MEEN 673
Nonlinear Finite Element Analysis
(with focus on solid and structural mechanics, heat transfer, and flows of viscous incompressible fluids)

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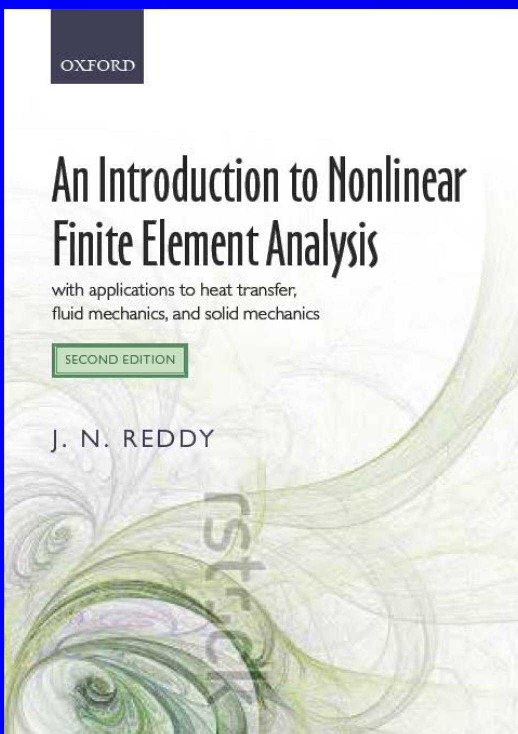
Fall 2016



GENERAL INTRODUCTION

Engineering **is a problem-solving discipline, and solution of a system requires an under-standing of the phenomena that occurs in the system.**

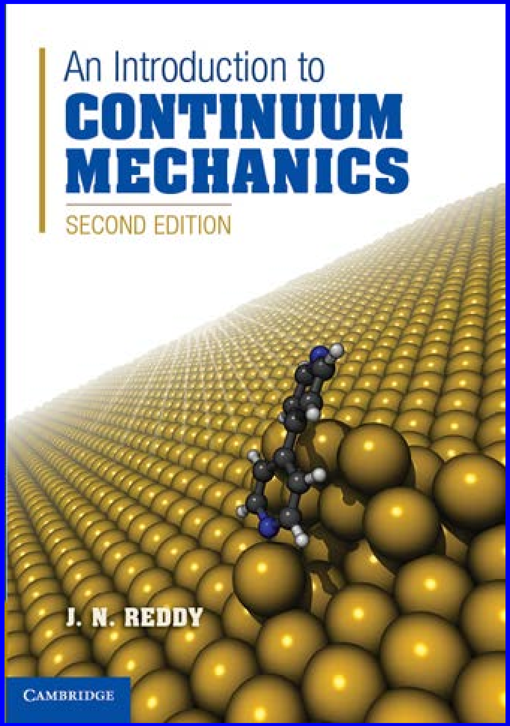
The study of natural phenomena involves

- **developing mathematical models,**
- **conducting physical experiments,**
- **carrying out numerical simulations, and**
- **designing and building systems.**

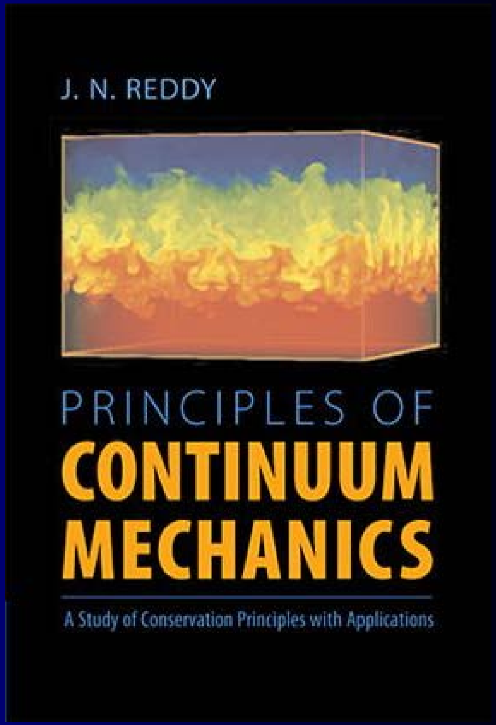
GENERAL INTRODUCTION

The mathematical description of physical phenomena requires mathematical tools such As vectors and tensors and the physical laws which govern the phenomena. Since this course is concerned with the numerical simulation of the physical phenomena (i.e. solving the equations by numerical methods), We review vectors and tensors and the Equations of mechanics first.

Chapter 1

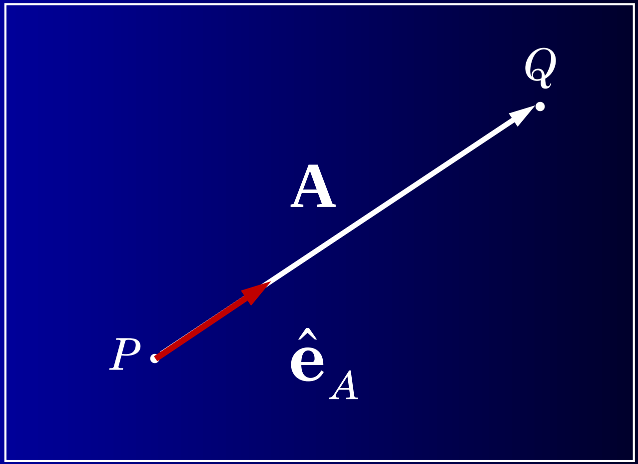


Much of the material included herein is taken from the instructor's two books exhibited here (both published by the Cambridge University Press)



Physical vector: A directed line segment with an arrow head.

