MEEN 673

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NONLINEAR FINITE ELEMENT ANALYSIS

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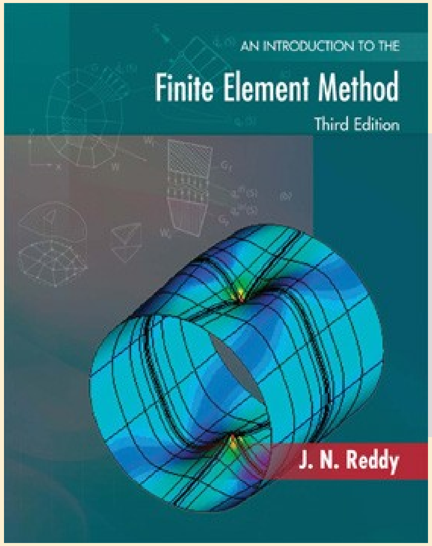

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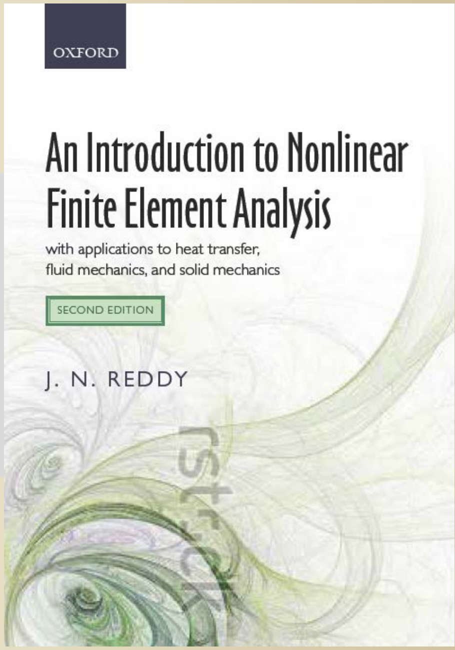


INTRODUCTORY REMARKS

- WHAT WE DO AS ENGINEERS?
- ROLE OF NUMERICAL SIMULATIONS
- CLASSICAL VARIATIONAL METHODS AND THEIR MAJOR DRAWBACK
- THE MAIN FEATURES OF FEM
- TERMINOLOGY USED

GENERAL INTRODUCTION
(MOTIVATION FOR THE COURSE)

Engineering is the discipline, art, and profession of acquiring and applying technical, scientific, and mathematical knowledge to design and implement materials, structures, machines, devices, systems, and processes that *safely realize a desired objective*.

GENERAL INTRODUCTION
(WHAT IS ENGINEERING?)

Engineering **is a problem-solving discipline, and solution requires an understanding of the phenomena that occurs in the system.**

The study of natural phenomena involves

- **developing mathematical models,**
- **conducting physical experiments,**
- **carrying out numerical simulations, and**
- **designing and building systems to achieve a *functionality* in *most economical way*.**

GENERAL INTRODUCTION

ROLE OF *ANALYSIS* IN DESIGN AND MANUFACTURING

Analysis **is an aid to design and manufacturing, and not an end in itself.**

Analysis steps:

- **identifying the problem and nature of the response to be determined,**
- **selecting the mathematical model,**
- **selecting a solution method, and**
- **evaluating the results in light of the design parameters.**

GENERAL INTRODUCTION

BENEFITS OF NUMERICAL SIMULATIONS

- ***Mathematical Model Development and Computer Simulations*** **continues to be a major component of engineering analysis, design and manufacturing (CAE and CAM).**

- **Computer (numerical) simulations reduce/replace prototype testing and hence reduce time and product costs**

