CONSTITUTIVE RELATIONS

CONTENTS

- General Introduction
- Solid mechanics: Generalized Hooke's law
- Fluid mechanics: viscous fluids
- Heat transfer: Fourier heat conduction law

GENERAL INTRODUCTION

- The kinematic relations and the conservation and balance principles are applicable to any continuum irrespective of its physical constitution.
- The kinematic variables such as strains and temperature gradient, and kinetic variables such as stresses and heat flux were introduced independently of each other.
- **Constitutive equations** are those relations that connect the *primary* field variables

$$(\rho, \theta, \nabla\theta, \mathbf{u}, \nabla\mathbf{u}, \mathbf{v}, \nabla\mathbf{v})$$

to the *secondary* field variables $(e, \eta, \mathbf{q}, \boldsymbol{\sigma})$,
and they involve the intrinsic physical properties of a continuum.

CONSTITUTIVE RELATIONS

- The main objective here is to study the most commonly known phenomenological constitutive equations that describe the macroscopic nature of the material response of idealized continua.
- Constitutive equations from solid mechanics, fluid mechanics, and heat transfer are discussed.
- We begin with certain terminologies that can be found in introductory texts on mechanics of materials and fluid mechanics.

SOME CONCEPTS from Constitutive Theory

- A continuum is said to be **homogeneous** if the material properties are the same throughout the continuum (i.e., material properties are independent of position). In a **heterogeneous** continuum, the material properties are a function of position.
- An **anisotropic** continuum is one that has different values of a material property in different directions at a point, that is, material properties are direction-dependent.
- An **isotropic** material is one for which a material property is the same in all directions at a point.

AXIOMS OF CONSTITUTIVE MODELS

Constitutive equations are often postulated based on experimental observations. Although experiments are necessary in the determination of various parameters (e.g., elastic constants, thermal conductivity, thermal coefficient of expansion, and coefficients of viscosity) appearing in the constitutive equations, the formulation of the constitutive equations for a given material is guided by certain rules. The approach typically involves assuming the form of the constitutive equation and then restricting the form to a specific one by appealing to certain physical requirements, and then measuring the parameters of the model through experiments.

AXIOMS OF CONSTITUTIVE MODELS

1. Physical admissibility. All constitutive equations must be consistent with the basic laws governing the continuum, such as the conservation of mass and balance of momenta and energy.

2. Determinism. The values of the constitutive variables (e.g., stress, heat flux, entropy, and internal energy) at a material point at any time are determined by the histories of motion and temperature of all points of the continuum.

3. Equipresence. A quantity appearing as an independent variable in one constitutive equation should appear in all constitutive equations.

4. Local action. The constitutive variables at a point **x** are not appreciably affected by the values of the dependent variables (e.g., displacements, strains, temperature, pressure, etc.) at points distant from **x**.

AXIOMS OF CONSTITUTIVE MODELS

5. Material frame indifference. The constitutive equations must be invariant with respect to observer transformations.

6. Material symmetry. The constitutive equations must be *form-invariant* with respect to transformations of the material frame of reference.

7. Dimensionality. The constitutive functionals should be dimensionally consistent in the sense that all terms appearing on either side of the constitutive equations should be the same.

8. Memory. The current values of the constitutive variables are not appreciably affected by their values at times distant from in the past. This is the time domain counterpart of of local action.

9. Causality. The variables entering the description of motion of a continuum and temperature are considered as the self-evident observable effects and the remaining quantities that enter the expression of entropy production are ``causes."

ELASTIC SOLIDS

- A material is called *Cauchy-elastic* or *elastic* if the stress field at time t depends only on the state of deformation and temperature at that time, and not on the history of these variables.

