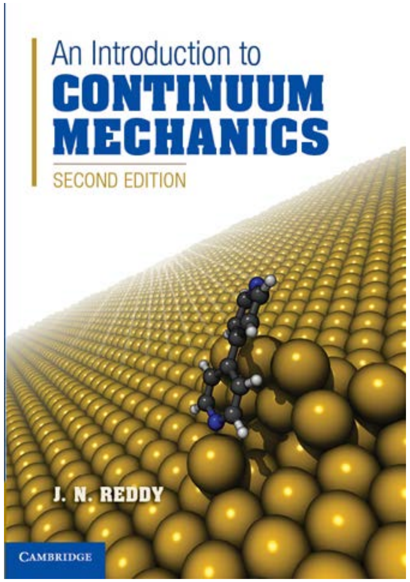
# APPLICATIONS: FLUID MECHANICS

Some simple applications from fluid mechanics are included here. Additional examples and exercises can be found in the author's text book , *An Introduction to Continuum Mechanics*, 2nd ed., Cambridge University Press, New York, 2013.




**More examples may be added at a later time.**

## CONTENTS

- **Parallel flows**
- **Couette and Poiseuille flows**
- **Hagen-Poiseulle flow**

# FLOWS OF Viscous Incompressible Fluids

**Conservation of mass**

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

***x*-momentum**

$$\mu\left[2\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\right]-\frac{\partial P}{\partial x}+\rho f_x$$

$$=\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)$$

***y*-momentum**

$$\mu\left[2\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\right]-\frac{\partial P}{\partial y}+\rho f_y$$

$$=\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)$$

***z*-momentum**

$$\mu\left[2\frac{\partial^2 v_z}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\right]-\frac{\partial P}{\partial z}+\rho f_z$$

$$=\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)$$

# PARALLEL FLOW: 1-D

**Suppose that** $v_y = v_z = 0$ **and that the body forces are negligible. Then we have**

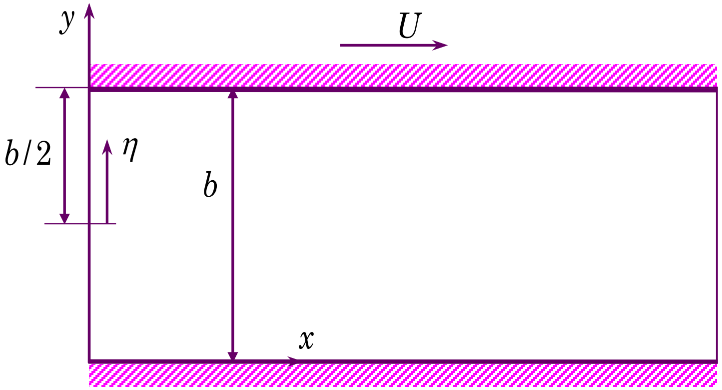
$$\frac{\partial v_x}{\partial x} = 0 \quad \rightarrow \quad v_x = v_x(y, z, t)$$

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = \rho \frac{\partial v_x}{\partial t}, \quad \frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 0.$$

## Steady Flow of Viscous Incompressible Fluid between Parallel Plates

$$\mu \frac{d^2 v_x}{dy^2} = \frac{dP}{dx}, \quad 0 < y < b$$

$$v_x(0) = 0, \quad v_x(b) = U$$



# ANALYTICAL SOLUTIONS: 1D (cont.)

**Couette Flow ( $U \neq 0$)**

$$v_x(y) = \frac{y}{b}U - \frac{b^2}{2\mu}\frac{dP}{dx}\frac{y}{b}\left(1 - \frac{y}{b}\right), \ 0 < y < b,$$