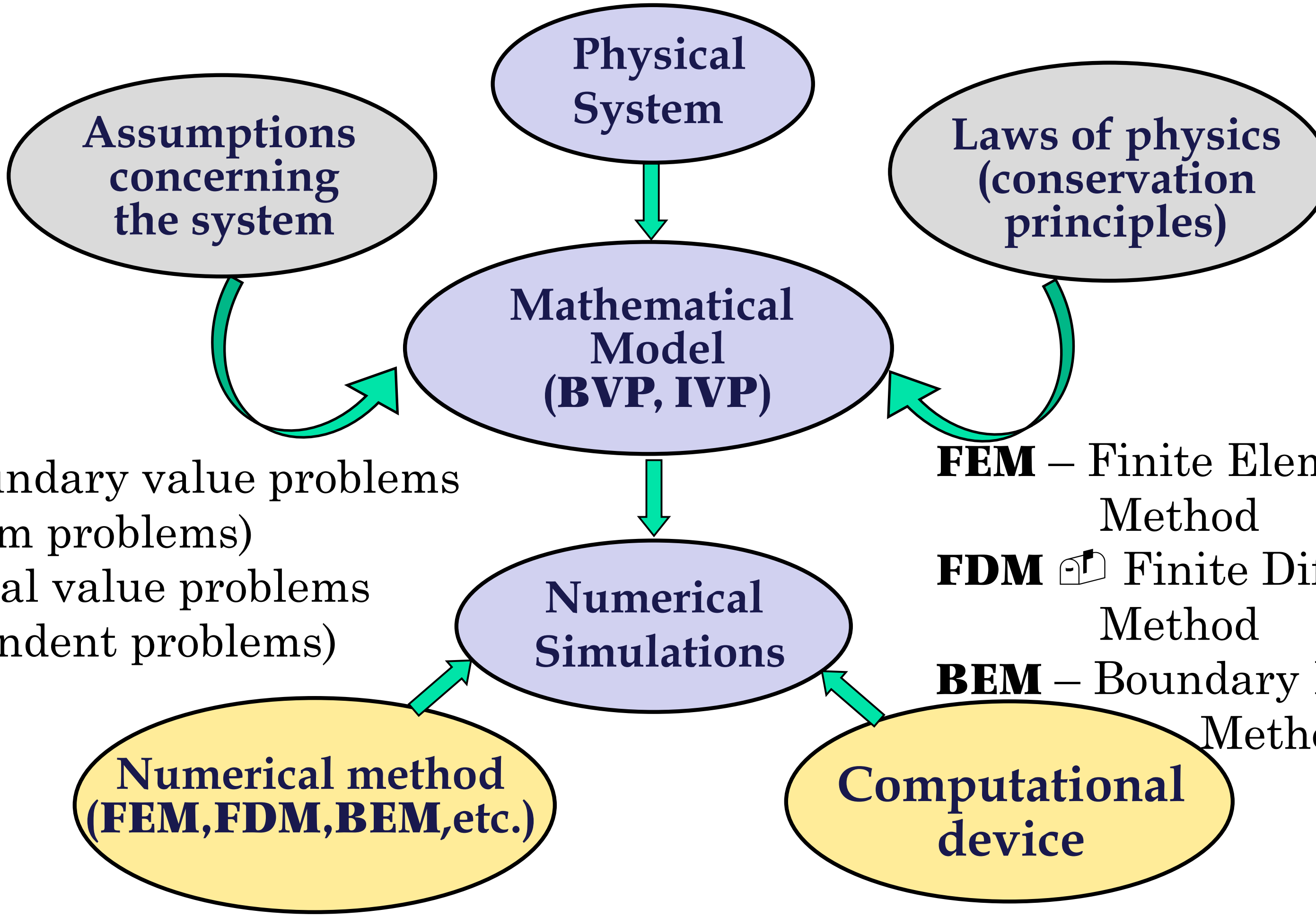
- **Numerical simulations facilitate investigations into the use of alternative materials and configurations.**

GENERAL INTRODUCTION

WHY SHOULD YOU TAKE THE COURSE?

- A good understanding of the phenomena modeled and the computer method used to simulate the process is essential for the analyst to aid the development and manufacturing of complex systems.
- Engineer-scientists with good background in engineering and science subjects as well as in computer modeling techniques will continue to have excellent opportunities to contribute to the science and technology for the benefit of the mankind.