- A *hyperelastic material*, also known as the *Green-elastic material*, is one for which there exists a Helmholtz free-energy potential $\Psi_0(\mathbf{E},\mathbf{T})$ whose derivative with respect to a strain gives the corresponding stress and whose derivative with respect to temperature gives the heat flux vector.

$$\Psi_0 = \Psi_0(\mathbf{E},\mathbf{T}), \quad \mathbf{S} = \rho_0 \frac{\partial \Psi_0(\mathbf{E},\mathbf{T})}{\partial \mathbf{E}}, \quad \mathbf{q} = \rho_0 \frac{\partial \Psi_0(\mathbf{E},\mathbf{T})}{\partial \mathbf{T}}$$

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial U_0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad \left(\sigma_{ij} = \rho_0 \frac{\partial U_0}{\partial \varepsilon_{ij}} \right)$$ **(for infinitesimal strains)**

LINEAR ELASTIC SOLIDS

$$\rho_0 U_0 = C_0 + C_{ij}\varepsilon_{ij} + \frac{1}{2}C_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}$$

$$\sigma_{mn} = \rho_0 \frac{\partial U_0}{\partial \varepsilon_{mn}} = C_{ij}\delta_{mi}\delta_{nj} + \frac{1}{2}C_{ijk\ell}\left(\varepsilon_{k\ell}\delta_{im}\delta_{jn} + \varepsilon_{ij}\delta_{km}\delta_{\ell n}\right)$$

$$= C_{mn} + \frac{1}{2}C_{mnk\ell}\varepsilon_{k\ell} + \frac{1}{2}C_{ijmn}\varepsilon_{ij} = C_{mn} + \frac{1}{2}\left(C_{mnij} + C_{ijmn}\right)\varepsilon_{ij}$$

$$= C_{mn} + C_{mnij}\varepsilon_{ij} = C_{mn} + C_{ijmn}\varepsilon_{ij}$$

Residual stress

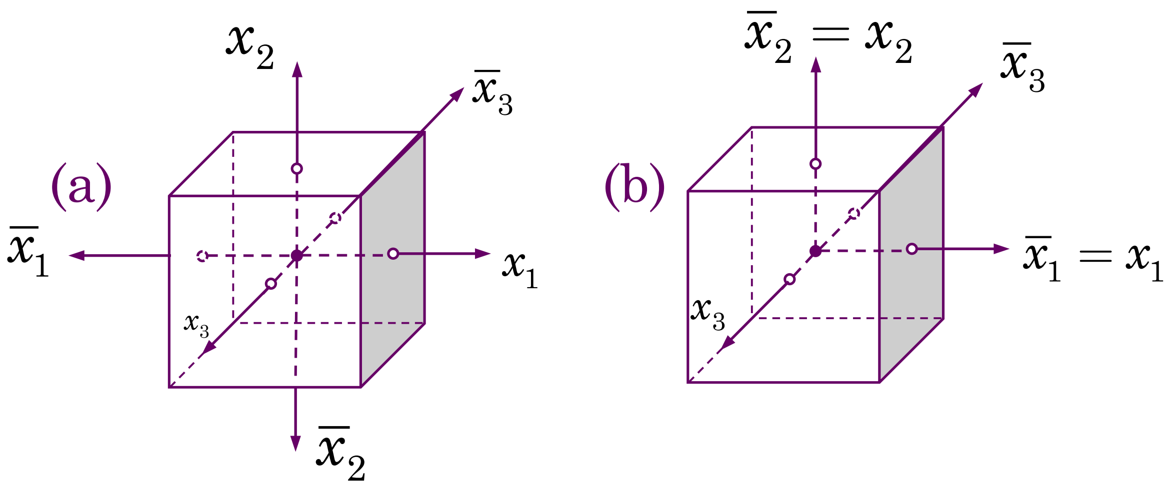
Symmetry of the coefficients:

$$C_{ijk\ell} = C_{k\ell ij},\ \ C_{ijk\ell} = C_{jik\ell},\ \ C_{ij\ell k} = C_{ijk\ell},\ \ C_{ijk\ell} = C_{ji\ell k}$$

LINEAR ELASTIC SOLIDS

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{2212} \\ & & & C_{2323} & C_{1313} & C_{2312} \\ & \text{sym.} & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

