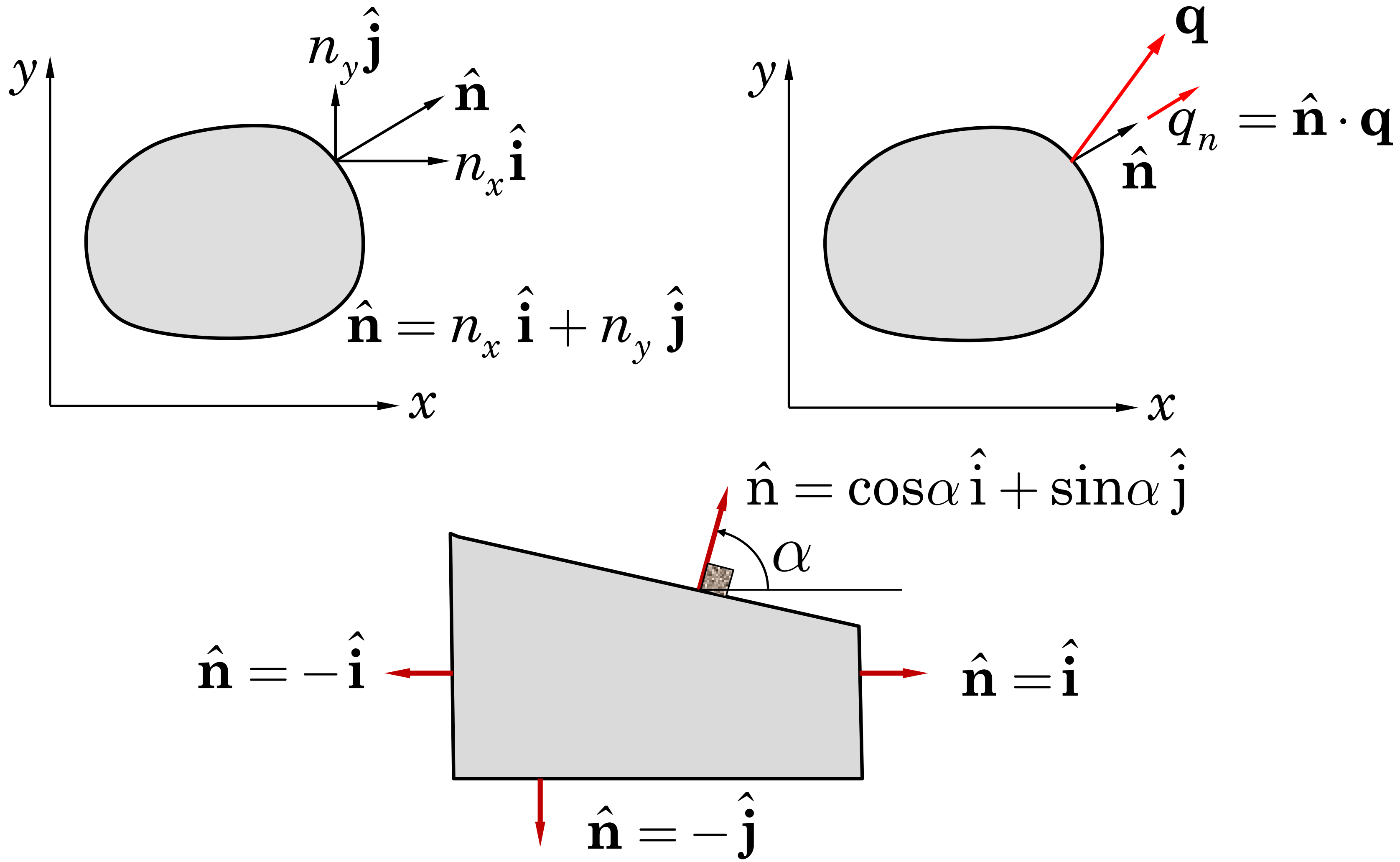
$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - w_i f \right] dxdy$$

Step 3

$$- \oint_{\Gamma^e} w_i \, q_n \, ds$$

$$q_n = \left(a_{11}\frac{\partial u_h}{\partial x}\right) n_x + \left(a_{22}\frac{\partial u_h}{\partial y}\right) n_y, \quad \text{flux normal to the boundary}$$

FINITE ELEMENT APPROXIMATION



FINITE ELEMENT APPROXIMATION

Weak form

$$0=\int_{\Omega^e}\left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right)+\frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right)-w_i f\right]dxdy-\oint_{\Gamma^e} w_i\, q_n\, ds$$

Approximation

$$u(x,y)\approx u_h(x,y)=c_1+c_2x+c_3y+c_4xy+\cdots\ (n \text{ terms})$$

$$=\sum_{j=1}^{n}u_j\psi_j(x,y)$$

Finite element model [the ith equation is obtained by replacing the weight function w by ψ_i $(i=1,2,\cdots,n)$]

$$0=\int_{\Omega^e}\left[\frac{\partial \psi_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right)+\frac{\partial \psi_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right)-\psi_i f\right]dxdy-\oint_{\Gamma^e} \psi_i\, q_n\, ds$$

FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$0 = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \left(a_{22} \frac{\partial u_h}{\partial y} \right) - \psi_i f \right] dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

$$0 = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \sum_{j=1}^{n} u_j \left(a_{11} \frac{\partial \psi_j}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \sum_{j=1}^{n} u_j \left(a_{22} \frac{\partial \psi_j}{\partial y} \right) \right] dxdy - \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

$$= \sum_{j=1}^{n} u_j \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy - \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$0=\sum_{j=1}^{n} u_j \int_{\Omega^e}\left(a_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+a_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy$$

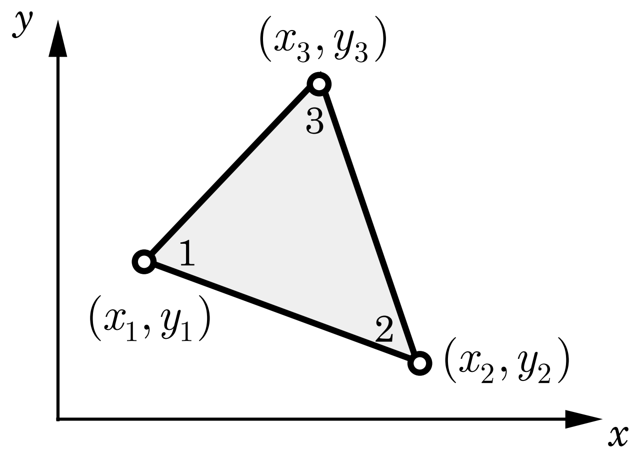
$$-\int_{\Omega^e}\psi_i f\,dxdy-\oint_{\Gamma^e}\psi_i\,q_n\,ds$$

$$=\sum_{j=1}^{n}K_{ij}^e u_j^e - f_i^e - Q_i^e = \sum_{j=1}^{n}K_{ij}^e u_j^e - F_i^e \quad \text{or} \quad \mathbf{K}^e\mathbf{u}^e=\mathbf{F}^e$$

$$K_{ij}^e=\int_{\Omega^e}\left(a_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+a_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy,$$

$$F_i^e=\int_{\Omega^e}\psi_i f\,dxdy+\oint_{\Gamma^e}\psi_i\,q_n\,ds$$

APPROXIMATION FUNCTIONS
Linear Triangular Element



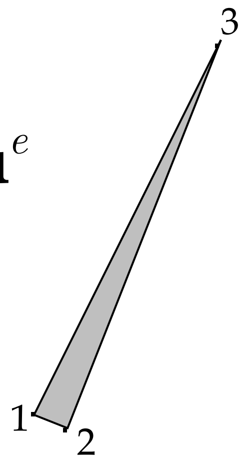
$$u_h^e(x,y) = c_1 + c_2 x + c_3 y$$

$$u_h^e(x_1,y_1) = c_1 + c_2 x_1 + c_3 y_1 \equiv u_1^e$$

$$u_h^e(x_2,y_2) = c_1 + c_2 x_2 + c_3 y_2 \equiv u_2^e$$

$$u_h^e(x_3,y_3) = c_1 + c_2 x_3 + c_3 y_3 \equiv u_3^e$$

$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \end{Bmatrix} \Rightarrow \mathbf{u}^e = \mathbf{A}^e \mathbf{c}^e \text{ or } \mathbf{c}^e = \mathbf{A}^{-1} \mathbf{u}^e$$



APPROXIMATION FUNCTIONS

Linear Triangular Element (continued)

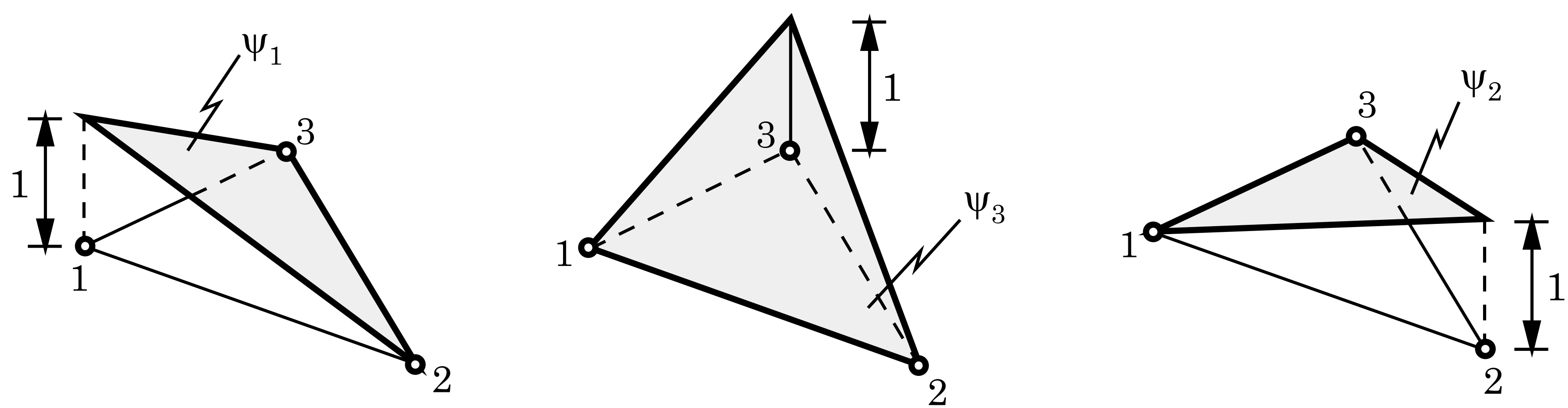
$$u_h^e(x,y) = c_1 + c_2 x + c_3 y = \left\{1 \;\; x \;\; y\right\}\left\{c^e\right\} = \left\{1 \;\; x \;\; y\right\}\left[A\right]^{-1}\left\{u^e\right\}$$

$$= \psi_1^e(x,y)u_1^e + \psi_2^e(x,y)u_2^e + \psi_3^e(x,y)u_3^e = \sum_{j=1}^{3} u_j^e \psi_j^e(x,y)$$

$$\psi_i^e(x,y) = \frac{1}{2\Delta^e}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