Examples: *force*, *displacement*, *velocity*, *weight*

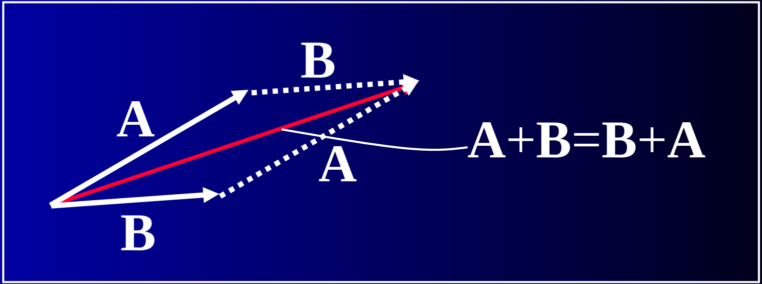


Unit vector along a given vector **A**:

The *unit vector*, $\hat{\mathbf{e}}_A \equiv \dfrac{\mathbf{A}}{A} \;\; (A \neq 0)$

is that vector which has the same direction as **A** but has a magnitude that is unity.

Rules or Axioms



Vector addition:

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B}+\mathbf{A}$ (commutative)

(ii) $(\mathbf{A} + \mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ (associative)

(iii) $\mathbf{A}+\mathbf{0}=\mathbf{A}$ (zero vector)

(iv) $\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$ (negative vector)

Scalar multiplication of a vector:

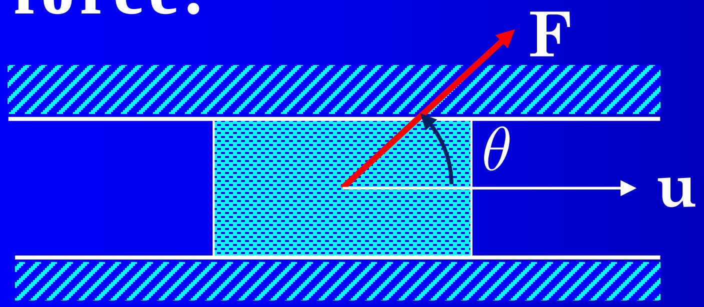
(i) $\alpha(\beta\mathbf{A})= \alpha\beta(\mathbf{A})$ (associative)

(ii) $(\alpha + \beta)\mathbf{A} = \alpha\,\mathbf{A}+ \beta\mathbf{A}$ (distributive w.r.t. scalar addition)

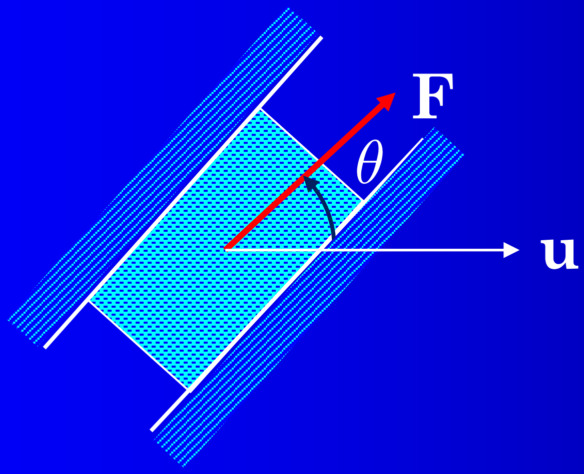
(iii) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$ (distributive w.r.t. vector addition)

(iv) $1.\,\mathbf{A}=\mathbf{A}\,.\,1$

Work done Magnitude of the force multiplied by the magnitude of the displacement in the direction of the force:



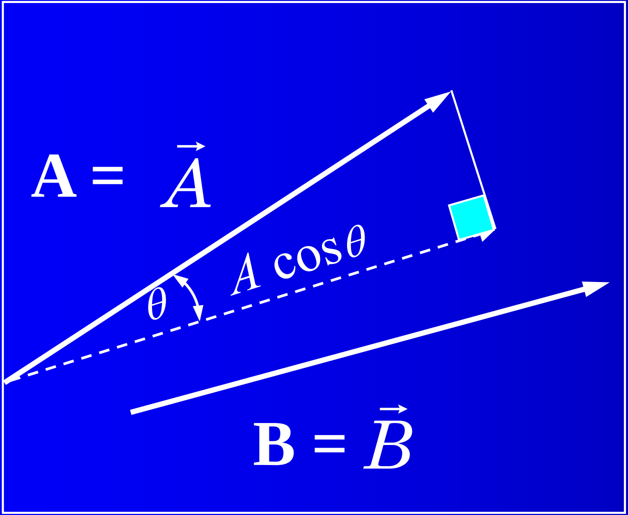
$$\mathrm{WD} = |\mathbf{F}| \cos\theta \times |\mathbf{u}|$$