NUMERICAL SIMULATION OF A TYPICAL PHYSICAL PROCESS



BVP – Boundary value problems (equilibrium problems)
IVP – Initial value problems (time-dependent problems)

FEM – Finite Element Method
FDM – Finite Difference Method
BEM – Boundary Element Method

BASIC FEATURES OF THE FINITE ELEMENT METHOD (FEM)

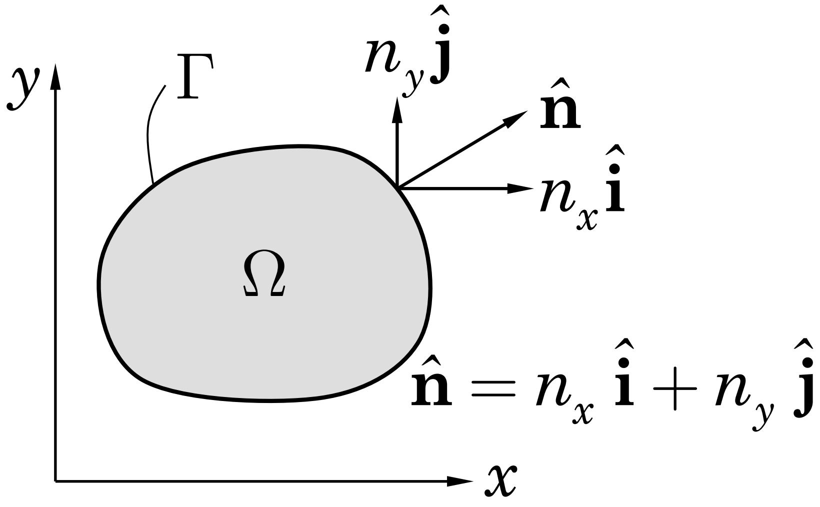
- **Divide whole into parts (*finite element mesh*)**
- **Set up the `problem' over a typical part**

  (derive a set of relationships between primary and secondary variables)
- **Assemble the parts to obtain the solution to the whole**

A BOUNDARY VALUE PROBLEM (BVP)

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0 \quad \text{in } \Omega$$

$$u=\hat{u} \quad \text{or} \quad \left(a_{11}\frac{\partial u}{\partial x}\right)n_x+\left(a_{22}\frac{\partial u}{\partial y}\right)n_y=\hat{q}_n \quad \text{on } \Gamma$$



Exact and Approximate Solutions

An ***exact solution*** satisfies (a) the differential equation at every point of the domain and (b) boundary conditions on the boundary. An ***approximate solution*** satisfies the differential equation as well as the boundary conditions in some "acceptable sense" (to be made clearer shortly).

We seek the approximate solution as a **linear combination** of unknown parameters c_i and known functions $\phi_i(x,y)$ that Satisfy the boundary conditions:

$$u(x,y) \approx u_N(x,y) = \sum_{i=1}^{N} c_i \phi_i(x,y)$$

We determine $u_N(x,y)$ such that the above expression satisfies the differential equation in a **weighted-residual sense.**

Determining Approximate Solutions

Suppose that ϕ_i is selected to satisfy the boundary conditions exactly. Then substitution of $u_N(x,y)$ into the differential equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u_N}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u_N}{\partial y}\right)-f\equiv R(x,y,c_i)\neq 0 \quad \text{in } \Omega$$

Then c_i are determined such that the **residual** (or error in the differential equation), $R(x,y,c_i)$, is zero in the weighted-residual sense:

$$0=\int_{\Omega} w_i R\,dxdy,\ \ i=1,2,\cdots,N$$

where w_i are linearly independent set of weight functions

WEIGHTED-INTEGRAL METHODS
for the Numerical Solution of Differential Eqs.

$$0 = \int_{\Omega} w_i R(x, y, c_1, c_2, \ldots, c_N)\, dxdy,\ \ i = 1, 2, \cdots, N$$

Collocation method:

$$0 = R(x_i, y_i) \ \Rightarrow \ w_i(x, y) = \delta(x - x_i, y - y_i) \ - \text{Dirac delta}$$