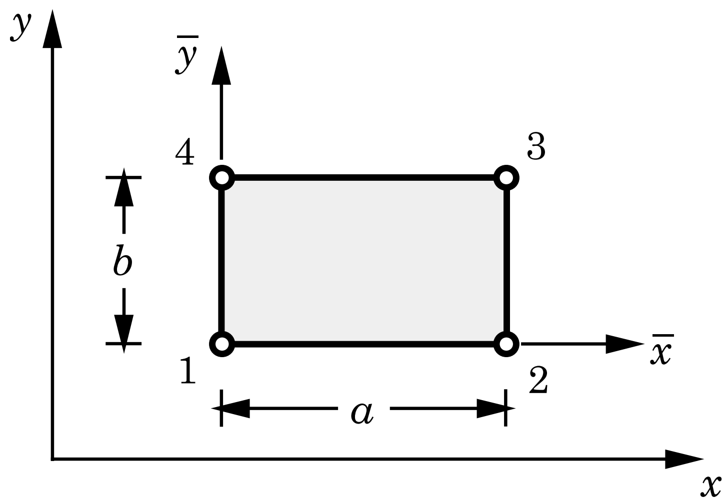
$$\alpha_i = x_j y_k - y_j x_k,\; \beta_i = y_j - y_k,\; \gamma_i = -(x_j - x_k)$$

$$\Delta^e = \text{Area of the triangle}, \quad 2\Delta^e = \text{Determinant of } [A]$$



APPROXIMATION FUNCTIONS

Linear Rectangular Element



$$u_h^e(\bar{x},\bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\,\bar{y}$$

$$u_h^e(\bar{x}_1,\bar{y}_1) = c_1 = u_1^e$$

$$u_h^e(\bar{x}_2,\bar{y}_2) = c_1 + c_2 a \equiv u_2^e$$

$$u_h^e(\bar{x}_3,\bar{y}_3) = c_1 + c_2 a + c_3 b + c_4 ab \equiv u_3^e$$

$$u_h^e(\bar{x}_4,\bar{y}_4) = c_1 + c_3 b \equiv u_4^e$$

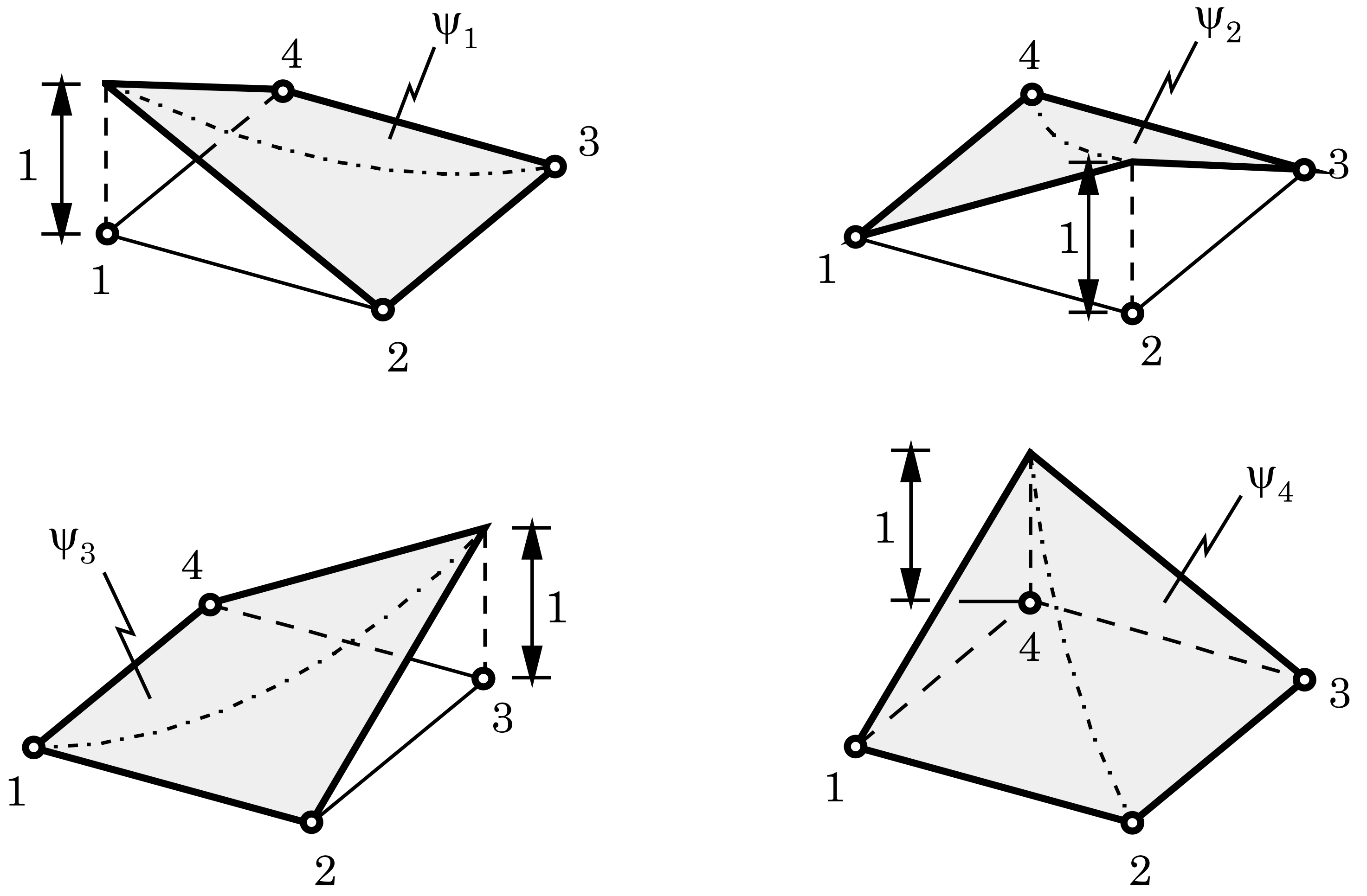
$$\psi_1^e(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right), \quad \psi_2^e(\bar{x},\bar{y}) = \frac{\bar{x}}{a}\left(1-\frac{\bar{y}}{b}\right),$$

$$\psi_3^e(\bar{x},\bar{y}) = \frac{\bar{x}}{a}\frac{\bar{y}}{b}, \quad \psi_4^e(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b},$$

$$\psi_j^e(x_i,y_i) = \delta_{ij}$$

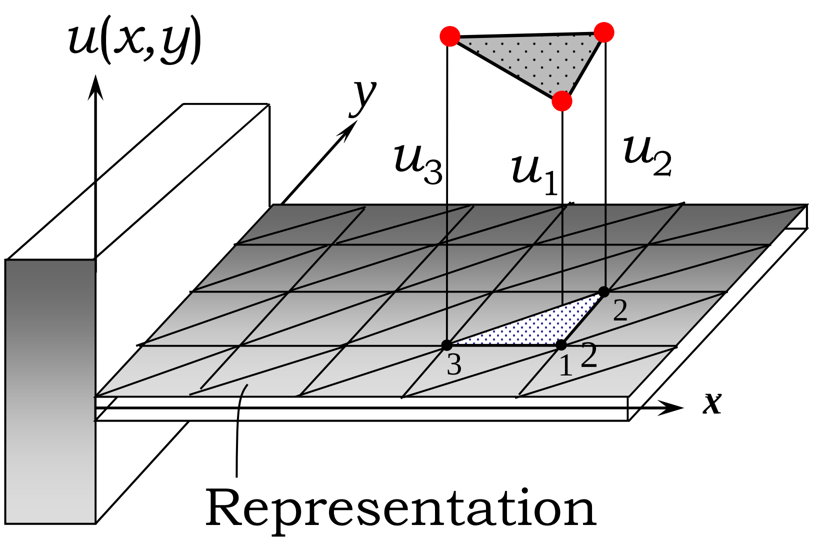
APPROXIMATION FUNCTIONS

Linear Rectangular Element (continued)

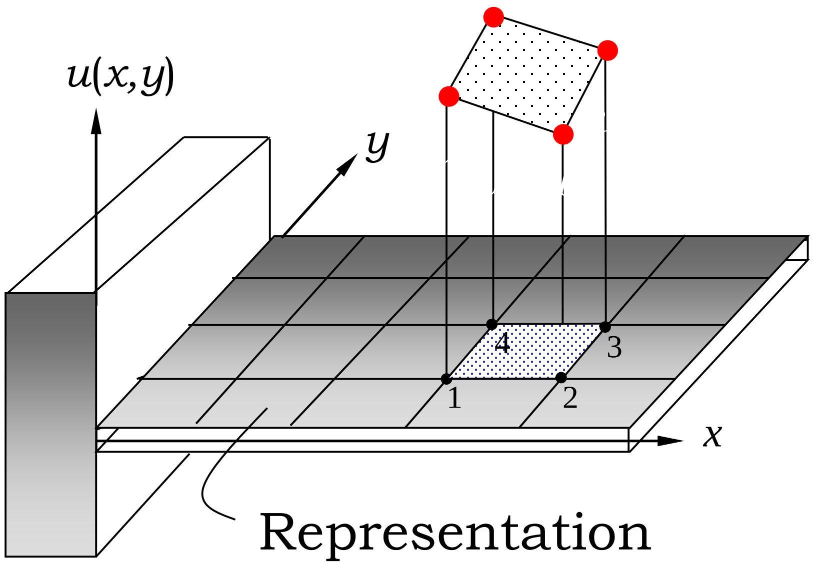


Interpolation Property of the Approximation Functions

$$\psi_j^e(x_i, y_i) = \begin{cases} 1, & i = j, \text{ a fixed value} \\ 0, & i \neq j, \text{ a fixed value} \end{cases} = \delta_{ij}; \quad \sum_{j=1}^{3} \psi_j^e(x, y) = 1$$



Representation of the *domain* by 3-node triangles



Representation of the *domain* by 4-node rectangles

NUMERICAL EVALUATION OF COEFFICIENT MATRICES

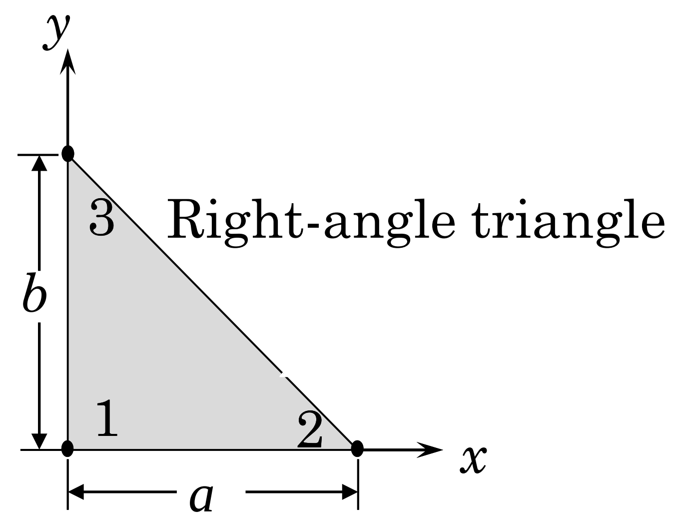
Linear Right-Angled Triangular Element

$$[K^e] = a_{11}^e \frac{b}{2a}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22}^e \frac{a}{2b}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