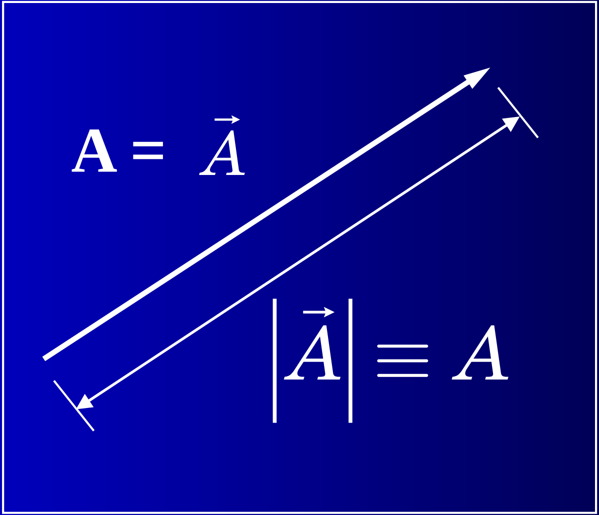


$$\mathrm{WD} = |\mathbf{F}| \times |\mathbf{u}| \cos\theta$$

VECTORS (continued)

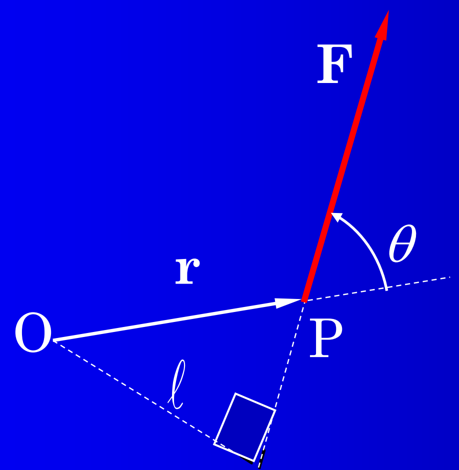
$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta = AB\cos\theta$$

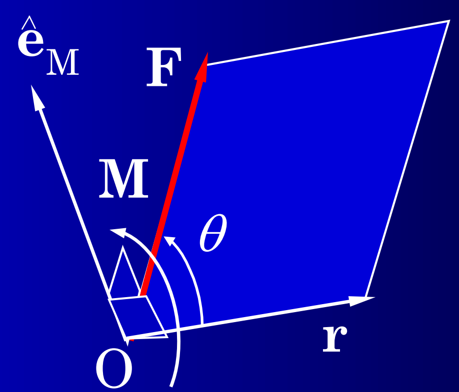




Moment of a force **Magnitude of the force multiplied by the magnitude of the perpendicular distance to the action of the force:**

$$\left|\mathbf{M}\right| = \ell F, \;\; \mathbf{M} = \left(r \sin\theta \times F\right) \hat{\mathbf{e}}_M = \mathbf{r} \times \mathbf{F}$$

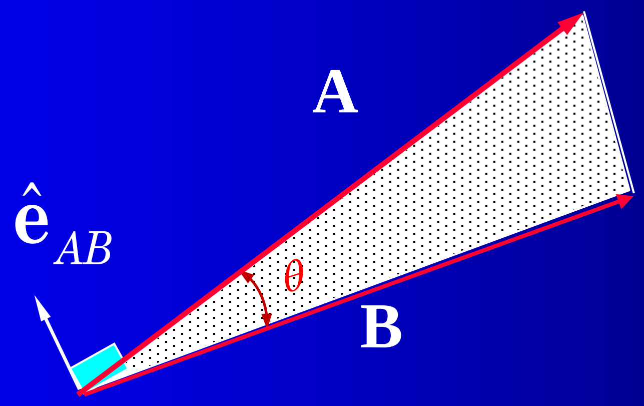




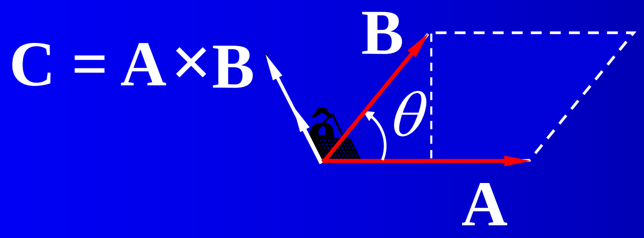
$$\ell = \left|\mathbf{r}\right| \sin\theta = r \sin\theta$$

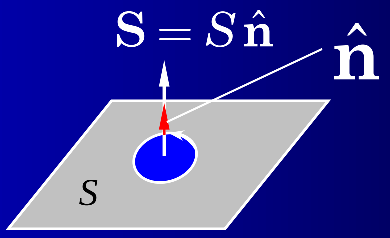
Vector product of two vectors is defined as

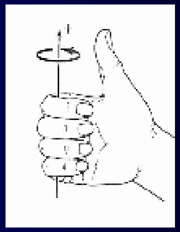
$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \, \hat{\mathbf{e}}_{AB} = AB \sin\theta \, \hat{\mathbf{e}}_{AB}$$



