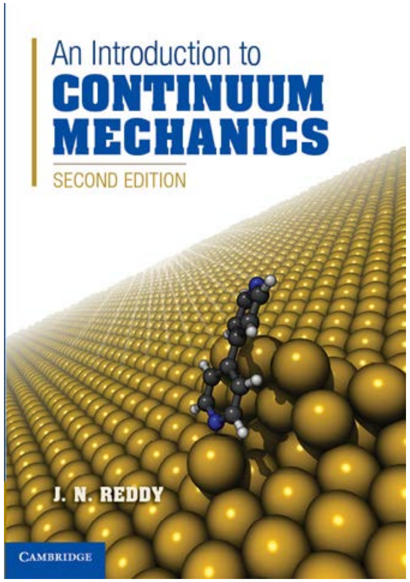
APPLICATIONS: HEAT TRANSFER

Some simple applications from heat transfer in solids are included here. Additional examples and exercises can be found in the author's text books shown on this page. **More examples may be added at a later time.**



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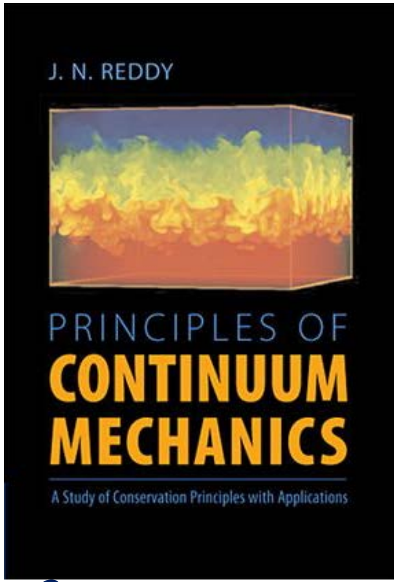


APPLICATIONS: HEAT TRANSFER

The *specific heat* of a substance is a measure of the variation of its stored energy with temperature. There are two specific heats in thermodynamics (J/kg-K):

Specific heat at constant volume: $c_v = \left.\frac{\partial e}{\partial T}\right|_{v=\text{const}}$

Specific heat at constant pressure: $c_p = \left.\frac{\partial H}{\partial T}\right|_{p=\text{const}}$

Here *e* denotes the internal energy per unit mass and *H* is the enthalpy per unit mass. For solids and liquids (not for gases), c_v and c_p are numerically equal. We have

$$\rho \frac{De}{Dt} = \rho\left(\frac{De}{DT}\frac{DT}{Dt}\right) = \rho c_v \frac{DT}{Dt}$$

GOVERNING EQUATIONS (1D, 2D, AND 3D)

The convective term is omitted for heat transfer in solids.

Vector form

$$\rho c_v \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + Q$$

3D

$$\rho c_v \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + Q$$

2D

$$\rho c_v \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + Q$$

1D

$$\frac{\partial}{\partial x}\left(kA \frac{\partial T}{\partial x}\right) + Ph(T - T_\infty) + QA = \rho A c_v \frac{\partial T}{\partial t}$$

B.C.

$$T = \hat{T}, \;\; kA \frac{\partial T}{\partial n} + h(T - T_\infty) = \hat{H}$$

ANALYTICAL SOLUTIONS: 1-D

Example 1: steady-state heat transfer in a cooling fin

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+Ph(T-T_\infty)+QA=0,\ \ 0<x<a$$

$$T(0)=T_0,\qquad \left[kA\frac{dT}{dx}+hA(T-T_\infty)\right]_{x=a}=0$$

Non-dimensional form

$$\theta=\frac{T-T_\infty}{T_0-T_\infty},\ \ \xi=\frac{x}{a},\ \ m^2=\frac{hPa^2}{kA},\ \ N=\frac{ha}{k}$$

$$\frac{d^2\theta}{d\xi^2}-m^2\theta=0,\quad \theta(0)=1,\ \left[\frac{d\theta}{d\xi}+N\theta\right]_{\xi=1}=0$$

ANALYTICAL SOLUTIONS: 1D (cont.)

General solution

$$\theta(\xi) = C_1 \cosh m\xi + C_2 \sinh m\xi, \;\; 0 < \xi < a$$

$$\frac{d\theta}{d\xi} = m\left(C_1 \sinh m\xi + C_2 \cosh m\xi\right)$$

$$\frac{d\theta}{d\xi} + N\theta = m\left(C_1 \sinh m\xi + C_2 \cosh m\xi\right)$$

$$+ N\left(C_1 \cosh m\xi + C_2 \sinh m\xi\right)$$

Determination of the constants of integration

$$\theta = 1 \text{ at } \xi = 0 \text{ gives } C_1 = 1$$

ANALYTICAL SOLUTIONS: 1D (cont.)