When $a_{11}^e = a_{22}^e = k_e$, we have

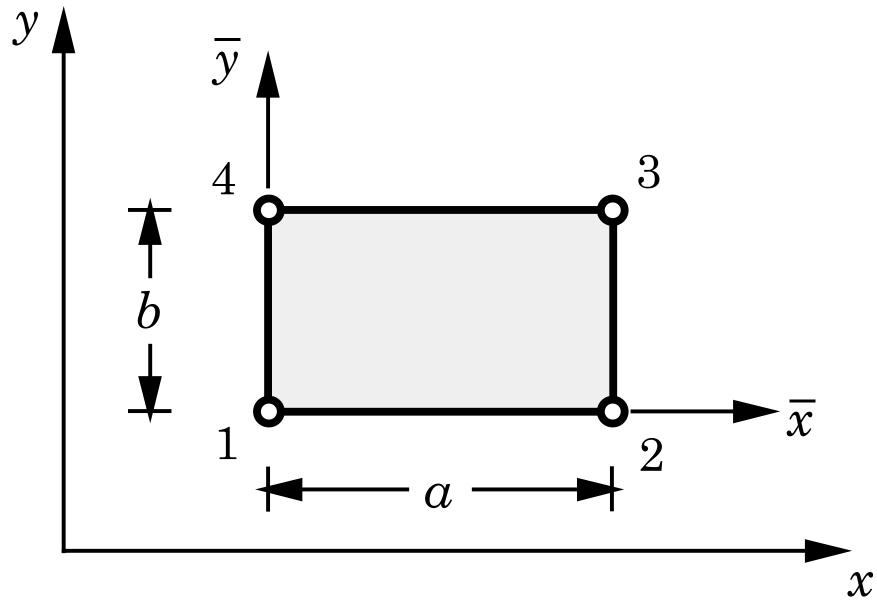
$$[K^e] = \frac{k_e}{2ab}\begin{bmatrix} a^2+b^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix}$$

$$\{f^e\} = \frac{f_e \Delta^e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



NUMERICAL EVALUATION OF COEFFICIENT MATRICES

Linear Rectangular Element



$$[K^e] = a_{11}^e \frac{b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + a_{22}^e \frac{a}{6b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}, \ \{f^e\} = \frac{f_e ab}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

PARAMETRIC FORMULATIONS (2-D)

Geometry: $$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi,\eta), \quad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi,\eta)$$

Solution: $$u(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi,\eta),y(\xi,\eta))$$

Thus, there are two meshes in finite element analysis.

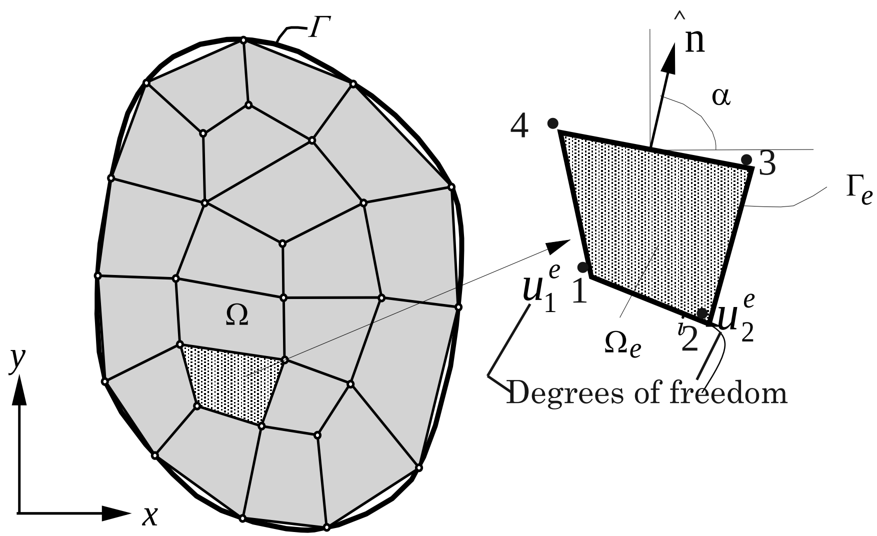
1. ***Superparametric* (*m* > *n*):** The polynomial degree of approximation used for the geometry is of higher order than that used for the dependent variable.

2. ***Isoparametric* (*m* = *n*):** Equal degree of approximation is used for both geometry and dependent variables.

3. ***Subparametric* (*m* < *n*):** Higher-order approximation of the dependent variable is used.

NUMERICAL EVALUATION OF INTEGRAL COEFFICIENTS

- **Transformation of the integrals posed on arbitrary-shaped element to the master element domain so that evaluation of the integrals is made easy.**
- **The Gauss integration rule that evaluates an integral expression as a linear sum of the integrand evaluated at certain points (Gauss points) and weights (Gauss weights) is used.**



$$u(x,y) \approx u_h^e(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x,y)$$

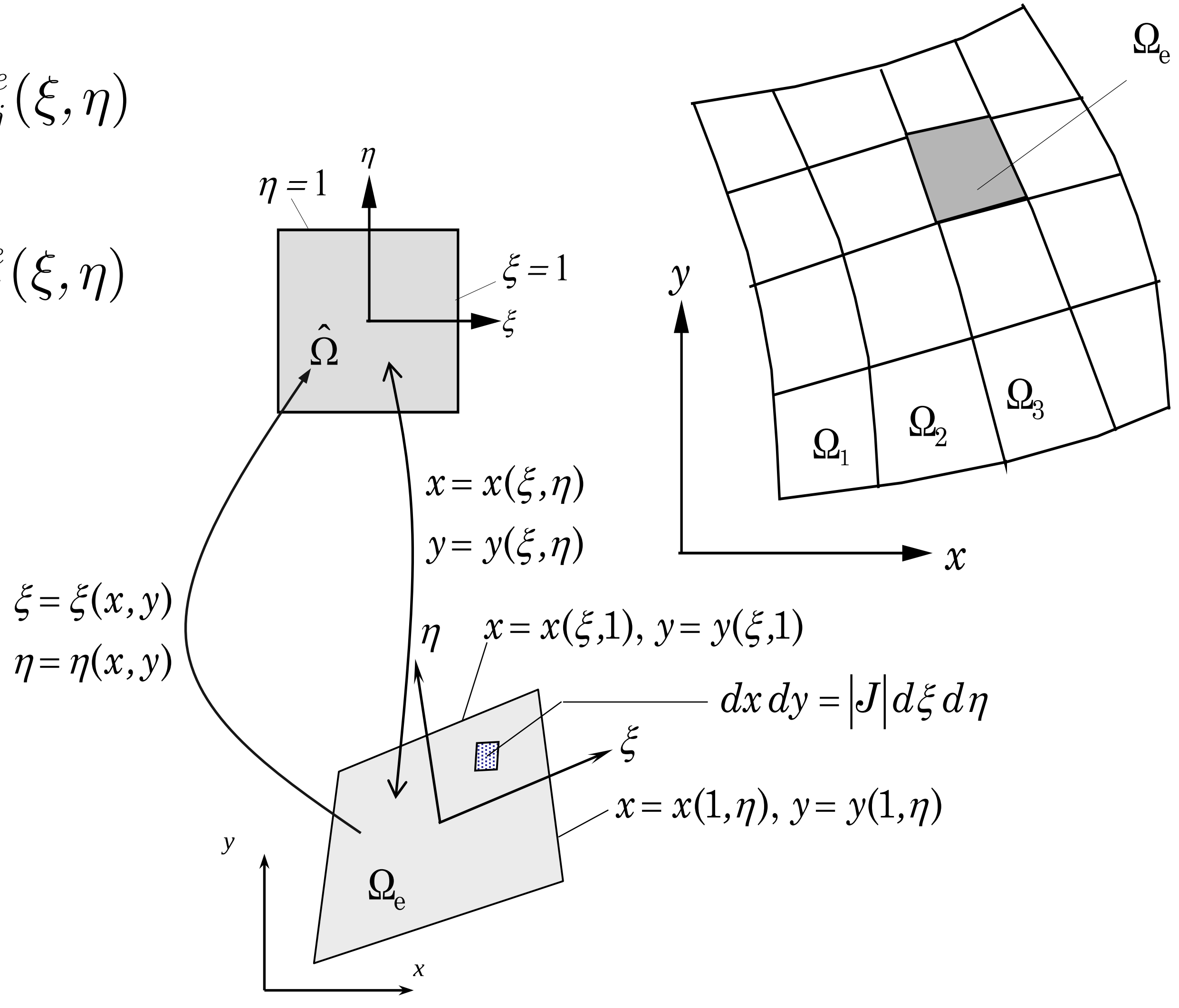
$$\approx u_h^e(x(\xi,\eta), y(\xi,\eta))$$

$$= \sum_{j=1}^{n} u_j^e \psi_j^e(\xi,\eta)$$

NUMERICAL INTEGRATION

$$x(\xi,\eta)=\sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi,\eta)$$

$$y(\xi,\eta)=\sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi,\eta)$$



NUMERICAL INTEGRATION

$$G_{ij} = \int_{x_a}^{x_b} F_{ij}(x)\,dx \approx \sum_{I=1}^{NPT} F_{ij}(x_I) W_I - \text{General}$$

$$K_{ij} = \int_{-1}^{+1} F_{ij}(\xi)\,d\xi \approx \sum_{I=1}^{NGP} F_{ij}(\xi_I) W_I - \text{Gauss rule}$$

$$NGP = \left[\frac{p+1}{2}\right],$$ Nearest larger integer equal to $(p+1)/2$

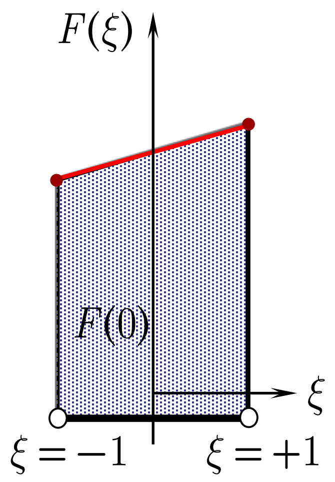
$$p = 1 \quad \Rightarrow \text{NGP} = 1$$

$$p = 2 \text{ or } 3 \Rightarrow \text{NGP} = 2$$

$$p = 4 \text{ or } 5 \Rightarrow \text{NGP} = 3$$

$$I_i = \int_{-1}^{+1} F(\xi)\,d\xi = \frac{h}{2}[F_1 + F_2] = 2\frac{F_1 + F_2}{2}$$

$$= 2 \times F(0) = F(\xi_1) W_1$$



TRANSFORMATION OF THE INTEGRAL

$$K_{ij}^e = \int_{\Omega_e} \left[a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dxdy$$

$$= \int_{\Omega_e} F_{ij}(x,y)\, dxdy = \int_{\hat{\Omega}} F_{ij}(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi d\eta$$

$$= \int_{\hat{\Omega}} F_{ij}(\xi,\eta)\, J\, d\xi d\eta \approx \sum_{I=1}^{NGP} \sum_{J=1}^{NGP} W_I\ W_J\ \hat{F}_{ij}(\xi_I, \eta_J)$$