PLANE AREA AS A VECTOR

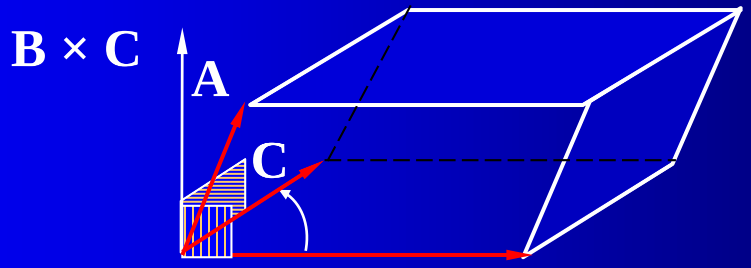






SCALAR TRIPLE PRODUCT

The product **A** . (**B** × **C**) is a scalar and it is termed *the scalar triple product*. It can be seen from the figure that the product **A** . (**B** × **C**) , except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A**, **B**, and **C**.



A . (**B** × **C**)

1. If two vectors are such that $\mathbf{A}\cdot\mathbf{B}=0$ what can we conclude?

2. If two vectors are such that $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ what can we conclude?

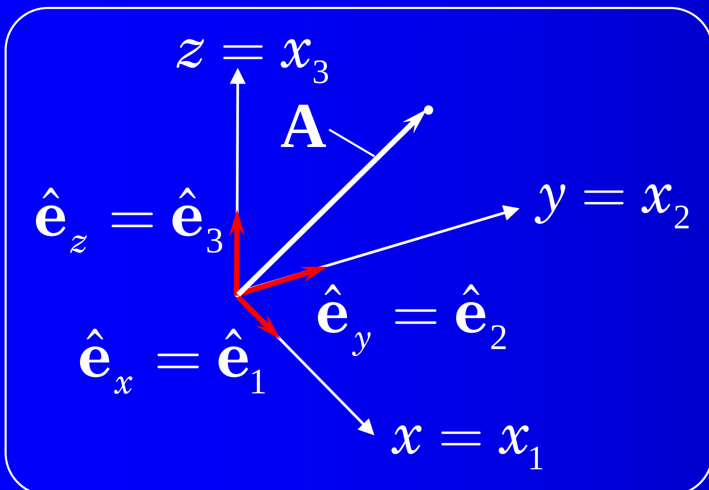
3. Prove that $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$

4. If three vectors are such that $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=0$ what can we conclude?

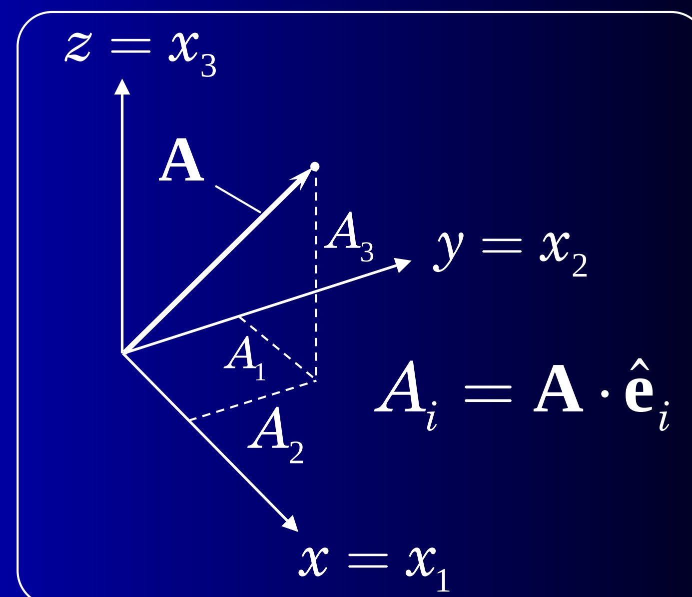
5. The velocity vector in a flow field is $\mathbf{v}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ (m/ s). Determine (a) the velocity vector $\mathbf{v}_n$ normal to the plane $\mathbf{n}=3\hat{\mathbf{i}}-4\hat{\mathbf{k}}$ passing through the point, (b) the angle between $\mathbf{v}$ and $\mathbf{v}_n$, (c) tangential velocity vector on the plane, and (d) The mass flow rate across the plane through an area $A=0.15\text{ m}^2$ if the fluid density is $\rho=10^3\text{ kg/ m}^3$ and the flow is uniform.