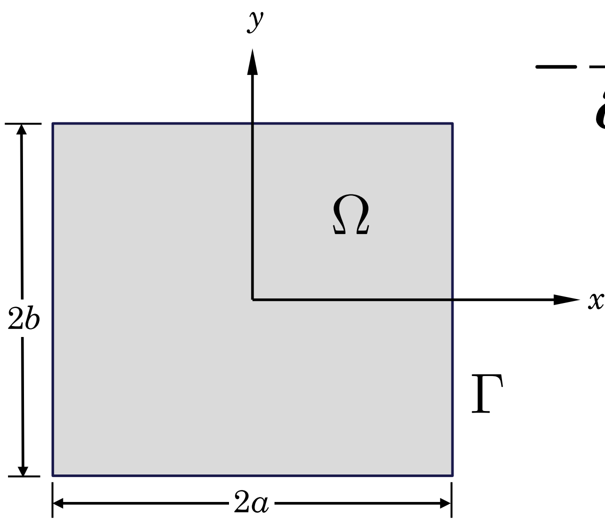
Least-squares method:

$$0 = \int_{\Omega} \frac{\partial R}{\partial c_i} R\, dxdy \Rightarrow w_i(x, y) = \frac{\partial R}{\partial c_i}$$

Galerkin Method:

$$0 = \int_{\Omega} \phi_i R\, dxdy \Rightarrow w_i(x, y) = \phi_i(x, y)$$

MAJOR DRAWBACK OF CONVENTIONAL VARIATIONAL METHODS

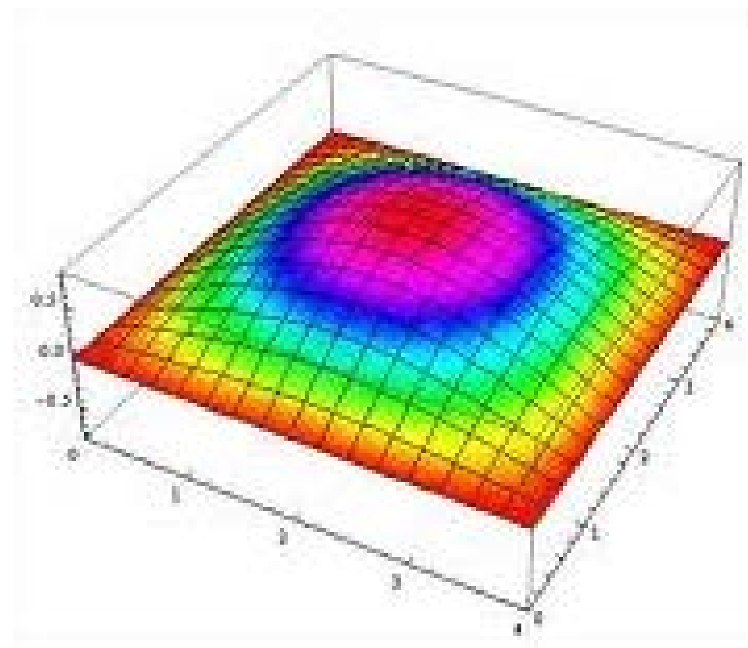


$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0 \quad \text{in } \Omega$$

$$u=0 \quad \text{on } \Gamma$$

Ω
(interior)

Γ
(the four line segments)



