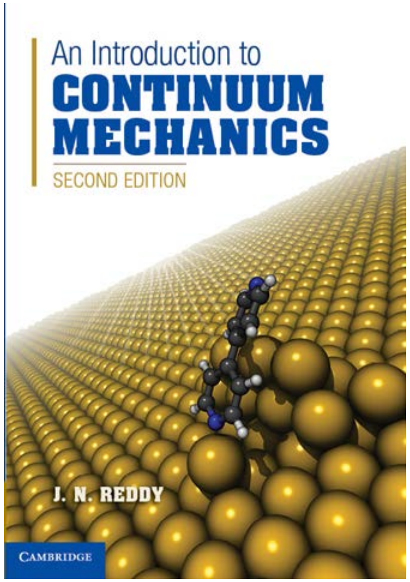
APPLICATIONS: SOLID MECHANICS

Only some applications from solid mechanics are included here. Additional examples and exercises can be found in the author's text book , *An Introduction to Continuum Mechanics*, 2nd ed., Cambridge University Press, New York, 2013.

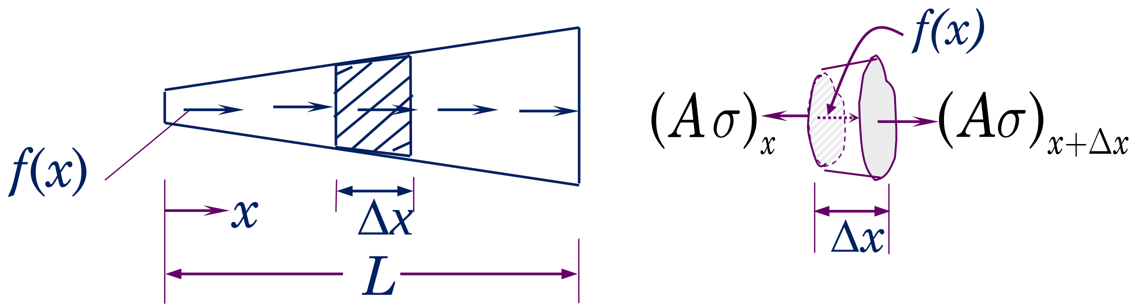


More examples may be added at a later time.

CONTENTS

- Bars
- Beams
- Plane elasticity

EXTENSIONAL DEFORMATIONS: BARS



$$\int_V \left(\frac{d\sigma_{xx}}{dx} + f_x \right) dV = 0 \Rightarrow \int_0^L \int_A \left(\frac{d\sigma_{xx}}{dx} + f_x \right) dA dx = 0$$

$$\Rightarrow \frac{dN}{dx} + f = 0, \quad 0 < x < L; \quad N = \int_A \sigma_{xx} dA = EA\frac{du}{dx}, \; f = \int_A f_x dA$$

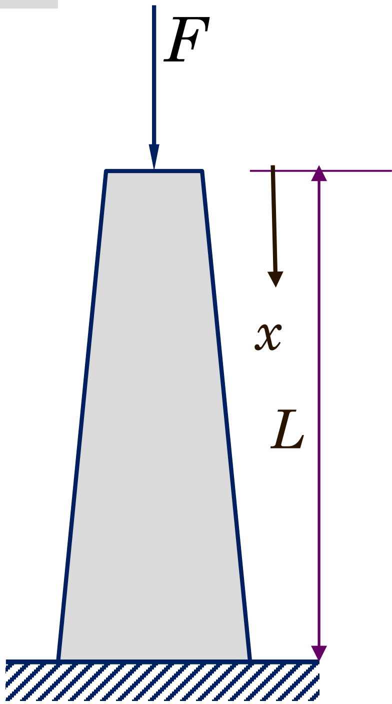
$$\frac{d}{dx}\left(EA\frac{du}{dx} \right) + f = 0, \quad 0 < x < L$$

$$EA\frac{du}{dx} = -\left(\int f(x)\, dx + C_1 \right)$$

$$u(x) = -\left\{ \int \left[\frac{1}{EA}\left(\int f(x)dx + C_1 \right) \right] dx + C_2 \right\}$$