Components of a vector

$$\mathbf{A} = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z$$
$$= A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{n}} = n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y + n_z\hat{\mathbf{e}}_z$$
$$= n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2 + n_3\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 = 0,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3 = 0,$$
$$\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 = 1,\quad \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3 = 0,\quad \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3 = 1,$$
$$\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 = 0,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3,$$
$$\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2$$



SUMMATION CONVENTION

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

Dummy index

$$= \sum_{i=1}^{3} A_i\hat{\mathbf{e}}_i = A_i\hat{\mathbf{e}}_i \text{ (summation convention)}$$

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_i\right)\hat{\mathbf{e}}_i = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_j\right)\hat{\mathbf{e}}_j$$

Dummy indices

Scalar product

$$\mathbf{A}\cdot\mathbf{B} = \left(A_i\hat{\mathbf{e}}_i\right)\cdot\left(B_j\hat{\mathbf{e}}_j\right)$$

$$= A_iB_j\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right)$$

$$= A_iB_j\delta_{ij} = A_iB_i$$

$$\delta_{ij} \equiv \left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right) = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

SUMMATION CONVENTION (continued)

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\ \hat{\mathbf{e}}_{AB}$$
$$= \left(A_i\hat{\mathbf{e}}_i\right)\times\left(B_j\hat{\mathbf{e}}_j\right) = A_iB_j\left(\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\right)$$
$$= A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j = \varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\varepsilon_{ijk} \equiv \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k = \begin{cases} \varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \\ 0, \text{ if any two indices are the same} \\ 1, \textit{ if } i\neq j\neq k, \text{ and they permute} \\ \qquad \text{in a natural order} \\ -1, \textit{ if } i\neq j\neq k, \text{and they permute} \\ \qquad \textit{opposite} \text{ to a natural order} \end{cases}$$

SUMMATION CONVENTION (continued)

Contraction of indices:

The Kronecker delta δ_{ij} modifies (or contracts) the subscripts in the coefficients of an expression in which it appears:

$$A_i \delta_{ij} = A_j, \quad A_i B_j \delta_{ij} = A_i B_i = A_j B_j, \quad \delta_{ij} \delta_{ik} = \delta_{jk}$$

Correct expressions:

$$F_i = A_i B_j C_j, \quad G_k = H_k (2 - 3 A_i B_i) + P_j Q_j F_k$$

Incorrect expressions:

$$A_i = B_j C_k, \; A_i = B_j \text{ and } F_k = A_i B_j C_k$$

SUMMATION CONVENTION (continued)

$$p_i = a_i b_j c_j \text{ and } c_k = d_i e_i q_k$$

$$a_i = \frac{p_i}{b_j c_j}$$

$$b_j c_j = \frac{p_i}{a_i}$$

$$b_j c_j = \frac{p_i}{a_i} = \frac{p_1}{a_1} + \frac{p_2}{a_2} + \frac{p_3}{a_3}$$

The permutation symbol and the Kronecker delta prove to be very useful in establishing vector identities. Since a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to establish it in one coordinate system. The following identity is useful:

ε-δ Identity:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

EXERCISES ON INDEX NOTATION

Exercise-1: Check which one of the following expressions are valid:

(a) $a_m b_s = c_m(d_r - f_r)$; (b) $a_m b_s = c_m(d_s - f_s)$

(c) $a_i = b_j c_i(d_i - f_i)$; (d) $x_m x_m = r^2$

(e) $a_i = 3$; (f) $\delta_{ij}\,\delta_{jk}\,\delta_{ki} = ?$

Exercise-2: Prove $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \varepsilon_{ijk}A_iB_jC_k = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$

Exercise-4: Simplify the expression $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$

Exercise-5: Rewrite the expression $\varepsilon_{mni}\,A_iB_jC_mD_n\hat{\mathbf{e}}_j$ in vector form

A second-order tensor is one that has two basis vectors standing next to each other, and they satisfy the same rules as those of a vector (hence, mathematically, tensors are also called vectors). A second-order tensor and its *transpose* can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{S} = S_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ji}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i; \qquad \mathbf{S}^{\mathrm{T}} = S_{ji}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ij}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i$$

A second-order tensor is symmetric only if

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \Leftrightarrow S_{ij} = S_{ji}$$

Second-order identity tensor has the form

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

We note that $\mathbf{S}\cdot\mathbf{T} \neq \mathbf{T}\cdot\mathbf{S}$ (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors) because

$$\mathbf{S}\cdot\mathbf{T} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(T_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{jl}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

$$\mathbf{T}\cdot\mathbf{S} = \left(T_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(S_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = T_{ij}S_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{jl}T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

We also note that (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors and $\mathbf{A}$ is a vector)

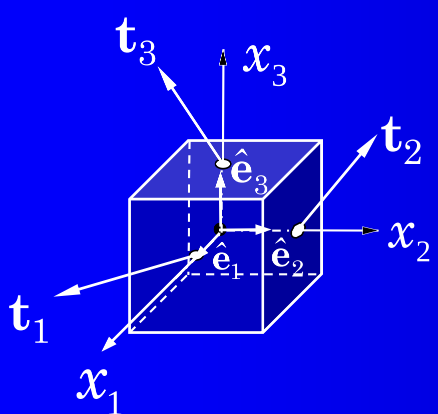
$$\mathbf{S}\times\mathbf{T} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(T_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{kl}\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p\hat{\mathbf{e}}_l$$

$$\mathbf{S}\cdot\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right) = S_{ij}A_j\hat{\mathbf{e}}_i$$

$$\mathbf{S}\times\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right) = S_{ij}A_k\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p$$