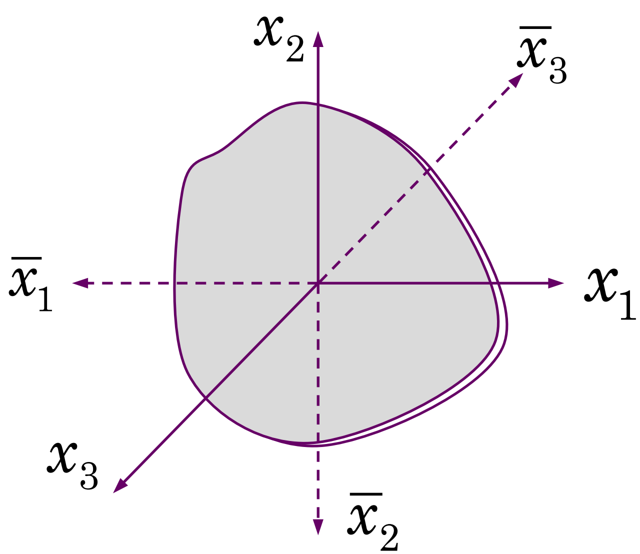
Material symmetry planes (1+2+3+4+5+6 = 21 constants)



Material Symmetry Planes and reduction of elastic constants

$$\bar{\sigma}_{ij} = \ell_{ip}\ell_{jq}\sigma_{pq}, \quad \bar{\varepsilon}_{ij} = \ell_{ip}\ell_{jq}\varepsilon_{pq}, \quad \bar{C}_{ijkl} = \ell_{ip}\ell_{jq}\ell_{kr}\ell_{ls}C_{pqrs}$$

(a)



x_3 is transverse to the plane of the body

Mirror reflection of each axis :

$$\bar{x}_1 = -x_1, \quad \bar{x}_2 = -x_2, \quad \bar{x}_3 = -x_3, \quad \hat{\bar{\mathbf{e}}}_i = -\hat{\mathbf{e}}_i, \quad \ell_{ij} = -\delta_{ij}$$

Material Symmetry Planes and reduction of elastic constants

$$[L] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

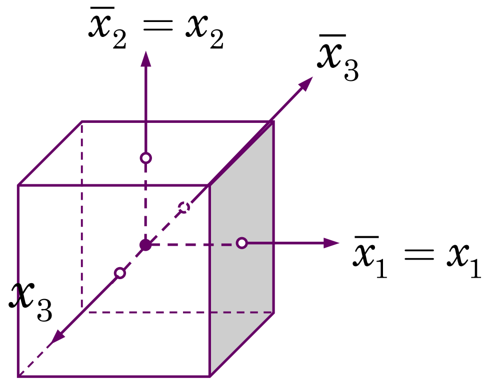
$$\bar{\sigma}_{ij} = (-\delta_{ip})(-\delta_{jq})\sigma_{pq} = \sigma_{ij}; \quad \bar{\varepsilon}_{ij} = (-\delta_{ip})(-\delta_{jq})\varepsilon_{pq} = \varepsilon_{ij},$$

$$\bar{C}_{ijk\ell} = (-1)^4 \delta_{ip}\delta_{jq}\delta_{kr}\delta_{\ell s} C_{pqrs} = C_{ijk\ell}$$

Thus, the transformation does not alter the constitutive relation of triclinic materials.

Material Symmetry Planes and reduction of elastic constants

(b)



$$\bar{x}_1 = x_1,\ \bar{x}_2 = x_2,\ \bar{x}_3 = -x_3$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[\bar{\sigma}] = [L][\sigma][L]^{\mathrm{T}}, \quad [\bar{\varepsilon}] = [L][\varepsilon][L]^{\mathrm{T}}$$

$$\bar{\sigma}_{ij} = \sigma_{ij},\ \text{except} \quad \bar{\sigma}_{13} = -\sigma_{13},\ \bar{\sigma}_{23} = -\sigma_{23},$$

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij},\ \text{except} \quad \bar{\varepsilon}_{13} = -\varepsilon_{13},\ \bar{\varepsilon}_{23} = -\varepsilon_{23}$$