W_I – Weights

(x_I, y_I) – Integration points

$$\frac{\partial \psi_i}{\partial \xi} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_i}{\partial \eta} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial \xi} \\ \dfrac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\ \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\ \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix}$$

NUMERICAL INTEGRATION

Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \xi} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \xi} \\ \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \eta} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \ldots & \frac{\partial \hat{\psi}_m}{\partial \xi} \\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \ldots & \frac{\partial \hat{\psi}_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

Global derivatives in terms of the local derivatives

$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = \mathbf{J}^{*} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}$$

ELEMENT CALCULATIONS

$$K_{ij}^e = \int_{\Omega_e} \left[a_{11} \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + a_{22}\frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right] dxdy$$

$$dxdy = J\, d\xi\, d\eta, \quad J = |\mathbf{J}|$$

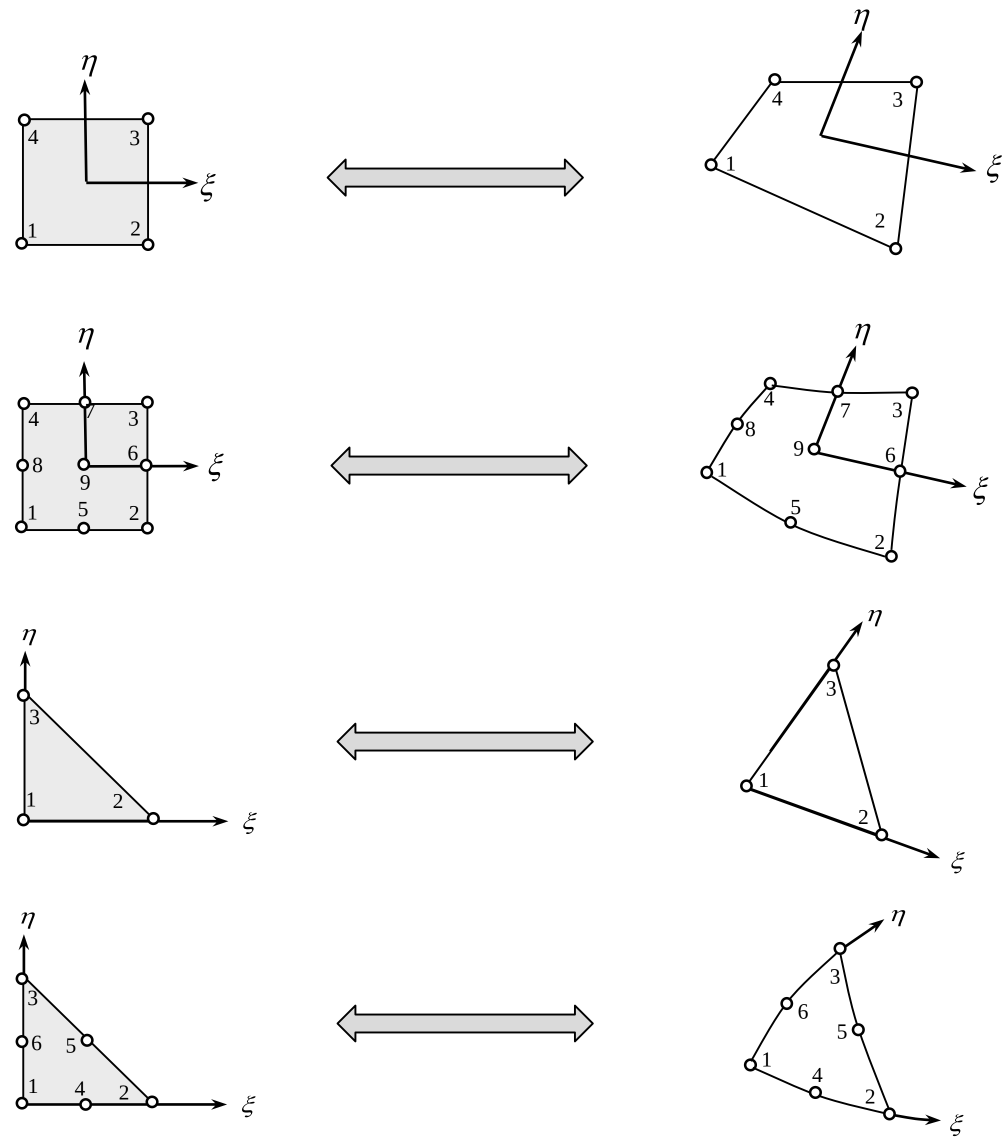
$$= \int_{\hat{\Omega}} \left\{ a_{11}(\xi,\eta)\left(J_{11}^* \frac{\partial \psi_i}{\partial \xi} + J_{12}^* \frac{\partial \psi_i}{\partial \eta}\right)\left(J_{11}^* \frac{\partial \psi_j}{\partial \xi} + J_{12}^* \frac{\partial \psi_j}{\partial \eta}\right) \right.$$

$$\left. + a_{22}(\xi,\eta)\left(J_{21}^* \frac{\partial \psi_i}{\partial \xi} + J_{22}^* \frac{\partial \psi_i}{\partial \eta}\right)\left(J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta}\right) \right\} J\, d\xi d\eta$$

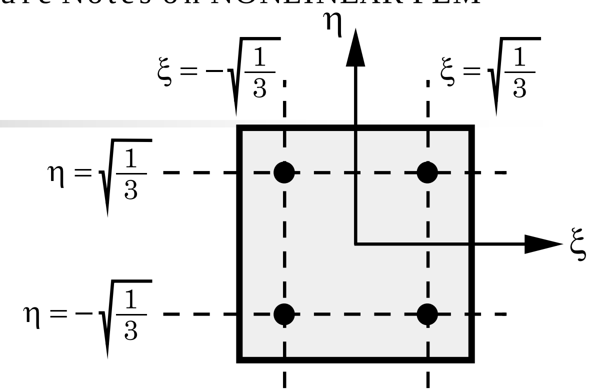
$$= \int_{\hat{\Omega}} \hat{F}_{ij}^e(\xi,\eta)\, d\xi\, d\eta = \int_{-1}^{1}\int_{-1}^{1} \hat{F}_{ij}^e(\xi,\eta)\, d\xi\, d\eta$$

$$\approx \sum_{I=1}^{NGP_\xi} \sum_{J=1}^{NGP_\eta} \hat{F}_{ij}^e(\xi_I, \eta_J)\, W_I W_J$$

Master element Actual element



GAUSS QUADRATURE



$$\int_{\Omega_e} F_{ij}(x,y)\,dxdy = \int_{\hat{\Omega}_e} \hat{F}_{ij}(\xi,\eta)\; d\xi\, d\eta$$

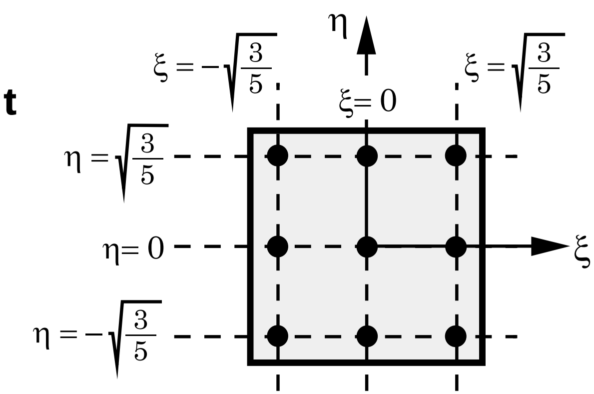
$$\approx \sum_{I,J=1}^{N} W_I W_J \hat{F}_{ij}(\xi_I, \eta_J)$$

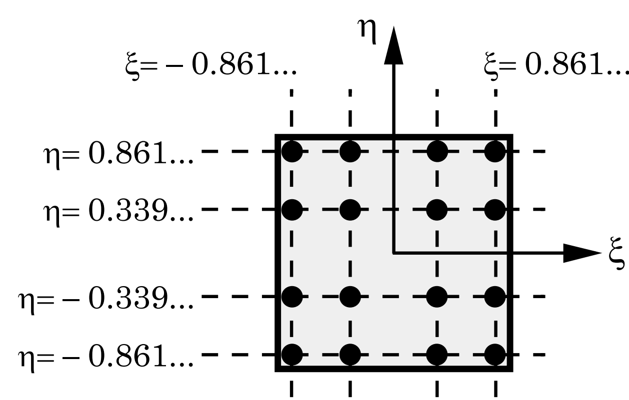
Domain of the *master* element

Domain of the *physical* element

Gauss *points*

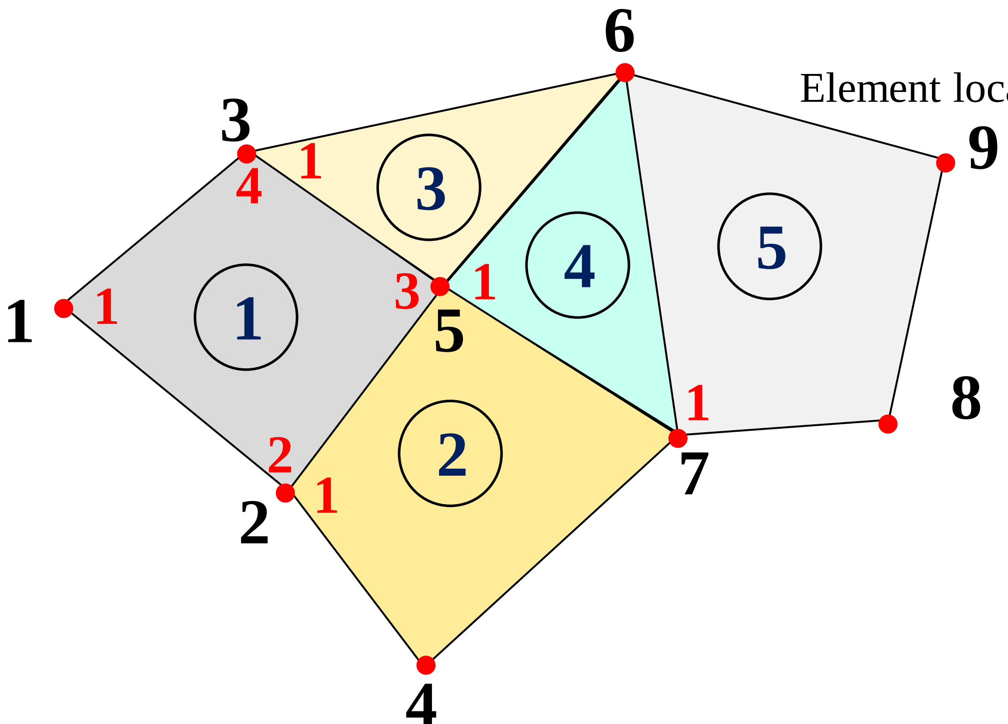
Gauss *weights*





ASSEMBLY OF ELEMENTS/EQUATIONS

(1 DoF per node)