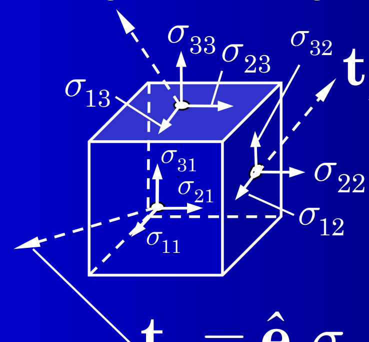
Stress tensor is a good example of a second-order tensor. The two basis vectors represent the direction and the plane on which they act. The Cauchy stress tensor is defined by the Cauchy formula (to be established in the sequel):

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \quad \text{or} \quad t_i = \sigma_{ij}\, n_j$$



$$\mathbf{t}_3 = \hat{\mathbf{e}}_1\sigma_{13} + \hat{\mathbf{e}}_2\sigma_{23} + \hat{\mathbf{e}}_3\sigma_{33}$$

$$\mathbf{t}_2 = \hat{\mathbf{e}}_1\sigma_{12} + \hat{\mathbf{e}}_2\sigma_{22} + \hat{\mathbf{e}}_3\sigma_{32}$$



$$\mathbf{t}_1 = \hat{\mathbf{e}}_1\sigma_{11} + \hat{\mathbf{e}}_2\sigma_{21} + \hat{\mathbf{e}}_3\sigma_{31}$$

$$\mathbf{t}_i = \hat{\mathbf{e}}_j\,\sigma_{ji} = \sigma_{ji}\hat{\mathbf{e}}_j$$

$$\sigma = \mathbf{t}_i\hat{\mathbf{e}}_i = \sigma_{ji}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

A n^{th}-order tensor is one that has n basis vectors standing next to each other, and they satisfy the same rules as those of a vector. A n^{th}-order tensor **T** can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{T} = T_{\underbrace{ijk\ldots p}_{n \text{ subs}}} \underbrace{\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \cdots \hat{\mathbf{e}}_p}_{n \text{ base vectors}};$$

The permutation tensor is a <u>third-order</u> tensor

$$\boldsymbol{\varepsilon} = \varepsilon_{ijk} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k$$

The elasticity tensor is a <u>fourth-order</u> tensor

$$\mathbf{C} = C_{ijkl} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \hat{\mathbf{e}}_l$$

A second-order Cartesian tensor **S** (i.e., tensor with Cartesian components) may be represented in barred $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and unbarred (x_1, x_2, x_3) Cartesian coordinate systems as

$$\mathbf{S} = s_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = \bar{s}_{mn}\hat{\bar{\mathbf{e}}}_m\hat{\bar{\mathbf{e}}}_n$$

The unit base vectors in the unbarred and barred systems are related by

$$\hat{\mathbf{e}}_j = \ell_{ij}\hat{\bar{\mathbf{e}}}_i \text{ and } \hat{\bar{\mathbf{e}}}_i = \ell_{ij}\hat{\mathbf{e}}_j, \quad \ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j$$

Thus the components of a second-order tensor transform according to

$$\boxed{\bar{s}_{ij} = \ell_{im}\ell_{jn}s_{mn}}$$

"Del" operator: $\nabla \equiv \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3}$

"Laplace" operator:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

"Gradient" operation:

$$\nabla F \equiv \hat{\mathbf{e}}_i \frac{\partial F}{\partial x_i}, \text{ where } F \text{ is a scalar function}$$

Grad F defines both the direction and magnitude of the maximum rate of increase of F at any point.

$$\nabla F = \hat{\mathbf{n}}\frac{\partial F}{\partial n},\ \text{where } \hat{\mathbf{n}} \text{ is a unit vector normal to the surface } F = \text{constant}$$

$$\text{We also have } \hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} \text{ and } \frac{\partial F}{\partial n} = \hat{\mathbf{n}}\cdot\nabla F$$

"Divergence" operation:

$$\nabla\cdot\mathbf{G} \equiv \left(\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\hat{\mathbf{e}}_j G_j\right) = \frac{\partial G_i}{\partial x_i},\ \text{where } \mathbf{G} \text{ is a } \textit{vector} \text{ function}$$

The divergence of a vector function represents the volume density of the outward flux of the vector field.

"Curl" operation:

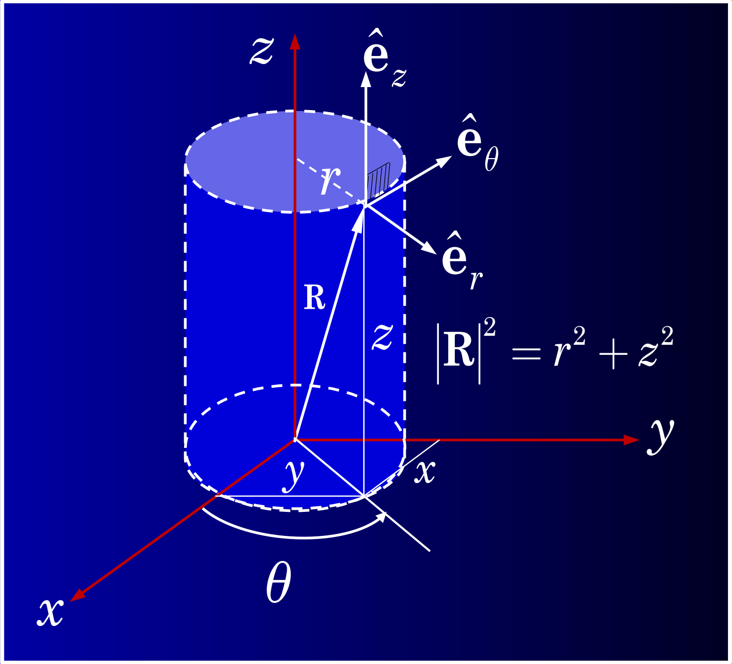
$$\nabla \times \mathbf{G} \equiv \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \times \left(\hat{\mathbf{e}}_j G_j \right) = \varepsilon_{ijk} \frac{\partial G_j}{\partial x_i} \hat{\mathbf{e}}_k,$$