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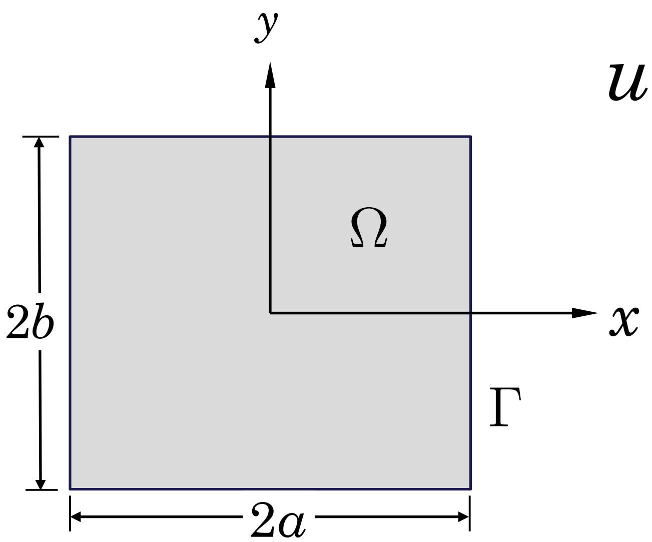
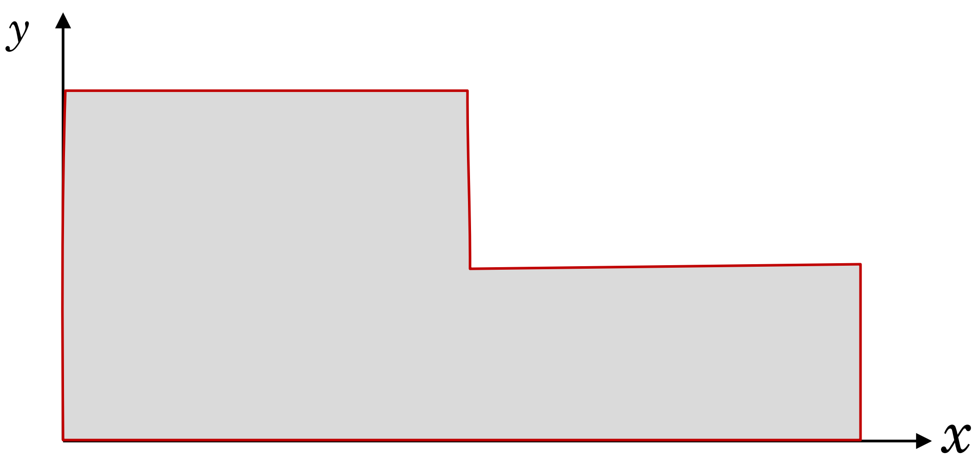