Material Symmetry Planes and reduction of elastic constants

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} + C_{1123}\varepsilon_{23} + C_{1113}\varepsilon_{13} + C_{1112}\varepsilon_{12}$$

$$\bar{\sigma}_{11} = C_{1111}\bar{\varepsilon}_{11} + C_{1122}\bar{\varepsilon}_{22} + C_{1133}\bar{\varepsilon}_{33} + C_{1123}\bar{\varepsilon}_{23} + C_{1113}\bar{\varepsilon}_{13} + C_{1112}\bar{\varepsilon}_{12}$$

$$= C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} - C_{1123}\varepsilon_{23} - C_{1113}\varepsilon_{13} + C_{1112}\varepsilon_{12}$$

Since $\sigma_{11} = \bar{\sigma}_{11}$, from the preceding two relations it follows that

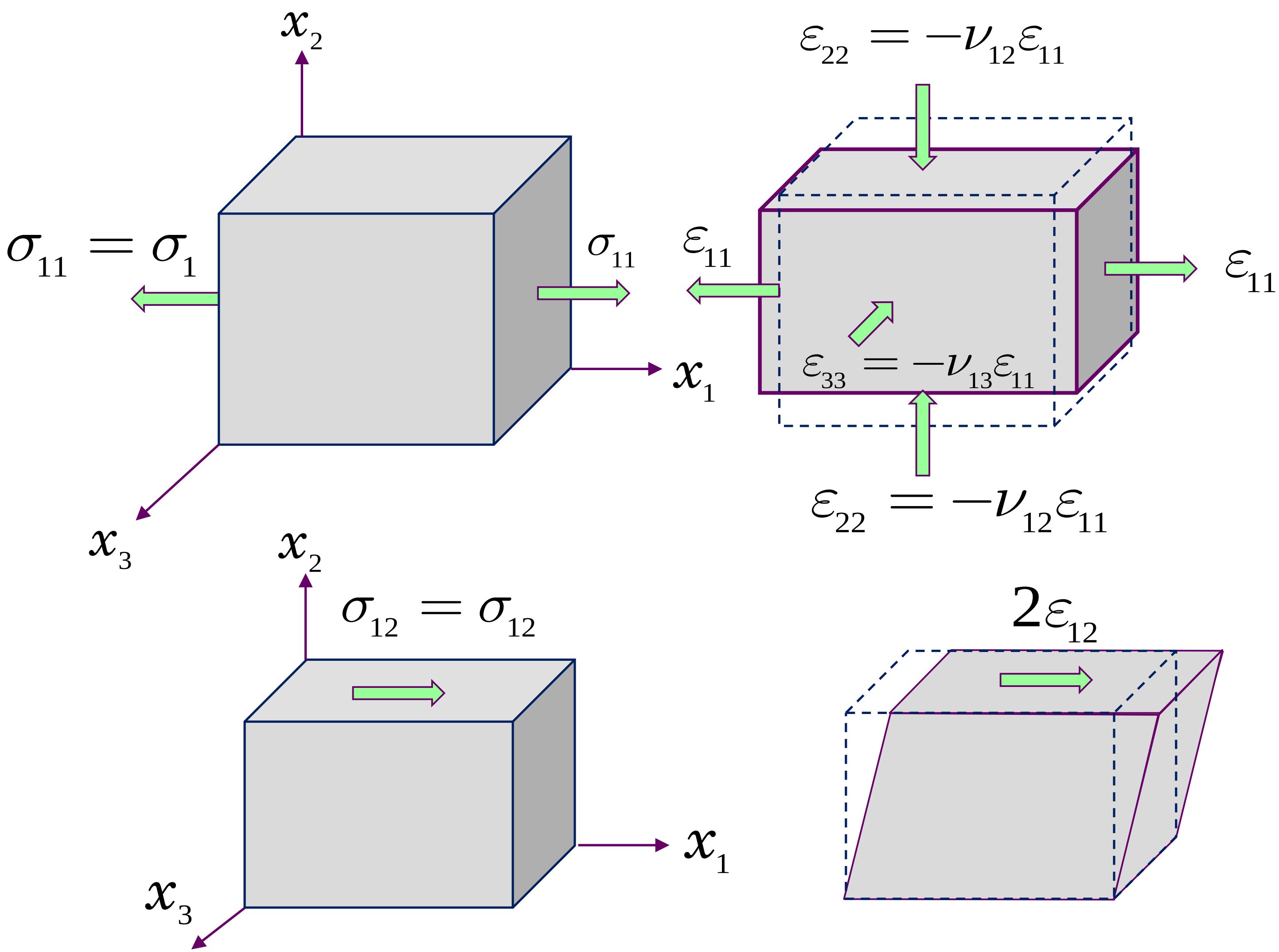
$$C_{1123}\varepsilon_{23} + C_{1113}\varepsilon_{13} = -C_{1123}\varepsilon_{23} - C_{1113}\varepsilon_{13},$$

which must hold for any independent set of strain components, ε_{23} and ε_{13}. This implies that $C_{1123} = 0$ and $C_{1113} = 0$.