where $\mathbf{G}$ is a *vector* function.

The curl of a vector function represents its rotation. If the vector field is the velocity of a fluid, curl of the velocity represents the rotation of the fluid at the point.

$$\begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$



$$\mathbf{A} = A_r \hat{\mathbf{e}}_r + A_\theta \hat{\mathbf{e}}_\theta + A_z \hat{\mathbf{e}}_z$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$

"Del" operator in cylindrical coordinates

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r} \hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\left[\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r\frac{\partial A_z}{\partial z}\right]$$

Here **A** is a vector:

$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$$

Verify these relations to yourself

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\right]$$

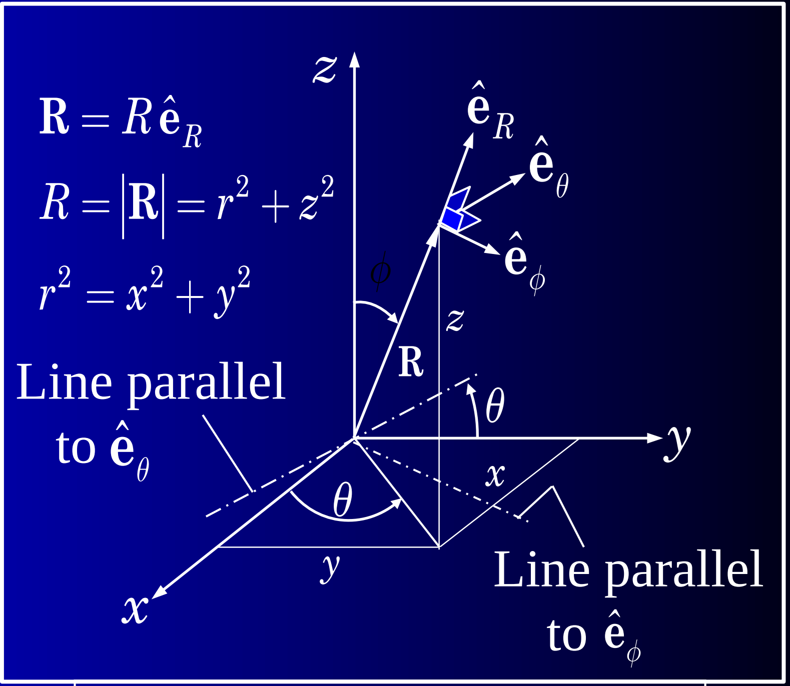
$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{e}}_z$$

$$\nabla \mathbf{A} = \frac{\partial A_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{\partial A_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - A_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \frac{\partial A_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \frac{\partial A_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r$$

$$+ \frac{1}{r}\left(A_r + \frac{\partial A_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial A_z}{\partial \theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial A_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial A_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix}$$



$$\boxed{\mathbf{A} = A_R\hat{\mathbf{e}}_R + A_\phi\hat{\mathbf{e}}_\phi + A_\theta\hat{\mathbf{e}}_\theta}$$

$$\frac{\partial \hat{\mathbf{e}}_R}{\partial \phi} = \hat{\mathbf{e}}_\phi, \quad \frac{\partial \hat{\mathbf{e}}_R}{\partial \theta} = \sin\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\hat{\mathbf{e}}_R, \quad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \theta} = \cos\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\sin\phi\,\hat{\mathbf{e}}_R - \cos\phi\,\hat{\mathbf{e}}_\phi$$

“Del” operator

$$\nabla = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial \phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right]$$

$$\nabla\cdot\mathbf{A} = \frac{2A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{A} = \frac{1}{R\sin\phi}\left[\frac{\partial(\sin\phi A_\theta)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_R + \left[\frac{1}{R\sin\phi}\frac{\partial A_R}{\partial\theta} - \frac{1}{R}\frac{\partial(RA_\theta)}{\partial R}\right]\hat{\mathbf{e}}_\phi + \frac{1}{R}\left[\frac{\partial(RA_\phi)}{\partial R} - \frac{\partial A_R}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$$

$$\nabla\mathbf{A} = \frac{\partial A_R}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_R + \frac{\partial A_\phi}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\phi + \frac{1}{R}\left(\frac{\partial A_R}{\partial\phi} - A_\phi\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_R + \frac{\partial A_\theta}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_R}{\partial\theta} - A_\theta\sin\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_R$$
$$+ \frac{1}{R}\left(A_R + \frac{\partial A_\phi}{\partial\phi}\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial A_\theta}{\partial\phi}\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_\phi}{\partial\theta} - A_\theta\cos\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\phi$$
$$+ \frac{1}{R\sin\phi}\left(A_R\sin\phi + A_\phi\cos\phi + \frac{\partial A_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$$