The second boundary condition gives

$$C_2 = -\left(\frac{m\sinh m + N\cosh m}{m\cosh m + N\sinh m}\right)$$

The final solution becomes

$$\theta(\xi) = \frac{m\cosh m(1-\xi) + N\sinh m(1-\xi)}{m\cosh m + N\sinh m}$$

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} \Rightarrow T(x) = \left(T_0 - T_\infty\right)\theta(\xi) + T_\infty,\ \xi = \frac{x}{a}$$

ANALYTICAL SOLUTIONS: 1D (cont.)

Example 2: steady-state heat transfer in surface-insulated rod

Governing equation (divided through out by *A*):

$$k\frac{d^2T}{dx^2} + Q = 0, \quad 0 < x < a$$

General solution:

$$T(x) = -\frac{1}{k}\int\left(\int Q(x)dx\right)dx + C_1 x + C_2$$

Boundary conditions:

$$\text{Case 1:} \quad T(0) = T_0, \quad T(a) = T_a$$

$$\text{Case 2:} \quad T(0) = T_0, \quad \left[kA\frac{dT}{dx} + h_a A(T - T_\infty^a)\right]_{x=a} = H_a$$

ANALYTICAL SOLUTIONS: 1D (cont.)

Boundary conditions:

$$\text{Case 3:} \quad \left[-kA\frac{dT}{dx} + h_0 A(T - T_\infty^0)\right]_{x=0} = H_0,$$

$$\left[kA\frac{dT}{dx} + h_a A(T - T_\infty^a)\right]_{x=a} = H_a$$

Solution for Case 3 (Q = constant):

$$T(x) = -\frac{Q}{2k}x^2 + C_1 x + C_2, \quad \frac{dT}{dx} = -\frac{Q}{k}x + C_1,$$

$$\left[-kA\frac{dT}{dx} + h_0 A(T - T_\infty^0)\right]_{x=0} = H_0 \;\Rightarrow -kAC_1 + h_0 A(C_2 - T_\infty^0) = H_0$$

$$\left[kA\frac{dT}{dx} + h_a A(T - T_\infty^a)\right]_{x=a} = H_a \;\Rightarrow -aAQ + kAC_1 + h_a A\left(-\frac{Q}{2k}a^2 + C_1 a + C_2 - T_\infty^a\right) = H_a$$

ANALYTICAL SOLUTIONS: 1D (cont.)

$$-\frac{k}{h_0}C_1 + C_2 = \frac{H_0}{Ah_0} + T_\infty^0, \quad \left(a + \frac{k}{h_a}\right)C_1 + C_2 = \frac{H_a}{Ah_a} + T_\infty^a + \left(\frac{a}{h_a} + \frac{a^2}{2k}\right)Q$$

$$\begin{bmatrix} -\beta_0 & 1 \\ a + \beta_a & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{H_0}{h_0 A} + T_\infty^0 \\ \dfrac{H_a}{h_a A} + T_\infty^a + Q\left(\dfrac{a}{h_a} + \dfrac{a^2}{2k}\right) \end{Bmatrix}, \quad \beta_0 = \frac{k}{h_0}, \quad \beta_a = \frac{k}{h_a}$$

$$C_1 = \frac{1}{(a + \beta_0 + \beta_a)}\left[T_\infty^a - T_\infty^0 + \frac{1}{A}\left(\frac{H_a}{h_a} - \frac{H_0}{h_0}\right) + Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\right]$$

$$C_2 = \frac{1}{(a + \beta_0 + \beta_a)}\left[\beta_0 T_\infty^a + \beta_a T_\infty^0 + \frac{1}{A}\left(\frac{\beta_0 H_a}{h_a} + \frac{\beta_a H_0}{h_0}\right) + aT_\infty^0 + \frac{a}{Ah_0}H_0 + \beta_0 Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\right]$$

ANALYTICAL SOLUTIONS: 1D (cont.)

Solution for Case 3 (Q = constant):

$$T(x) = -\frac{Q}{2k}x^2 + C_1 x + C_2$$

$$C_1 = \frac{1}{(a+\beta_0+\beta_a)}\left[T_\infty^a - T_\infty^0 + \frac{1}{A}\left(\frac{H_a}{h_a} - \frac{H_0}{h_0}\right) + Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\right]$$

$$C_2 = \frac{1}{(a+\beta_0+\beta_a)}\left[\begin{array}{l}\beta_0 T_\infty^a + \beta_a T_\infty^0 + \frac{1}{A}\left(\frac{\beta_0 H_a}{h_a} + \frac{\beta_a H_0}{h_0}\right) \\ +aT_\infty^0 + \frac{a}{Ah_0}H_0 + \beta_0 Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\end{array}\right]$$

Solution for Case 1:

Let h_0 and $h_a \to \infty$ (or β_0 and $\beta_a \to 0$) and set $T_\infty^0 = T_0$

and $T_\infty^a = T_a$ to obtain $C_1 = \dfrac{T_a - T_0}{a} + \dfrac{Qa}{2k}$ and $C_2 = T_0$

ANALYTICAL SOLUTIONS: 1D (cont.)