Connectivity matrix

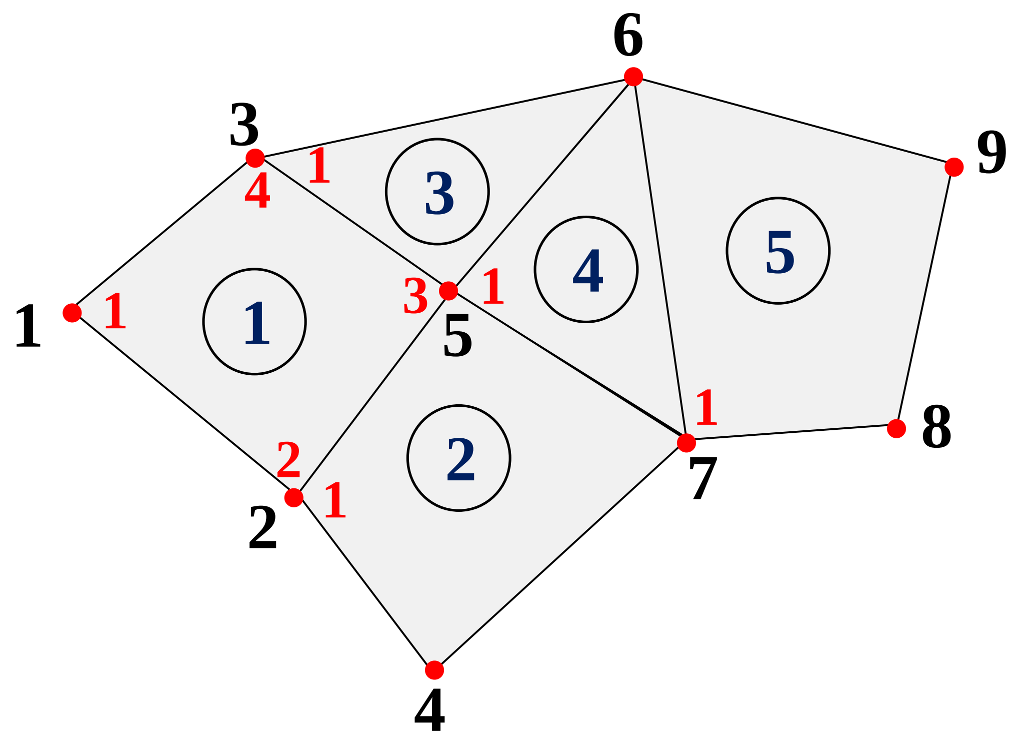
Element local node numbers → $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

$$[B] = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 2 & 4 & 7 & 5 \\ 3 & 5 & 6 & \times \\ 5 & 7 & 6 & \times \\ 7 & 8 & 9 & 6 \end{bmatrix}$$

K_{IJ} = a relationship between certain property of global node I and global node J.
= 0, if node I and J do not belong to the same element

$$K_{11} = K_{11}^{(1)},\ \ K_{13} = K_{14}^{(1)},\ \ K_{15} = K_{13}^{(1)},\ \ K_{14} = 0,$$
$$K_{22} = K_{22}^{(1)} + K_{11}^{(2)},\ K_{25} = K_{23}^{(1)} + K_{14}^{(2)},\ \ K_{26} = 0,$$
$$K_{55} = K_{33}^{(1)} + K_{44}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)},$$
$$K_{56} = K_{23}^{(3)} + K_{13}^{(4)},\ F_1 = F_1^{(1)},\ F_2 = F_2^{(1)} + F_1^{(2)},$$
$$F_5 = F_3^{(1)} + F_4^{(2)} + F_2^{(3)} + F_1^{(4)}$$

POST-COMPUTATION OF VARIABLES



$$u_h^e(x,y) = \sum_{j=1}^{n} u_j^{(e)} \psi_j^{(e)}(x,y), \quad (x,y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x}, \quad \frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y}, \quad (x,y) \in \Omega^e$$

Derivatives of the Solution

Linear triangular element

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e \beta_j^e, \quad (x,y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e \gamma_j^e, \quad (x,y) \in \Omega^e$$

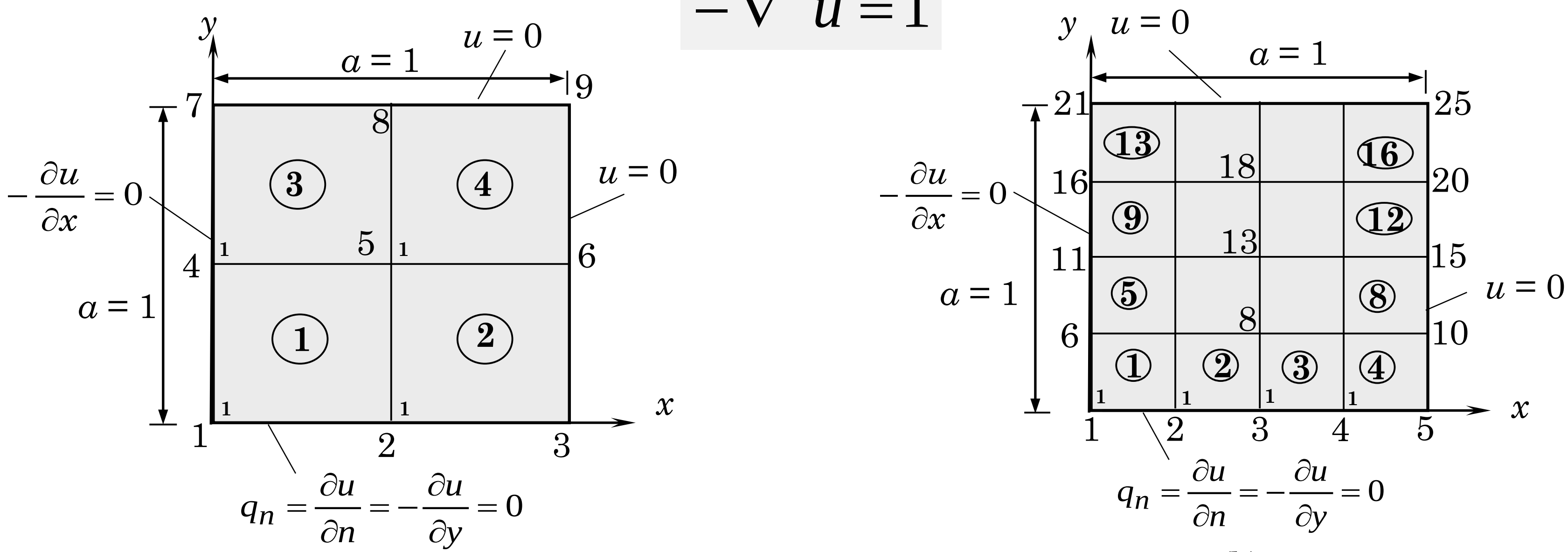
Linear Rectangular element

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e (\beta_j^e + \mu_j^e y), \quad (x,y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e (\gamma_j^e + \mu_j^e x), \quad (x,y) \in \Omega^e$$

Example 8.3.1 from the book

2×2 mesh $-\nabla^2 u = 1$ **4×4 mesh**



(a) (b)

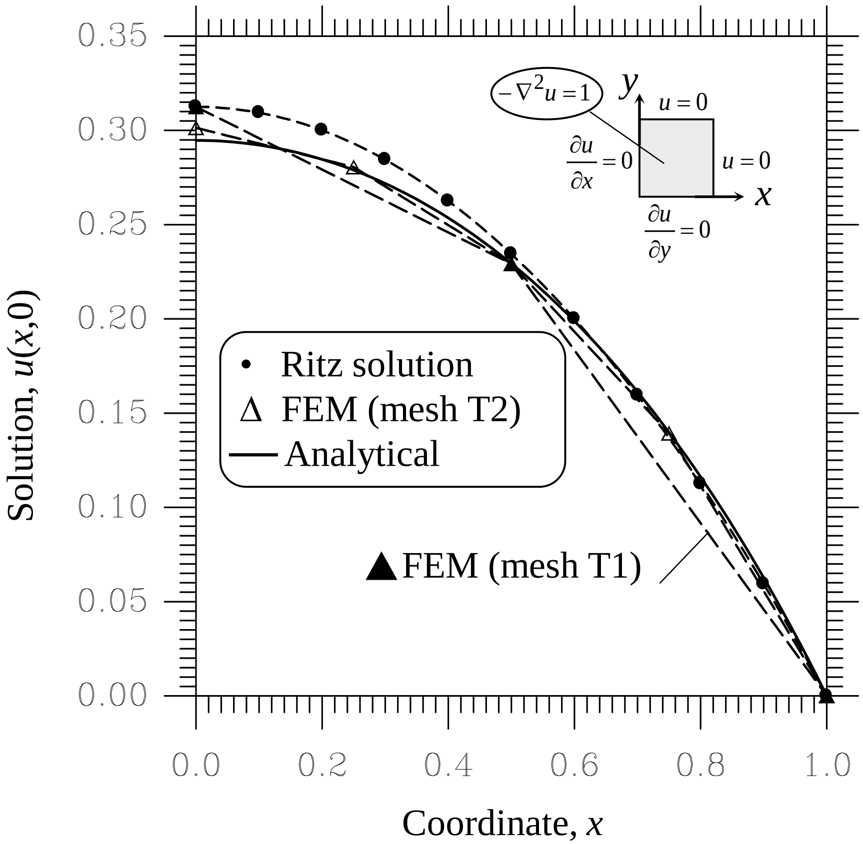
For example, equation for node 1 is:

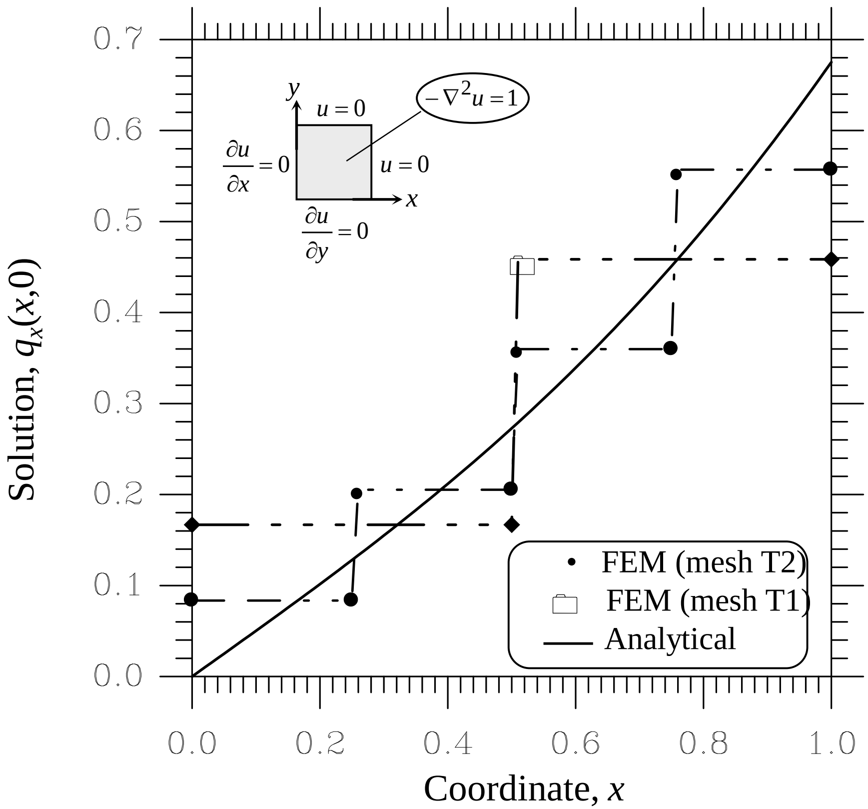
$$K_{11}U_1 + K_{12}U_2 + K_{14}U_4 + K_{15}U_5 = F_1$$