BAR ANALYTICAL SOLUTIONS

Example 1: Solve the bar equation



$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x)$$

with the following data:

$$E = 28\times10^{9}\,(\text{N/m}),\ A(x) = \frac{1}{4}(1+x)\ (\text{m}^2),\ F = 5\times10^{3}\ (\text{N})$$

$$f(x) = 6.25(1+x)10^{3}\ (\text{N/m})$$

Solution: We have

$$\int f(x)\,dx = 3.125(1+x)^2 10^3$$

$$\int\left[\frac{1}{EA}\left(\int f(x)\,dx + C_1\right)\right]dx = \frac{12.5\times10^3}{E}\int\frac{(1+x)^2}{(1+x)}\,dx + \frac{4C_1}{E}\int\frac{1}{1+x}\,dx$$

$$= \frac{6.25\times10^3}{E}(1+x)^2 + \frac{4C_1}{E}\ln(1+x)$$

BAR ANALYTICAL SOLUTIONS (cont.)

$$EA\frac{du}{dx} = -\left[3.125(1+x)^2 10^3 + C_1\right]$$

$$u(x) = -\frac{1}{E}\left\{\left[6.25\times 10^3(1+x)^2 + 4C_1 \ln(1+x)\right] + C_2\right\}$$

Use of the boundary conditions yields ($\left(EA\dfrac{du}{dx}\right)_{x=0} = F,\; u(L)=0$)

$$C_1 = 1.875\times 10^3, \quad C_2 = -56.25\times 10^3 - 4C_1 \ln 3$$

The solution is

$$u(x) = \frac{10^3}{E}\left\{56.25 - 6.25(1+x)^2 + 7.5\left[\ln(1+x) - \ln 3\right]\right\} \text{ (m)},$$

$$\varepsilon(x) = \frac{du}{dx} = \frac{10^3}{E}\left[-12.5(1+x) + \frac{7.5}{1+x}\right] \text{ (m/m)},$$

$$\sigma(x) = E\varepsilon = 10^3\left[-12.5(1+x) + \frac{7.5}{1+x}\right] \text{ (N/m}^2\text{)}.$$

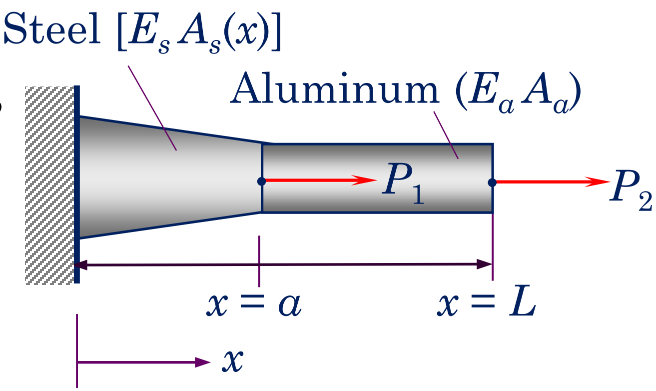
BAR ANALYTICAL SOLUTIONS (cont.)

Example 2: Determine the displacements in the composite bar using the following data:

$$E_1 = 200 \text{ GPa}, \; A_1 = \left(1.5 - \frac{x}{1.92}\right)^2 \times 10^{-4} \text{ m}^2,$$

$$E_2 = 73 \text{ GPa}, A_2 = 10^{-4} \text{ m}^2, a = 0.96 \text{ m},$$

$$L = 2.16 \text{ m}, \; P_1 = 2{,}000 \text{ N}, P_2 = 1{,}000 \text{ N}$$



Solution: The governing equations in each part are

$$\frac{d}{dx}\left(E_1 A_1 \frac{du_1}{dx}\right) = 0, \;\; 0 < x < a, \qquad \frac{d}{dx}\left(E_2 A_2 \frac{du_2}{dx}\right) = 0, \;\; a < x < L$$

The solutions of these equations are

$$u_1(x) = \frac{C_1}{(2.88 - x)} + C_2, \;\; 0 < x < a, \quad u_2(x) = C_3 x + C_4, \;\; a < x < L$$

BAR ANALYTICAL SOLUTIONS (cont.)

The boundary conditions are

$$u_1(0) = 0, \quad \left(E_2 A_2 \frac{du_2}{dx}\right)_{x=2.16} = P_2$$

The continuity of the displacements and balance of forces require

$$u_1(a) = u_2(a), \quad \left[\left(E_1 A_1 \frac{du_1}{dx}\right)_{x=0.96^+} - \left(E_2 A_2 \frac{du_2}{dx}\right)_{x=0.96^-}\right] = P_1$$

$$u_1(0) = 0 \Rightarrow C_1 = -2.88 C_2 \text{ and } \left(E_2 A_2 \frac{du_2}{dx}\right)_{x=2.16} = P_2 \Rightarrow C_3 = \frac{P_2}{E_2 A_2} = \frac{10^{-2}}{73}$$

The continuity and balance conditions give

$$\frac{C_1}{1.92} + C_2 = 0.96 C_3 + C_4 \text{ or } \left(1 - \frac{2.88}{1.92}\right) C_2 = 0.96 \frac{P_2}{E_2 A_2} + C_4$$

$$E_1 C_1 \frac{1}{1.92} - E_2 C_3 = 10^4 P_1 \quad \text{or} \quad -2.88 E_1 C_2 \frac{10^{-4}}{(1.92)^2} = P_1 + P_2$$

BAR ANALYTICAL SOLUTIONS (cont.)