Solution for Case 1:

$$T(x) = -\frac{Q}{2k}x^2 + \left(\frac{T_a - T_0}{a} + \frac{Qa}{2k}\right)x + T_0$$

Solution for Case 2:

Let $h_0 \to \infty$ (or $\beta_0 \to 0$) and set $T_\infty^0 = T_0$ and

$$C_1 = \frac{1}{(a+\beta_a)}\left[T_\infty^a - T_0 + \frac{H_a}{Ah_a} + Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\right], \quad C_2 = T_0$$

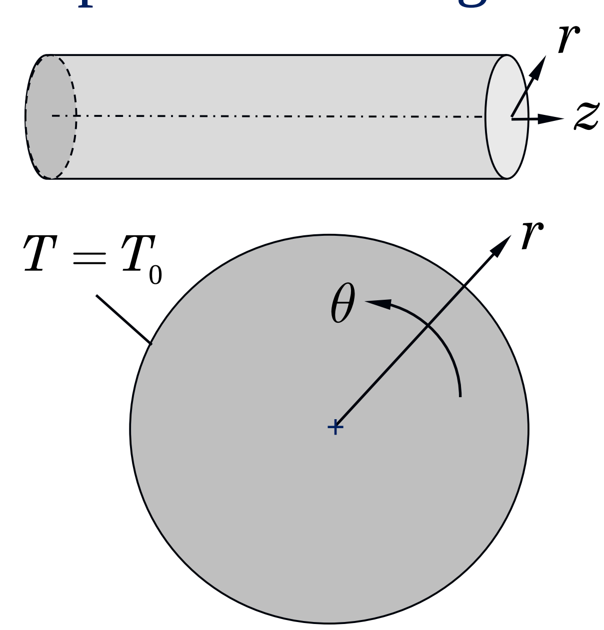
$$T(x) = -\frac{Q}{2k}x^2 + \frac{1}{(a+\beta_a)}\left[T_\infty^a - T_0 + \frac{H_a}{Ah_a} + Q\left(\frac{a}{h_a} + \frac{a^2}{2k}\right)\right]x + T_0$$

These solutions are valid for a plane wall with $A = 1$.

ANALYTICAL SOLUTIONS: 1D (cont.)

Example 3: steady-state, *axisymmetric* heat transfer in a *long* cylinder of radius *R* with internal heat generation *Q*.

If the material and boundary conditions are independent of the circumference and the cylinder is long, the problem is reduced to 1-D problem of finding the temperature along any radial line (see figure).



The governing equation is given by

$$\frac{1}{r}\left[\frac{\partial(rq_r)}{\partial r} + \cancelto{0}{\frac{\partial q_\theta}{\partial \theta}} + r\cancelto{0}{\frac{\partial q_z}{\partial z}}\right] = Q$$

$$\frac{1}{r}\left[\frac{d}{dr}(rq_r)\right] = Q, \quad q_r = -k\frac{dT}{dr}$$

ANALYTICAL SOLUTIONS: 1D (cont.)

The boundary-value problem becomes

$$-\frac{1}{r}\left[\frac{d}{dr}\left(rk\frac{dT}{dr}\right)\right]=Q,\ \left(rq_r\right)_{r=0}=0,\ T(R)=T_0$$

The solution to this equation is

$$T(r)=-\frac{Q}{4k}r^2+c_1\log r+c_2$$

The boundary conditions give $c_1=0,\ c_2=\frac{Q}{4k}R^2+T_0$

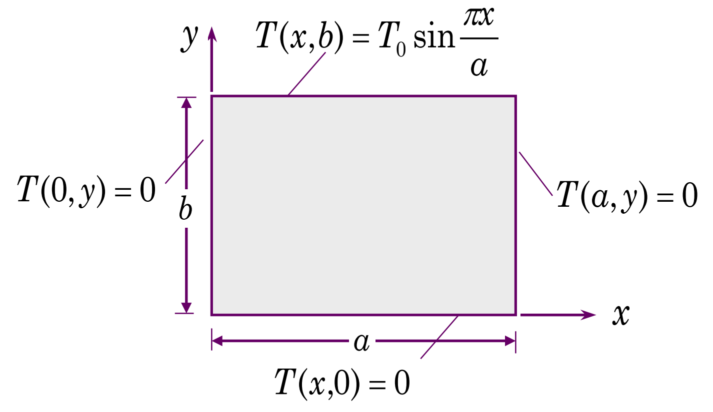
The solution becomes

$$T(r)-T_0=\frac{QR^2}{4k}\left(1-\frac{r^2}{R^2}\right)$$

TWO-DIMENSIONAL HEAT TRANSFER

Steady-state heat conduction in a rectangular domain

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0$$



Boundary conditions:

$$T(x,0)=0,\ \ T(0,y)=0,\ \ T(a,y)=0,\ \ T(x,b)=T_0\sin\frac{\pi x}{a}$$

Solution: Separation of variables method

$$T(x,y)=X(x)Y(y)\ \ \Rightarrow\ \frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\frac{d^2X}{dx^2}+\lambda^2X=0,\quad \frac{d^2Y}{dy^2}-\lambda^2Y=0$$

TWO-DIMENSIONAL HEAT TRANSFER

General solution:

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x, \quad Y(y) = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

$$T(x,y) = \left(C_1 \cos\lambda x + C_2 \sin\lambda x\right)\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right)$$

Determination of the constants:

$$T(x,0) = 0 \Rightarrow \left(C_1 \cos\lambda x + C_2 \sin\lambda x\right)\left(C_3 + C_4\right) = 0 \Rightarrow C_3 = -C_4$$

$$T(0,y) \ = 0 \Rightarrow C_1\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right) = 0 \rightarrow C_1 = 0$$

$$T(a,y) = 0 \Rightarrow C_2 \sin\lambda a\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right) = 0 \Rightarrow \sin\lambda a = 0$$

$$\sin\lambda a = 0 \Rightarrow \lambda a = n\pi \ \text{ or } \ \lambda_n = \frac{n\pi}{a}$$

TWO-DIMENSIONAL HEAT TRANSFER

The solution becomes:

$$T(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Determination of A_n:

$$T(x,b)=T_0\sin\frac{\pi x}{a}=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$\int_0^a\sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx=\begin{cases}0, & m\neq n\\ \dfrac{a}{2}, & m=n\end{cases}$$

$$A_1=\frac{T_0}{\sinh\frac{n\pi b}{a}},\quad A_n=0\text{ for }n\neq 1$$

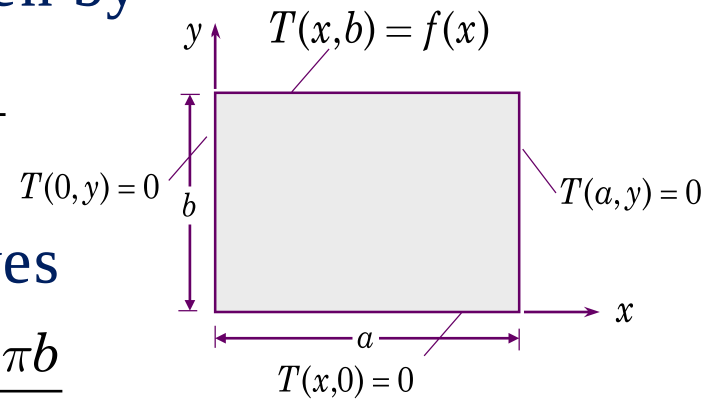
The final solution:

$$T(x,y)=T_0\frac{\sinh\frac{\pi y}{a}}{\sinh\frac{\pi b}{a}}\sin\left(\frac{\pi x}{a}\right)$$

TWO-DIMENSIONAL HEAT TRANSFER

The solution to this problem is given by

$$T(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$



The boundary condition at *y*=*b* gives

$$T(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

and $A_n = \dfrac{2}{a \sinh\dfrac{n\pi b}{a}} \displaystyle\int_0^a \sin\frac{n\pi x}{a} f(x)\, dx$

For $f(x) = T_0$, we have $A_n = \dfrac{2T_0}{a \sinh\dfrac{n\pi b}{a}} \displaystyle\int_0^a \sin\frac{n\pi x}{a}\, dx = \dfrac{2T_0(1-\cos n\pi)}{n\pi \sinh\dfrac{n\pi b}{a}}$

$$T(x,y) = \frac{4T_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \frac{\sinh\dfrac{n\pi y}{a}}{\sinh\dfrac{n\pi b}{a}} \sin\frac{n\pi x}{a}$$

EXERCISES

1. Solve for the temperature distribution in a long hollow cylinder (Example 3) with inner radius a and outer radius b and the inner and outer sides are maintained at temperatures T_i and T_o.

2. Determine the temperature distribution in a rectangular domain with the boundary conditions shown. You may use the principle of superposition