Similarly, we can show that

$$C_{2223} = 0,\ C_{2213} = 0,\ C_{3323} = 0,\ C_{3313} = 0,\ C_{1223} = 0,\ C_{1213} = 0$$

GENERALIZED HOOKE'S LAW
For orthotropic materials



GENERALIZED HOOKE'S LAW

Strain-Stress Relations for an orthotropic material

The simultaneous application of $(\sigma_{11}, \sigma_{22}, \sigma_{33})$, assuming the principle of superposition holds, yields the relations

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

$$\nu_{ij} E_j = \nu_{ji} E_i \text{ (no sum on } i, j\text{); for fixed } i \neq j$$

There are nine independent material constants:

$$E_1,\ E_2,\ E_3,\ G_{23},\ G_{13},\ G_{12},\ \nu_{12},\ \nu_{13},\ \nu_{23}$$

GENERALIZED HOOKE'S LAW

Linear Stress-Strain Relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

$$C_{11} = \frac{1-\nu_{23}\nu_{23}}{E_2E_3\Delta}, \quad C_{12} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta}$$

$$C_{13} = \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta}, \quad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta}$$

$$C_{23} = \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_3\Delta}, \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta}$$

$$C_{44} = G_{23}, \quad C_{55} = G_{31}, \quad C_{66} = G_{12}$$

$$\Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

Constitutive Equations for Isotropic Material

$$E_1 = E_2 = E_3 = E,\ G_{12} = G_{13} = G_{23} = G = \frac{E}{2(1+\nu)},$$

$$\nu_{12} = \nu_{23} = \nu_{13} = \nu \qquad ([\Delta = (1+\nu)^2(1-2\nu)]$$

$$C_{11} = C_{22} = C_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)},$$

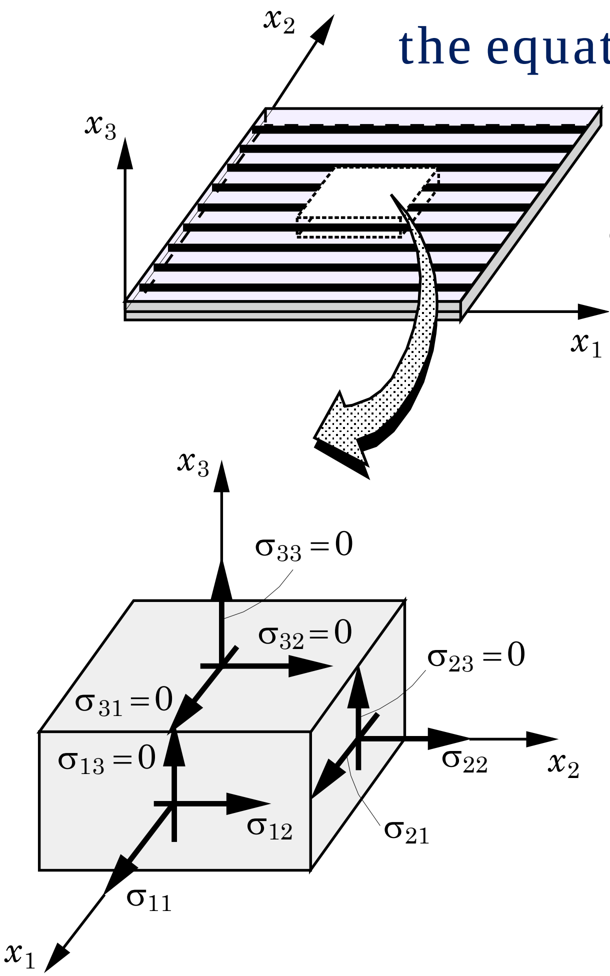
$$C_{12} = C_{13} = C_{23} = \frac{\nu E}{(1+\nu)(1-2\nu)},\ C_{44} = C_{55} = C_{66} = G$$