The constants of integration are

$$C_1 = 5.5296 \times 10^{-4},\ C_2 = -1.92 \times 10^{-4},\ C_3 = 1.36986 \times 10^{-4},\ C_4 = -0.94685 \times 10^{-4}$$

The complete displacement, strains, and stresses are

$$u(x) = \begin{cases} \left(\dfrac{1.92x}{2.88 - x} \right) 10^{-4}\ \text{m}, & 0 \leq x \leq 0.96, \\ (1.36986x - 0.94685) 10^{-4}\ \text{m}, & 0.96 \leq x \leq 2.16 \end{cases}$$

$$\varepsilon(x) = \frac{du}{dx} = \begin{cases} \dfrac{5.5296}{(2.88 - x)^2} 10^{-4}\ \text{m/ m}, & 0 \leq x \leq 0.96, \\ 1.36986 \times 10^{-4}\ \text{m/ m}, & 0.96 \leq x \leq 2.16, \end{cases}$$

$$\sigma(x) = E\varepsilon = \begin{cases} \dfrac{11.0592}{(2.88 - x)^2} 10^{7}\ \text{N/m}^2, & 0 \leq x \leq 0.96 \\ 10^{7}\ \text{N/m}^2, & 0.96 \leq x \leq 2.16 \end{cases}$$

APPLICATIONS: SOLID MECHANICS

BEAMS:

Governing equations

$$\frac{dV}{dx} + q = 0, \quad \frac{dM}{dx} - V = 0 \Rightarrow \quad \frac{d^2M}{dx^2} + q = 0, \quad 0 < x < L$$

$$M = \int_A z\sigma_{xx}\, dA = \int_A z(E\varepsilon_{xx})dA = -E\frac{d^2w}{dx^2}\left(\int_A z^2 dA\right) = -EI\frac{d^2w}{dx^2}$$

$$V = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x), \; 0 < x < L$$

BEAM ANALYTICAL SOLUTIONS

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = \int q(x)\,dx + C_1,$$

$$\frac{d^2w}{dx^2} = \frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx + C_1x + C_2\right]$$

$$\frac{dw}{dx} = \int\frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx\right]dx + C_1\int\frac{x}{EI}dx$$

$$+ C_2\int\frac{1}{EI}dx + C_3$$

$$w(x) = \int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)dx\right)dx\right]dx\right\}dx + C_1\int\left(\int\frac{x}{EI}dx\right)dx$$

$$+ C_2\int\left(\int\frac{1}{EI}dx\right)dx + C_3x + C_4$$

BEAM SOLUTIONS (*EI* = const.)

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = \int q(x)\,dx + C_1,$$

$$\frac{d^2w}{dx^2} = \frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx + C_1x + C_2\right]$$

$$\frac{dw}{dx} = \frac{1}{EI}\int\left[\int\left(\int q(x)\,dx\right)dx\right]dx + C_1\frac{x^2}{2EI} + C_2\frac{x}{EI} + C_3$$

$$w(x) = \frac{1}{EI}\int\left\{\int\left[\int\left(\int q(x)dx\right)dx\right]dx\right\}dx + C_1\frac{x^3}{6EI}$$

$$+ C_2\frac{x^2}{2EI} + C_3x + C_4$$

BEAM ANALYTICAL SOLUTIONS (cont.)

Simply-supported (at both ends) beams

$$w(0) = w(L) = 0;\ M(0) = M(L) = 0$$

$$-M(x) = EI\frac{d^2w}{dx^2} = \int\left(\int q(x)\,dx\right)dx + C_1x + C_2$$

$$M(0) = 0 \Rightarrow C_2 = 0;\ \ M(L) = 0 \Rightarrow C_1 = -\frac{1}{L}\left[\int\left(\int q(x)\,dx\right)dx\right]_{x=L}$$

$$w(x) = \int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)dx\right)dx\right]dx\right\}dx + C_1\int\left(\int\frac{x}{EI}dx\right)dx + C_3x + C_4$$

$$w(0) = 0 \Rightarrow C_4 = 0$$

$$w(L) = 0 \Rightarrow C_3 = -\frac{1}{L}\left[\int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)dx\right)dx\right]dx\right\}dx + C_1\int\left(\int\frac{x}{EI}dx\right)dx\right]_{x=L}$$

BEAM ANALYTICAL SOLUTIONS (cont.)