or

$$K_{11}^{(1)}U_1 + K_{12}^{(1)}U_2 + K_{14}^{(1)}U_4 + K_{13}^{(1)}U_5 = f_1^{(1)} + Q_1^{(1)} = f_1^{(1)}$$

Finite Element, Ritz, and Exact Solutions





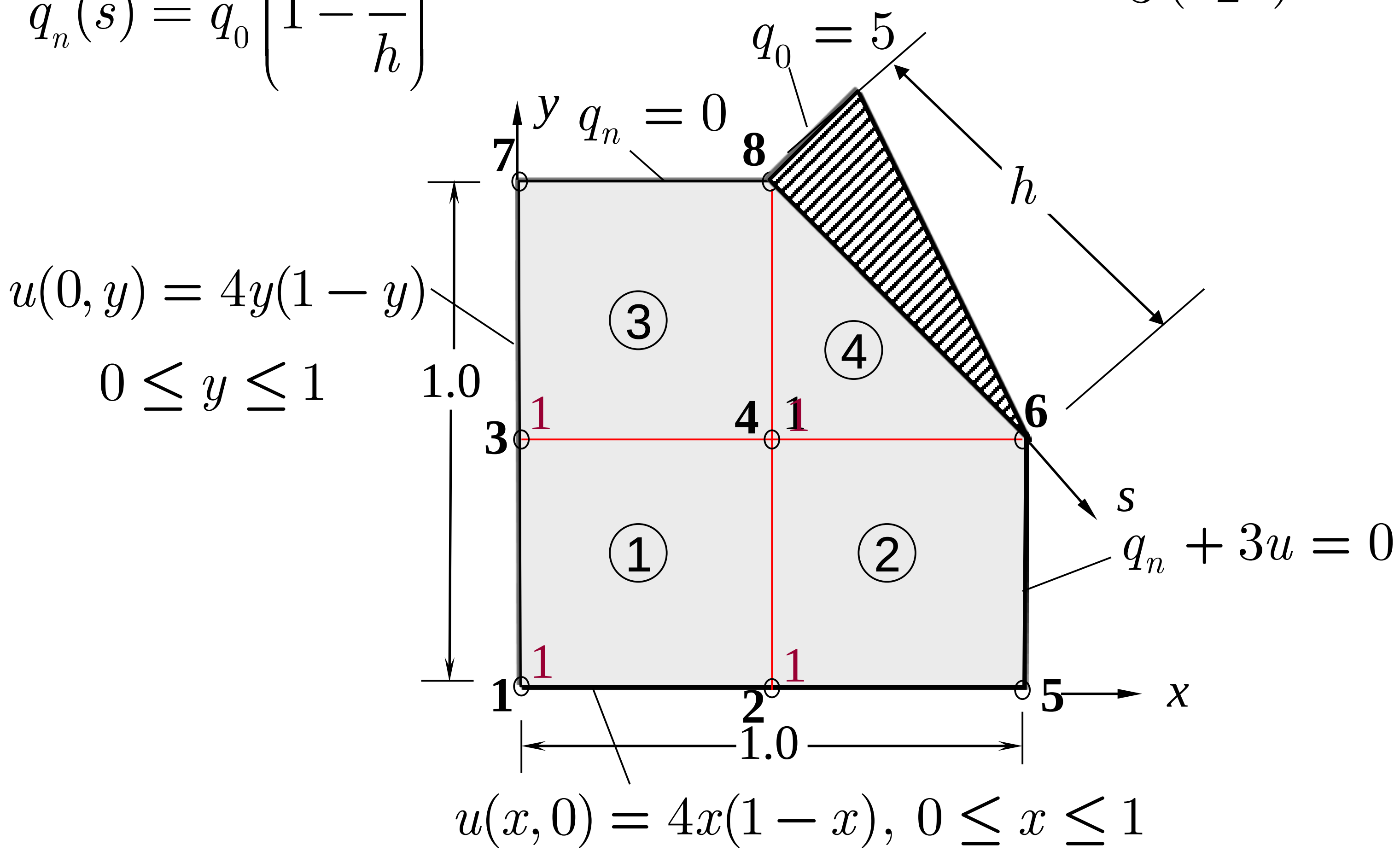
DISCUSSION (distributed boundary source and convection type boundary condition)

$$Q_i = \int_0^h q_n(s)\,\psi_i(s)\,ds$$

$$q_n(s) = q_0\left(1 - \frac{s}{h}\right)$$

For linear elements

$$Q_8 = \frac{2}{3}\left(\frac{q_n h}{2}\right), \quad Q_6 = \frac{1}{3}\left(\frac{q_n h}{2}\right)$$



DISCUSSION

(computation of convection type BC)

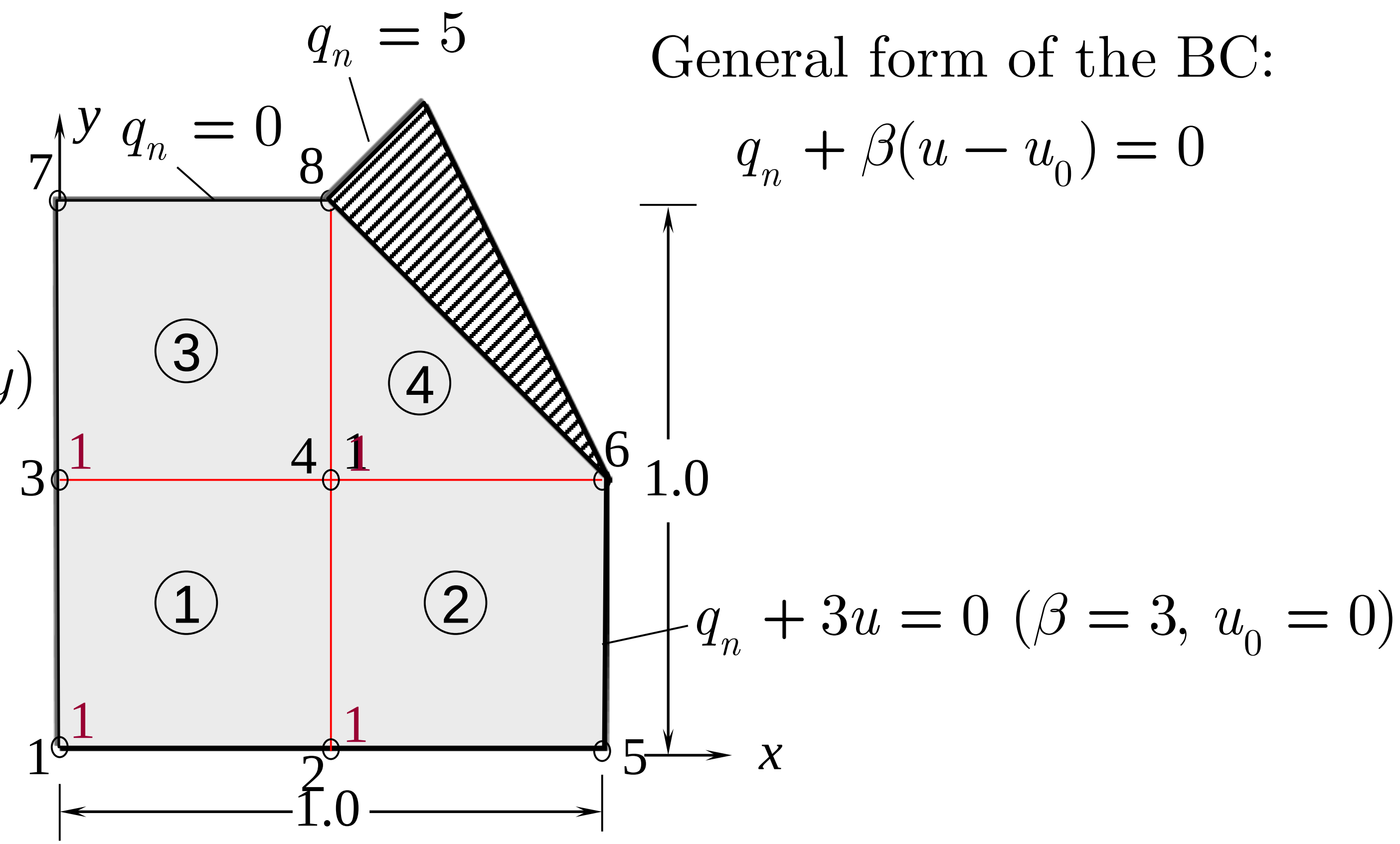
$$Q_i = \int_0^h q_n(s)\,\psi_i(s)\,ds = \int_0^h -\beta(u-u_0)\,\psi_i(s)\,ds$$

$$= \int_0^h -\beta\Big(\sum_{j=1}^n u_j\psi_j - u_0\Big)\,\psi_i(s)\,ds$$

$$= -\beta\sum_{j=1}^n u_j\int_0^h \psi_i\psi_j\,ds + \beta\int_0^h u_0\psi_i\,ds$$

General form of the BC:

$$q_n + \beta(u-u_0) = 0$$



$u(0,y) = 4y(1-y)$

$0 \le y \le 1$

$u(x,0) = 4x(1-x),\ 0 \le x \le 1$

Transient Analysis of 2-D Problems

Model Governing Differential Equation

$$c_1 \frac{\partial u}{\partial t} + c_2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a_{11} \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22} \frac{\partial u}{\partial y}\right) = f(x, y, t)$$

Weak formulation

$$\begin{aligned} 0 &= \int_{\Omega_e} w_i \left[c_1 \frac{\partial u_h}{\partial t} + c_2 \frac{\partial^2 u_h}{\partial t^2} - \frac{\partial}{\partial x}\left(a_{11} \frac{\partial u_h}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22} \frac{\partial u_h}{\partial y}\right) \right. \\ &\qquad \left. - f(x, y, t) \right] dxdy \\ &= \int_{\Omega_e} \left[w_i c_1 \frac{\partial u_h}{\partial t} + w_i c_2 \frac{\partial^2 u_h}{\partial t^2} + a_{11} \frac{\partial w_i}{\partial x} \frac{\partial u_h}{\partial x} + a_{22} \frac{\partial w_i}{\partial y} \frac{\partial u_h}{\partial y} \right. \\ &\qquad \left. - w_i f \right] dxdy - \oint_{\Gamma_e} q w_i \, ds \end{aligned}$$

SPATIAL DISCRETIZATION: Finite Element Model

Approximation

$$u(x,y,t) \approx u_h^e(x,y,t) = \sum_{j=1}^{n} u_j^e(t)\psi_j^e(x,y)$$

Finite element model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

$$C_{ij}^e = \int_{\Omega_e} c_1 \psi_i \psi_j \; dxdy, \quad M_{ij}^e = \int_{\Omega_e} c_2 \psi_i \psi_j \; dxdy$$

$$K_{ij}^e = \int_{\Omega_e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$F_i^e = \int_{\Omega_e} f \, \psi_i \; dxdy + \oint_{\Gamma_e} q \, \psi_i \; ds$$