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Plane-Stress Reduced State

Begin with the **strain-stress** relations and set to the transverse stress components to zero. Then invert the equations to obtain the stress-strain relations.



$$\sigma_3 = 0,\ \sigma_4 = 0,\ \sigma_5 = 0 \quad (\sigma_{33} = 0,\ \sigma_{32} = 0,\ \sigma_{31} = 0)$$

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2, \quad \varepsilon_4 = 0,\ \varepsilon_5 = 0$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Constitutive Equations of the Plane-Stress State

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

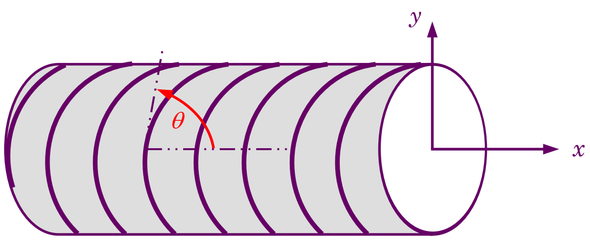
$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

$$E_1, \quad E_2, \quad \nu_{12}, \quad G_{12}$$

EXERCISE PROBLEM

Consider a thin, filament-wound, closed cylindrical pressure vessel shown. The vessel is of 63.5 cm (25 in.) internal diameter, 2 cm thickness (0.7874 in.) and pressurized to 1.379 MPa (200 psi). Determine (a) stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ in the vessel, (b) stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ in the material coordinates (x_1, x_2, x_3) with x_1 being along the filament direction, (c) strains $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ in the material coordinates, and (d) strains $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ in the vessel. Assume a filament winding angle of $\theta = 53.125^{\circ}$ from the longitudinal axis of the pressure vessel, and use the following material properties, typical of graphite-epoxy material:

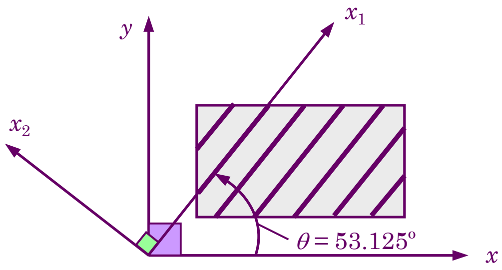


$$E_1 = 140 \text{ GPa } (20.3 \text{ Msi}),$$

$$E_2 = 10 \text{ GPa } (1.45 \text{ Msi})$$

$$G_{12} = 7 \text{ GPa } (1.02 \text{ Msi}),$$

$$\nu_{12} = 0.3$$



SOLUTION TO THE EXERCISE PROBLEM

a) The equations of equilibrium of forces in a structure do not depend on the material properties. Hence, equations derived for the longitudinal (σ_{xx}) and circumferential (σ_{yy}) stresses in a thin-walled cylindrical pressure vessel are valid here:

$$\sigma_{xx} = \frac{pD_i}{4h}, \quad \sigma_{yy} = \frac{pD_i}{2h}$$

$$\sigma_{xx} = \frac{1.379 \times 0.635}{4h} = \frac{0.2189}{h} \text{ MPa}, \quad \sigma_{yy} = \frac{1.379 \times 0.635}{2h} = \frac{0.4378}{h} \text{ MPa}$$

b) Relationships between the local and global stresses:

$$\sigma_{11} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\sigma_{xy} \cos\theta \sin\theta$$

$$\sigma_{22} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \cos\theta \sin\theta$$

$$\sigma_{12} = -\sigma_{xx} \sin\theta \cos\theta + \sigma_{yy} \cos\theta \sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