(a) Simply-supported beam under uniformly distributed load (downward), $q(x) = -q_0$

$$C_1 = -\frac{1}{L}\left[\int\left(\int q(x)\,dx\right)dx\right]_{x=L} = \frac{q_0 L}{2}$$

$$C_3 = -\frac{1}{L}\left[\int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)dx\right)dx\right]dx\right\}dx + C_1\int\left(\int\frac{x}{EI}dx\right)dx\right]_{x=L}$$

$$= \frac{q_0 L^3}{24EI} + C_1\frac{q_0 L^2}{6EI} = -\frac{q_0 L^3}{24EI}$$

ANALYTICAL SOLUTIONS (cont.)

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = -\frac{q_0}{2L}(2x-L) = -V(x),$$

$$EI\frac{d^2w}{dx^2} = -\frac{q_0 x}{2}(x-L) = -M(x)$$

$$\frac{dw}{dx} = -\frac{q_0x^3}{6EI} + \frac{q_0x^2L}{4EI} - \frac{q_0L^3}{24EI} = -\frac{q_0L^3}{24EI}\left(1-6\frac{x^2}{L^2}+4\frac{x^3}{L^3}\right) = -\theta(x)$$

$$w(x) = -\frac{q_0L^4}{24EI}\left(\frac{x}{L}-2\frac{x^3}{L^2}+\frac{x^4}{L^4}\right)$$

$$V_{\max} = V(0) = -0.5q_0;\quad M_{\max} = M(0.5L) = -\frac{q_0L^2}{8}$$

$$\theta_{\max} = \theta(0) = \frac{q_0L^3}{24EI};\quad w_{\max} = w(0.5L) = -\frac{5q_0L^4}{384EI}$$

ANALYTICAL SOLUTIONS (cont.)

Clamped beams under uniform load (upward)

$$w(0) = w(L) = 0;\ \theta(0) = \theta(L) = 0$$

$$w(x) = \int \left\{ \int \frac{1}{EI} \left[\int \left(\int q(x)dx \right) dx \right] dx \right\} dx + C_1 \int \left(\int \frac{x}{EI} dx \right) dx$$

$$+ C_2 \int \left(\int \frac{1}{EI} dx \right) dx + C_3 x + C_4$$

$$w(x) = \frac{q_0 x^4}{24EI} + C_1 \frac{x^3}{6EI} + C_2 \frac{x^2}{2EI} + C_3 x + C_4$$

$$w(0) = 0 \Rightarrow C_4 = 0;\ w(L) = 0 \Rightarrow C_3 + \left(\frac{q_0 L^3}{24EI} + C_1 \frac{L^2}{6EI} + C_2 \frac{L}{2EI} \right) = 0$$

ANALYTICAL SOLUTIONS (cont.)

Clamped beams under uniform load (upward)

$$\frac{dw}{dx} = \frac{q_0 x^3}{6EI} + C_1 \frac{x^2}{2EI} + C_2 \frac{x}{EI} + C_3$$

$$\left.\frac{dw}{dx}\right|_{x=0} = 0 \Rightarrow C_3 = 0; \quad \left.\frac{dw}{dx}\right|_{x=L} = 0 \Rightarrow C_2 = -\frac{q_0 L^2}{6} - C_1 \frac{L}{2}$$

$$\frac{q_0 L^3}{24EI} + C_1 \frac{L^2}{6EI} - \frac{L}{2EI}\left(\frac{q_0 L^2}{6} + C_1 \frac{L}{2}\right) = 0 \Rightarrow C_1 = -\frac{q_0 L}{2}, \; C_2 = \frac{q_0 L^2}{12}$$

$$V(x) = \frac{q_0 L}{2}\left(1 - \frac{2x}{L}\right); \quad M(x) = \frac{q_0 x}{2}(L - x) - \frac{q_0 L^2}{12}$$

$$\theta(x) = -\frac{dw}{dx} = -\frac{q_0 x^3}{6EI} + \frac{q_0 L x^2}{4EI} + \frac{q_0 x L^2}{12EI}$$

$$w(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} + \frac{q_0 L x^2}{24EI} = \frac{q_0 L^4}{24EI}\frac{x^2}{L^2}\left(1 - \frac{x}{L}\right)^2$$

BEAM SOLUTIONS (cont.)