TIME APPROXIMATIONS (Parabolic)

Semidiscrete FE model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}, \quad 0 < t < T$$

NOTE: From here, the procedure is the same as in 1-D

Alfa-family of approximation

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \Delta t_{s+1}\left[\alpha \dot{\mathbf{u}}_{s+1} + (1-\alpha)\dot{\mathbf{u}}_s\right]$$

$$\mathbf{C}\mathbf{u}_{s+1} = \mathbf{C}\mathbf{u}_s + \Delta t_{s+1}\left[\alpha \mathbf{C}\dot{\mathbf{u}}_{s+1} + (1-\alpha)\mathbf{C}\dot{\mathbf{u}}_s\right]$$

$$\mathbf{C}\dot{\mathbf{u}}_{s+1} = \mathbf{F}_{s+1} - \mathbf{K}_{s+1}\mathbf{u}_{s+1} \qquad \mathbf{C}\dot{\mathbf{u}}_s = \mathbf{F}_s - \mathbf{K}_s\mathbf{u}_s$$

$$\left(\mathbf{C} + \alpha\Delta t_{s+1}\mathbf{K}_{s+1}\right)\mathbf{u}_{s+1} = \left(\mathbf{C} - (1-\alpha)\Delta t_s\mathbf{K}_s\right)\mathbf{u}_s + \Delta t_{s+1}\left[\alpha\mathbf{F}_{s+1} + (1-\alpha)\mathbf{F}_s\right]$$

Fully discretized model

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1}, \qquad \hat{\mathbf{K}}_{s+1} = \alpha\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C},$$

$$\hat{\mathbf{F}}_{s+1} = \left[(1-\alpha)\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C}\right]\mathbf{u}_s + \Delta t_{s+1}\left[\alpha\mathbf{F}_{s+1} + (1-\alpha)\mathbf{F}_s\right]$$

TIME APPROXIMATIONS (Hyperbolic)

Semidiscrete FE model

$$\mathbf{C}^e\dot{\mathbf{u}}^e + \mathbf{M}^e\ddot{\mathbf{u}}^e + \mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e$$

Newmark scheme (second-order equations)

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \Delta t\,\dot{\mathbf{u}}_s + \tfrac{1}{2}(\Delta t)^2\,\ddot{\mathbf{u}}_{s,\gamma}$$

$$\dot{\mathbf{u}}_{s+1} = \dot{\mathbf{u}}_s + \Delta t\,\ddot{\mathbf{u}}_{s,\alpha},\quad \ddot{\mathbf{u}}_{s,\alpha} = (1-\alpha)\ddot{\mathbf{u}}_s + \alpha\,\ddot{\mathbf{u}}_{s+1}$$

Fully discretized model

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1},\qquad \hat{\mathbf{K}}_{s+1} = \mathbf{K}_{s+1} + a_3\mathbf{M}_{s+1} + a_5\mathbf{C}_{s+1}$$

$$\hat{\mathbf{F}}_{s+1} = \mathbf{F}_{s+1} + \mathbf{M}_{s+1}\left(a_3\mathbf{u}_s + a_4\dot{\mathbf{u}}_s + a_5\ddot{\mathbf{u}}_s\right) + \mathbf{C}_{s+1}\left(a_5\mathbf{u}_s + a_6\dot{\mathbf{u}}_s + a_7\ddot{\mathbf{u}}_s\right)$$

An Example: Transient Heat Conduction Problem

Governing equation

$$c_1 \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = f(x,y,t)$$

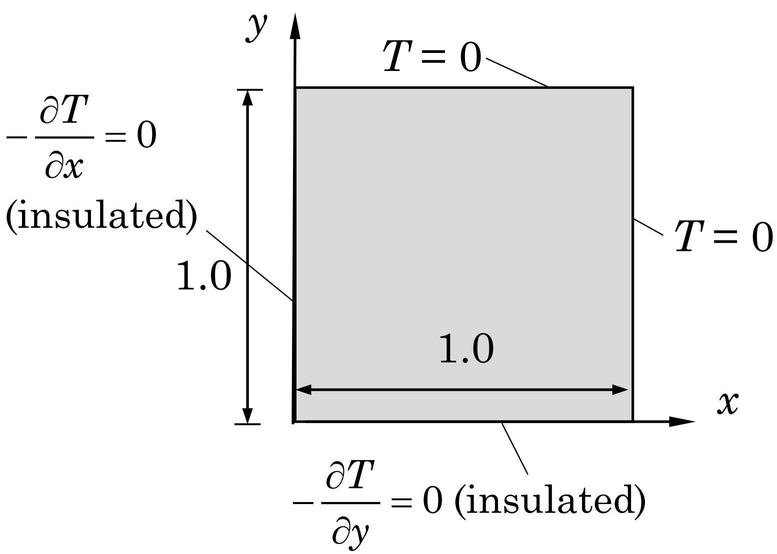
Boundary conditions

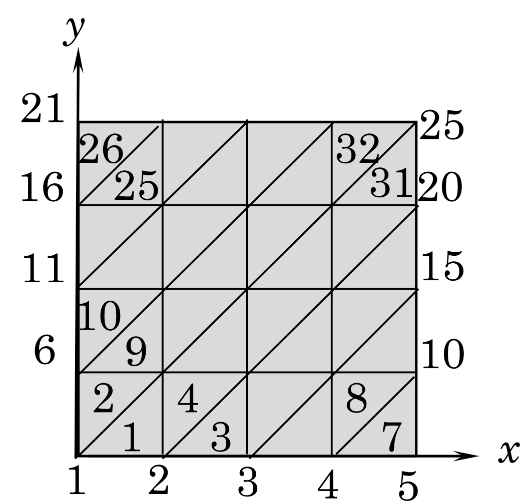
$$k\left(\frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y\right) = 0 \text{ on } x = 0 \text{ and } y = 0$$