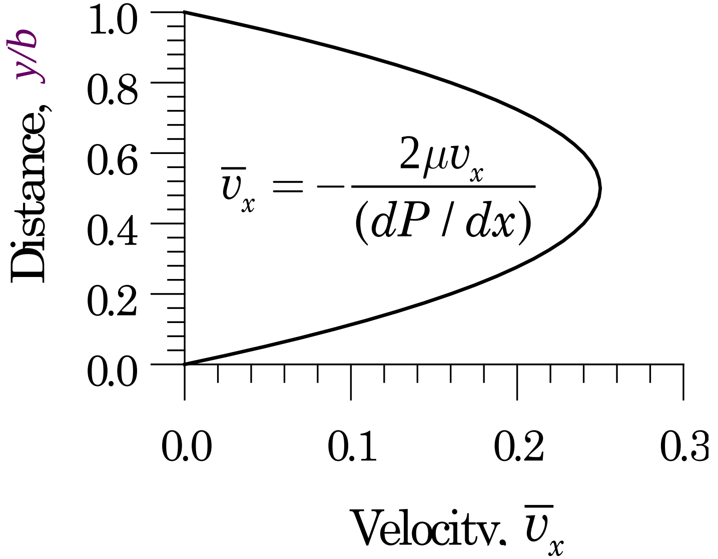
$$\overline{v}_x(\overline{y}) = \overline{y} + f\overline{y}\left(1 - \overline{y}\right), \ \overline{v}_x = \frac{v_x}{U}, \ \overline{y} = \frac{y}{b}, \ f = -\frac{b^2}{2\mu U}\frac{dP}{dx}$$

**Poiseuille Flow ($U = 0$ )**

$$v_x(y) = -\frac{b^2}{2\mu}\frac{dP}{dx}\frac{y}{b}\left(1 - \frac{y}{b}\right), \ 0 < y < b,$$

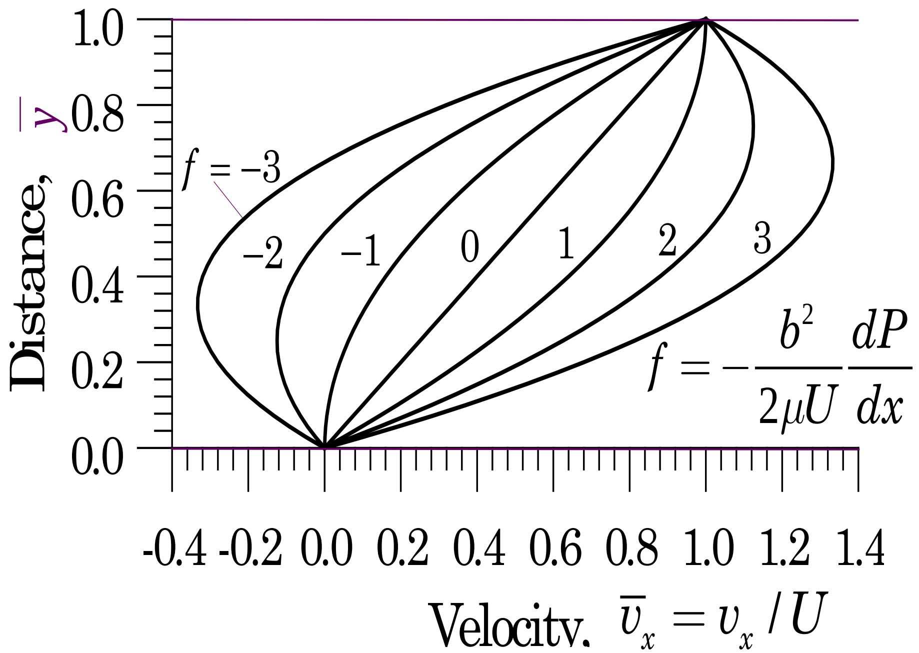
$$v_x(\eta) = -\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{b^2}{4} - \eta^2\right), \ \eta = y - \frac{b}{2}, \ -\frac{b}{2} < \eta < \frac{b}{2}$$