Establish the following identities (using rectangular Cartesian components and index notation):

1. $\nabla(r) = \dfrac{\mathbf{r}}{r}$

2. $\nabla(r^n) = nr^{n-2}\mathbf{r}$

3. $\nabla \times (\nabla F) = \mathbf{0}$

4. $\nabla \cdot (\nabla F \times \nabla G) = 0$

5. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

6. $\nabla \cdot \left(\mathbf{A} \times \mathbf{B}\right) = \nabla \times \mathbf{A} \cdot \mathbf{B} - \nabla \times \mathbf{B} \cdot \mathbf{A}$

7. $\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A} \cdot \nabla \mathbf{A}$.

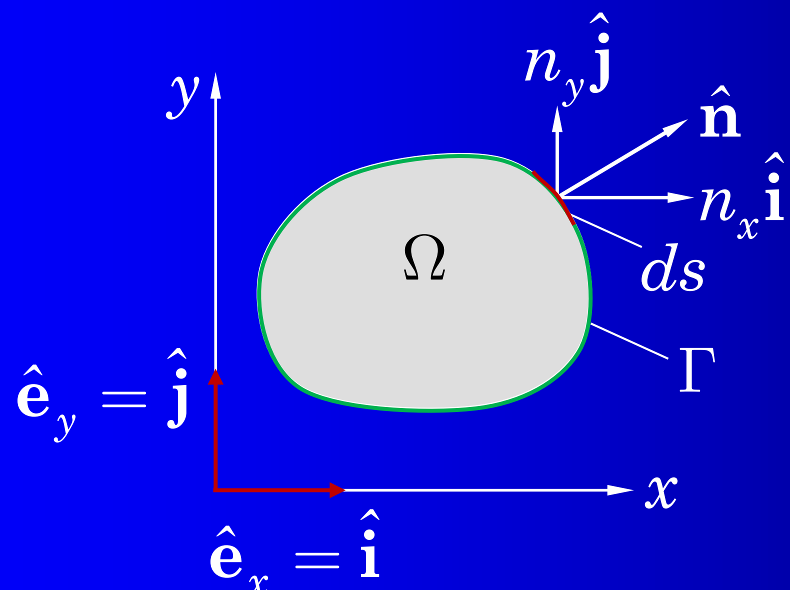
f – scalar; $\mathbf{F}$ – vector

Quantity	Vector	Scalar	Nonsense
$\nabla \times (\nabla f)$	☐	☐	☐
$\nabla \cdot (\nabla \times \mathbf{F})$	☐	☐	☐
$\nabla \cdot (\nabla \times f)$	☐	☐	☐
$\nabla \times (\nabla \times \mathbf{F})$	☐	☐	☐
$\nabla (\nabla \cdot \mathbf{F})$	☐	☐	☐
$\nabla \times (\nabla \times f)$	☐	☐	☐

$$\int_\Omega \nabla\phi \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\phi \, ds \quad \text{(Gradient theorem)}$$

$$\int_\Omega \nabla\cdot\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{A} \, ds \quad \text{(Divergence theorem)}$$

$$\int_\Omega \nabla\times\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\times\mathbf{A} \, ds \quad \text{(Curl theorem)}$$



$$\begin{aligned}\hat{\mathbf{n}} &= n_x \, \hat{\mathbf{i}} + n_y \, \hat{\mathbf{j}} \\ &= n_x \, \hat{\mathbf{e}}_x + n_y \, \hat{\mathbf{e}}_y \\ &= n_1 \, \hat{\mathbf{e}}_1 + n_2 \, \hat{\mathbf{e}}_2\end{aligned}$$

Establish the following identities using the integral theorems:

1. $$\text{volume} = \tfrac{1}{6}\oint_{\Gamma} \nabla(r^2)\cdot\hat{\mathbf{n}}\, d\Gamma = \tfrac{1}{3}\oint_{\Gamma} \mathbf{r}\cdot\hat{\mathbf{n}}\, d\Gamma$$

2. $$\int_{\Omega} \nabla^2\phi\, d\Omega = \oint_{\Gamma} \frac{\partial\phi}{\partial n}\, d\Gamma$$

3. $$\int_{\Omega} \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right) d\Omega = \oint_{\Gamma} \phi\frac{\partial\psi}{\partial n}\, d\Gamma$$

4. $$\int_{\Omega} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\Omega = \oint_{\Gamma} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\Gamma$$

5. $$\int_{\Omega} \left(\phi\nabla^4\psi - \nabla^2\phi\nabla^2\psi\right) d\Omega = \oint_{\Gamma} \left[\phi\frac{\partial}{\partial n}(\nabla^2\psi) - \nabla^2\psi\frac{\partial\phi}{\partial n}\right] d\Gamma$$