$$T = 0 \text{ on } y = 1 \text{ and } x = 1$$

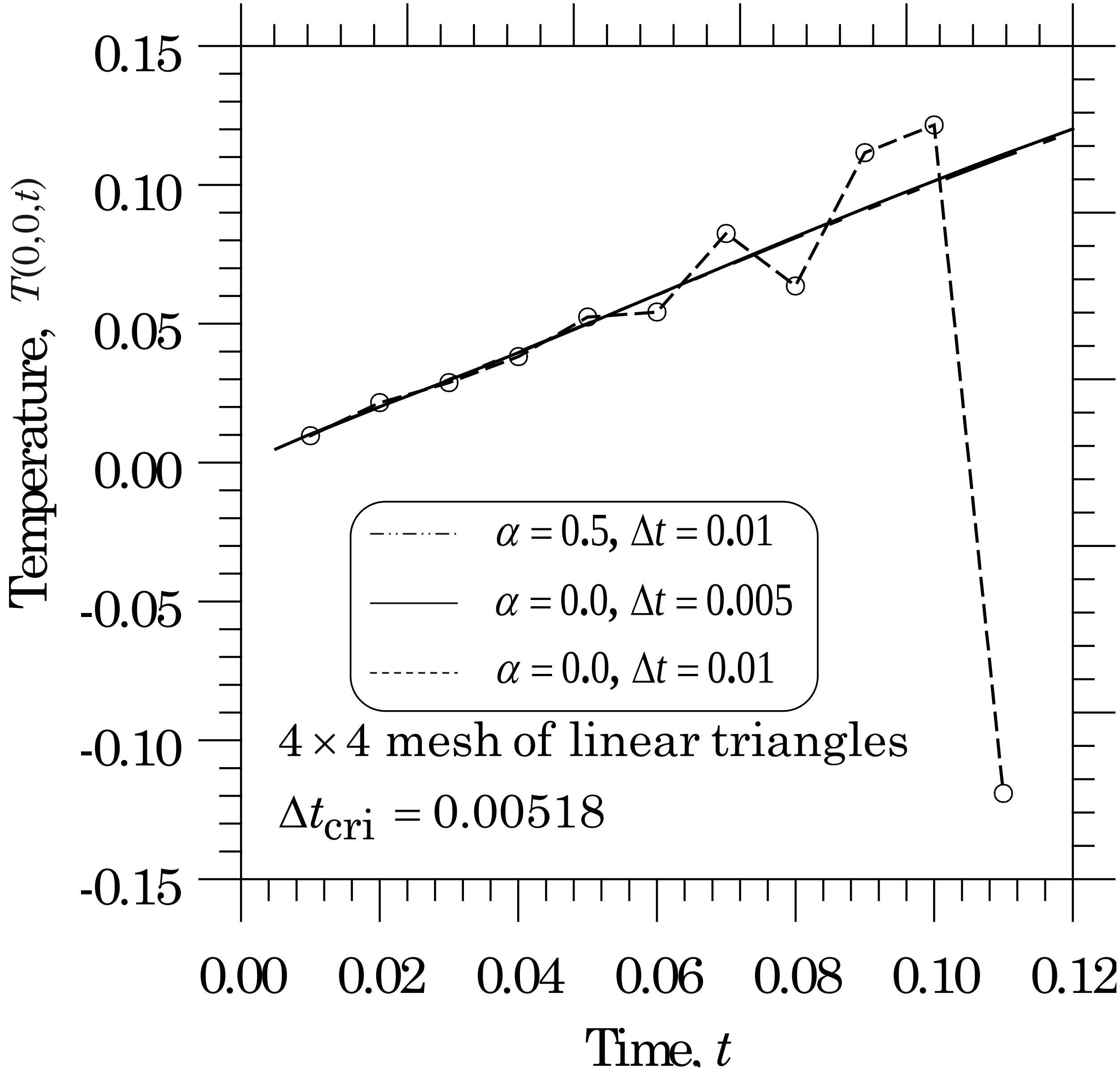
Initial condition

$$T(x,y,0) = 1$$

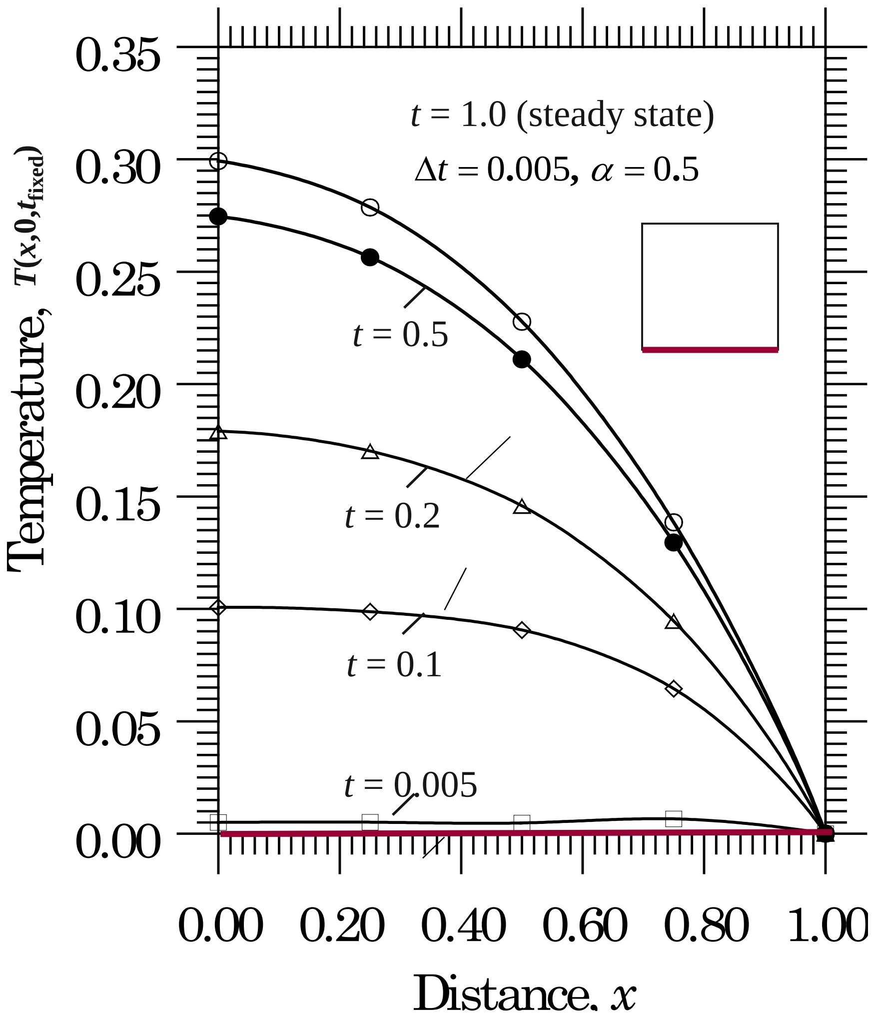


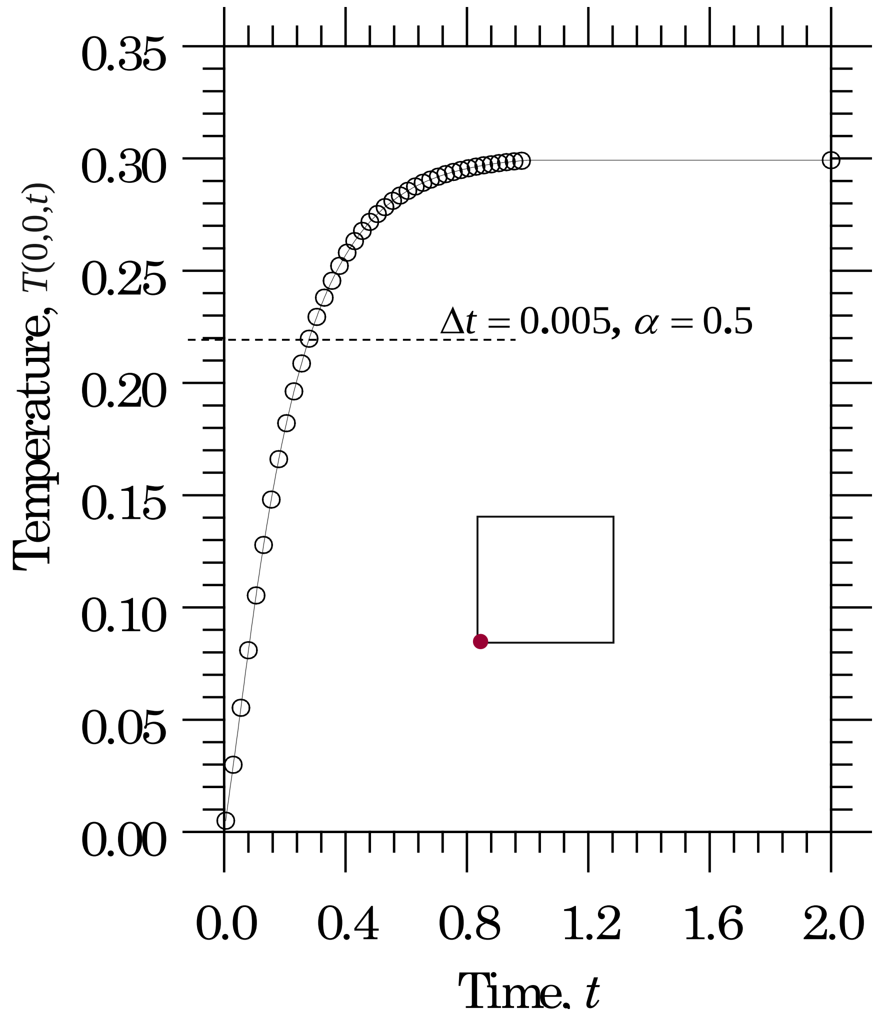


Stability Characteristics

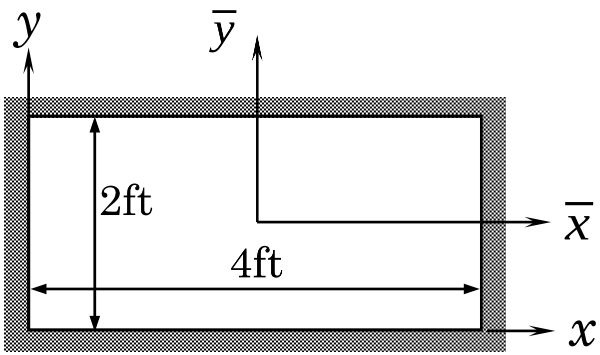


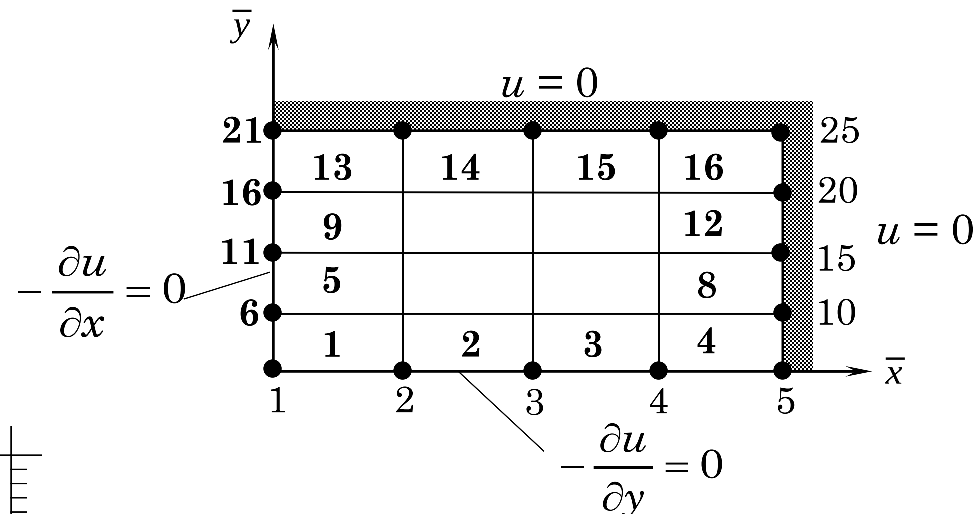
A NUMERICAL EXAMPLE (Parabolic)

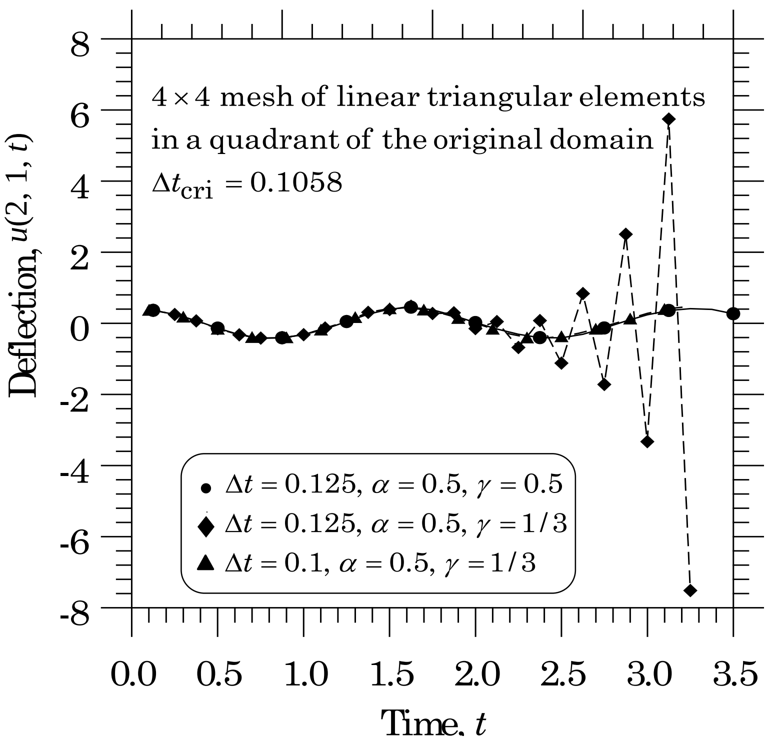


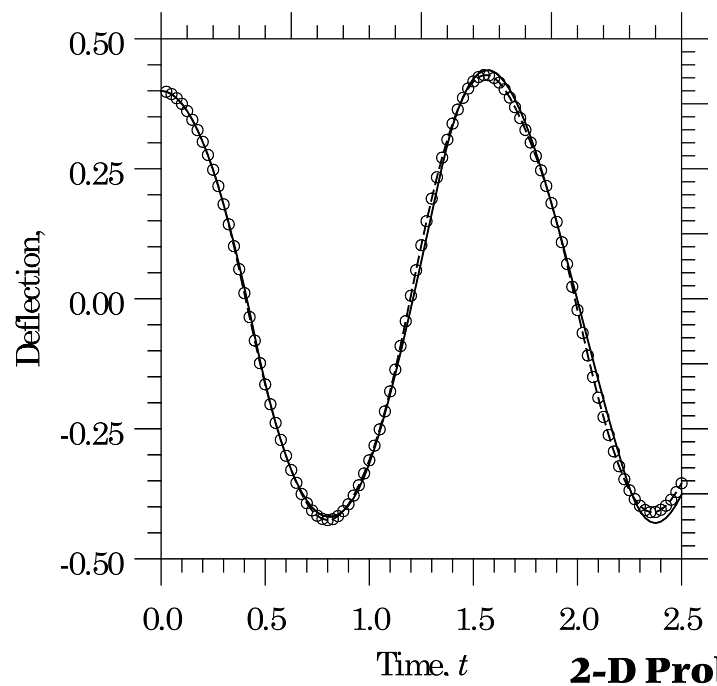


TRANSIENT RESPONSE OF A MEMBRANE









SUMMARY

1. **Starting point is a model differential equation in *u***
2. **Construct an integral statement – weak form,**

   which has three steps. Integration by parts that (a) relaxes ("weakens" differentiability on the variable *u*, and (b) brings in secondary variable into the integral form.
3. **Substitute suitable approximation for *u* and obtain the finite element model (i.e., a set of algebraic relations between nodal values of *u* and *Q*).**
4. **Numerical evaluation of coefficients** K_{ij} and f_i
5. **Assemble equations, impose boundary conditions, and solve the assembled equations.**
6. **Post-computation of variables follows same procedure as in 1-D FEM.**
7. **Reviewed time-dependent problems.**