$$V_{\max} = V(0) = \frac{q_0 L}{2}\left(1 - \frac{2x}{L}\right) \; M$$

$$M_{\max} = M(0) = -\frac{q_0 L^2}{12}$$

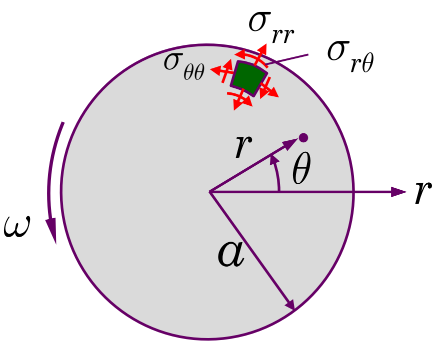
$$w_{\max} = w(0.5L) = \frac{q_0 L^4}{384EI}$$

PLANE ELASTICITY
(Analytical solutions)

Problem: Consider a thin, uniform, solid circular disk of radius *a*, spinning at a constant angular velocity of ω, as shown in the figure below. Use the semi-inverse method to determine the displacements, strains, and stresses in the disk.

Solution: First, we set up the polar cylindrical coordinate system, (r,θ) with the origin at the center of the disk, *r* being the radial coordinate. We begin with the observation

$$u_r(0,\theta) = \text{finite},\ u_\theta = 0 \Rightarrow\ u_r(r) = U(r)$$



$$\varepsilon_{rr} = \frac{dU}{dr},\ \varepsilon_{\theta\theta} = \frac{U}{r},\ \varepsilon_{r\theta} = 0$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) = \frac{E}{1-\nu^2}\left(\frac{dU}{dr} + \nu\frac{U}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}) = \frac{E}{1-\nu^2}\left(\nu\frac{dU}{dr} + \frac{U}{r}\right)$$

PLANE ELASTICITY

(Analytical solutions)

Equation of equilibrium

$$\frac{d\sigma_{rr}}{dr}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})+\rho\omega^2 r=0$$

$$\frac{E}{1-\nu^2}\left[\frac{d^2U}{dr^2}+\nu\frac{d}{dr}\left(\frac{U}{r}\right)+\frac{1-\nu}{r}\left(\frac{dU}{dr}-\frac{U}{r}\right)\right]+\rho\omega^2 r=0$$

Note that

$$\frac{1}{r}\left(\frac{dU}{dr}-\frac{U}{r}\right)=\frac{d}{dr}\left(\frac{U}{r}\right),\quad \frac{d}{dr}\left(\frac{dU}{dr}-\frac{U}{r}\right)=\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rU)\right]$$

Hence, we have

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rU)\right]=-\alpha r,\quad \alpha=\frac{1-\nu^2}{E}\rho\,\omega^2$$

whose solution is

$$U(r)=\frac{c_1}{2}r+\frac{c_2}{r}-\frac{\alpha}{8}r^3$$

PLANE ELASTICITY

(Analytical solutions)

The constants of integration are determined using the fact that U is finite at $r=0$ and the stress boundary condition $\sigma_{rr}(a,\theta)=0$. We obtain $c_2=0$. The stress is

$$\sigma_{rr}(r)=\frac{E}{1-\nu^2}\left(\frac{dU}{dr}+\nu\frac{U}{r}\right)=\frac{E\alpha}{(1-\nu^2)}\left[-\frac{3+\nu}{8}\alpha r^2+\frac{1+\nu}{2}c_1\right]$$

$$\sigma_{rr}(a)=0\Rightarrow-\frac{3+\nu}{8}\alpha a^2+\frac{1+\nu}{2}c_1=0\quad\text{or}\quad c_1=\frac{1}{4}\left(\frac{3+\nu}{1+\nu}\right)\alpha a^2$$

Hence, the displacement and stress fields are

$$u(r)=\frac{1}{4}\left(\frac{3+\nu}{1+\nu}\right)\alpha a^2 r-\frac{\alpha}{8}r^3=\frac{(1-\nu)}{8E}\left[2(3+\nu)a^2-(1+\nu)r^2\right]\rho_0\omega^2 r$$

$$\sigma_{rr}(r)=\frac{(3+\nu)}{8}\left(a^2-r^2\right)\rho_0\omega^2,\ \sigma_{\theta\theta}(r)=\frac{1}{8}\left[(3+\nu)a^2-(1+3\nu)r^2\right]\rho_0\omega^2$$

$$u_{\max}=u_r(a)=\frac{(1-\nu)(5+\nu)}{8E}\rho_0\omega^2a^3,\ \sigma_{\max}=\sigma_{rr}(0)=\sigma_{\theta\theta}(0)=\frac{(3+\nu)}{8}\rho_0\omega^2a^2$$