$$u(x,y) \approx u_N(x,y) = \sum_{i=1}^{N} c_i \phi_i(x,y)$$

$$\phi_1(x,y) = (a^2 - x^2)(b^2 - y^2)$$

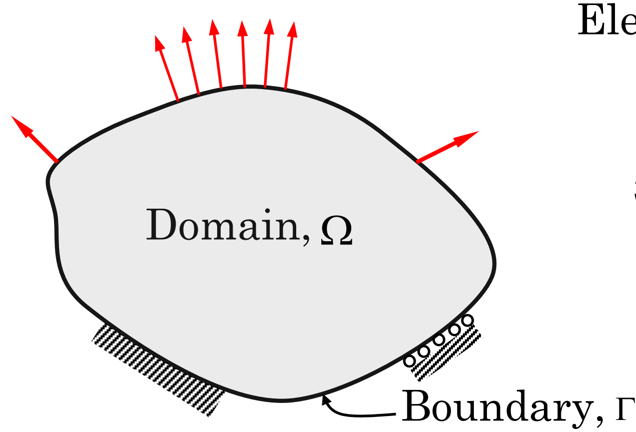
or

$$\phi_1(x,y) = \cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b}$$

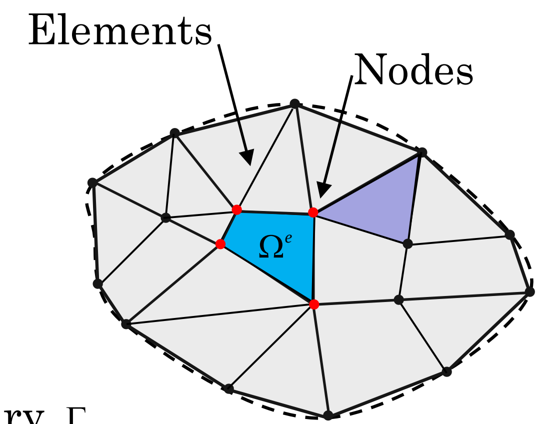


$$\phi_1(x,y) = ?$$

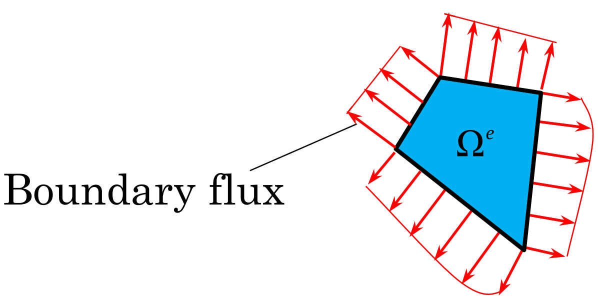
Finite Element Discretization



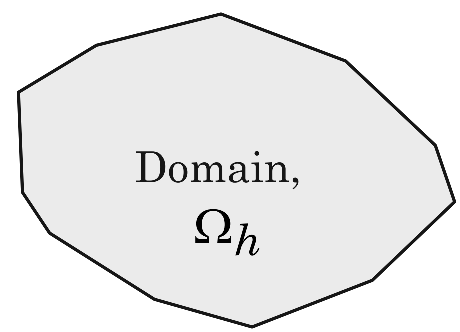
(*a*) Given domain



(*b*) Finite element mesh

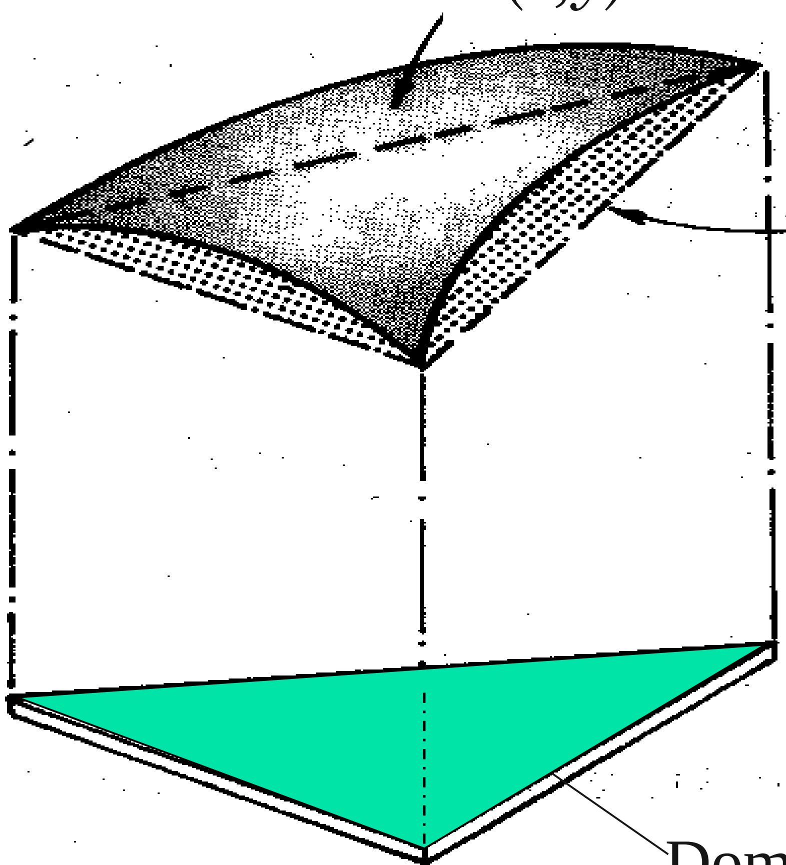


(*c*) Typical element with boundary fluxes



(*d*) Discretized domain

Approximation of a curved surface with a plane



FEM Terminology

- ***Element*** A geometric sub-domain of the region being simulated, with the property that it allows a unique (1) representation of its geometry and (2) derivation of the approximation (interpolation) functions.

- ***Node*** A geometric location in the element which plays a role in the derivation of the interpolation functions and it is the point at which solution is sought.

- ***Mesh*** A collection of elements (or nodes) that replaces the actual domain.

FEM Terminology (continued)

- ***Finite Element Model*** A set of algebraic equations relating the nodal values of the primary variables (e.g., displacements) to the nodal values of the secondary variables (e.g., forces) in an element.
- ***Finite element model*** is NOT the same as the *finite element method.* There is only one finite element method but there can be more than one finite element model of a problem (or mathematical model).
- ***Numerical Simulation*** Evaluation of the mathematical model (i.e., solution of the governing equations) using a numerical method and computer.

Major Steps of Finite Element Model Development

- Begin with the *governing equations* of the problem
- Develop its ***weak form*** over a typical finite element
- *Approximate* the solution over each finite element
- Obtain algebraic relations among the *quantities of interest* over each finite element (i.e., finite element model)