# ANALYTICAL SOLUTIONS: 1D (cont.)




**Poiseuille Flow**

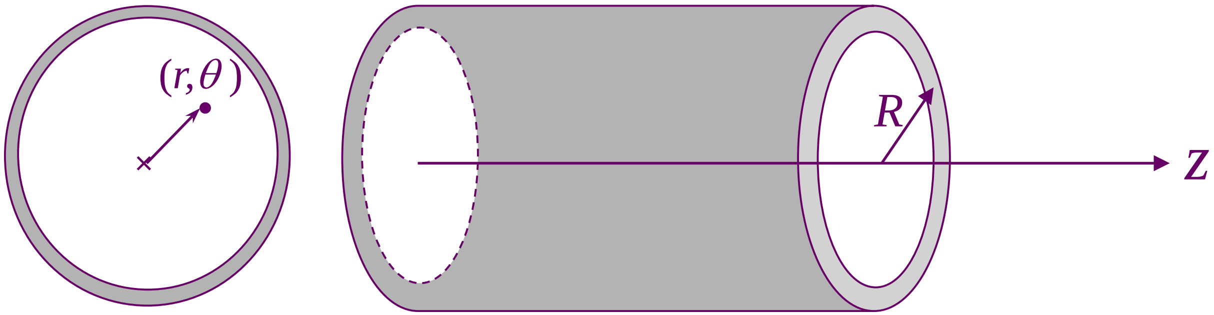
**Couette Flow**



# ANALYTICAL SOLUTIONS: 1D (cont.)

**Problem:**

Use the continuity equation and linear momentum equation for axisymmetric Hagen-Poiseulle flow through a circular pipe to derive the expression for the velocity field.




**Solution:**

We have $v_r = v_\theta = 0$ and $v_z = v_z(r)$

## Hagen-Poiseulle flow through a circular pipe (cont.)

**The governing equations are**

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

$$\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2}-2\frac{\partial v_\theta}{\partial \theta}\right)+\frac{\partial^2 v_r}{\partial z^2}\right]-\frac{\partial P}{\partial r}+\rho f_r$$

$$=\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2}+2\frac{\partial v_r}{\partial \theta}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right]-\frac{\partial P}{\partial \theta}+\rho f_\theta$$

$$=\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]-\frac{\partial P}{\partial z}+\rho f_z$$

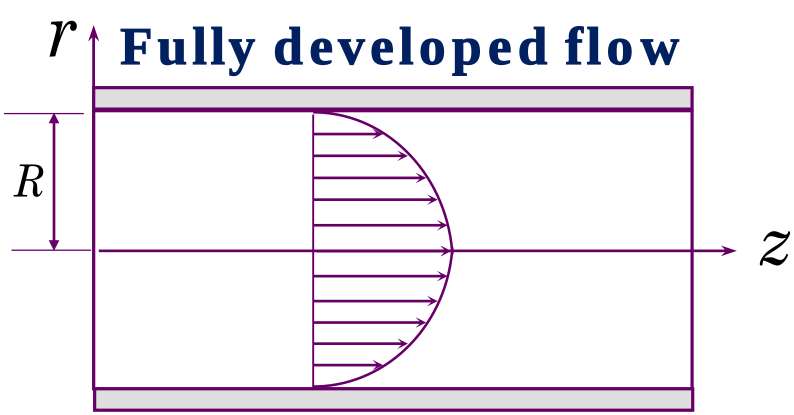
$$=\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

# HAGEN-POISEULLE FLOW (cont.)

**Simplified equations** $(v_r = 0,\ v_\theta = 0)$

$$\frac{\partial P}{\partial \theta} = 0, \quad \frac{\partial P}{\partial r} = 0, \quad \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dP}{dz}$$

$$v_z(r) = \frac{r^2}{4\mu}\frac{dP}{dz} + A\log r + B$$




**Boundary conditions**

$$r\tau_{rz} \equiv r\mu\left(\cancelto{0}{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r}\right) = 0 \text{ at } r = 0; \quad \text{and} \quad v_z = 0 \text{ at } r = R$$

$$A = 0, \quad B = -\frac{R^2}{4\mu}\frac{dP}{dz} \quad \Rightarrow \quad v_z(r) = -\frac{1}{4\mu}\frac{dP}{dz}\left(R^2 - r^2\right)$$

$$(v_z)_{\max} = v_z(0) = -\frac{R^2}{4\mu}\frac{dP}{dz}, \quad \tau_w = -\mu\left(\frac{dv_z}{dr}\right)_{r=R} = \frac{R}{2}\frac{dP}{dz}$$