Solution (continued)

$$\sigma_{11} = \frac{0.2189}{h}(0.6)^2 + \frac{0.4378}{h}(0.8)^2 = \frac{0.3590}{h}\text{ MPa},$$

$$\sigma_{22} = \frac{0.2189}{h}(0.8)^2 + \frac{0.4378}{h}(0.6)^2 = \frac{0.2977}{h}\text{ MPa},$$

$$\sigma_{12} = \left(\frac{0.4378}{h} - \frac{0.2189}{h}\right)\times 0.6\times 0.8 = \frac{0.1051}{h}\text{ MPa}$$

$$\sigma_{11} = 17.95\text{ MPa}, \sigma_{22} = 14.885\text{ MPa}, \ \sigma_{12} = 5.255\text{ MPa}$$

c) The local strains are:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \frac{\sigma_{22}\nu_{12}}{E_1} = \frac{17.95}{140\times 10^3} - \frac{14.885\times 0.3}{140\times 10^3} = 0.0963\times 10^{-3}\text{ m/m},$$

$$\varepsilon_{22} = -\frac{\sigma_{11}\nu_{12}}{E_1} + \frac{\sigma_{22}}{E_2} = -\frac{17.95\times 0.3}{140\times 10^3} + \frac{14.885}{10\times 10^3} = 1.45\times 10^{-3}\text{m/m},$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G_{12}} = \frac{5.255}{2\times 7} = 0.3757\times 10^{-3}$$

Solution (continued)

d) The global strains are related to the local strains by

$$\varepsilon_{xx} = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta - 2\varepsilon_{12}\cos\theta\sin\theta$$

$$\varepsilon_{yy} = \varepsilon_{11}\sin^2\theta + \varepsilon_{22}\cos^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta$$

$$\varepsilon_{xy} = \left(\varepsilon_{11} - \varepsilon_{22}\right)\cos\theta\sin\theta + \varepsilon_{12}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\varepsilon_{xx} = 10^{-3}\left[0.0963 \times (0.6)^2 + 1.45 \times (0.8)^2 - 0.3757 \times 0.6 \times 0.8\right]$$

$$= 0.782 \times 10^{-3}\,\text{m/m}$$

$$\varepsilon_{yy} = 10^{-3}\left[0.0963 \times (0.8)^2 + 1.45 \times (0.6)^2 + 0.3757 \times 0.6 \times 0.8\right]$$

$$= 0.764 \times 10^{-3}\ \text{m/m}$$

$$\varepsilon_{xy} = 10^{-3}\left\{2(0.0963 - 1.45) \times (0.6) \times 0.8 + 0.3757[(0.6)^2 - (0.8)^2]\right\}$$

$$= -1.405 \times 10^{-3}$$

VISCOUS FLUIDS

The stress in a fluid is proportional to the time rate of strain (i.e., time rate of deformation). The proportionality parameter is known as the **viscosity**. It is a measure of the intermolecular forces exerted as layers of fluid attempt to slide past one another.

An ideal fluid

$$\boldsymbol{\sigma} = -p(\rho, T)\mathbf{I}, \qquad p = R\rho T / m$$

Viscous, Newtonian, compressible fluid

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - p\mathbf{I}, \quad \boldsymbol{\tau} = \mathbf{F}(\mathbf{D}), \qquad \boldsymbol{\tau} = 2\mu\mathbf{D} + \lambda(\mathrm{tr}\mathbf{D})\mathbf{I}$$

$$\boldsymbol{\sigma} = 2\mu\mathbf{D} + \lambda(\mathrm{tr}\mathbf{D})\mathbf{I} - p\mathbf{I}, \quad \sigma_{ij} = 2\mu D_{ij} + \left(\lambda D_{kk} - p\right)\delta_{ij}$$

VISCOUS FLUIDS

Viscous, Newtonian, compressible fluid

Stokes's hypothesis:

$$2\mu + 3\lambda = 0 \;\Rightarrow\; \lambda = -\frac{2}{3}\mu$$

$$\boldsymbol{\sigma} = 2\mu\mathbf{D} - \frac{2}{3}\mu(\mathrm{tr}\mathbf{D})\mathbf{I} - p\mathbf{I}, \quad \sigma_{ij} = 2\mu D_{ij} - \left(\frac{2}{3}\mu D_{kk} + p\right)\delta_{ij}$$

Viscous, Newtonian, incompressible fluid
($D_{kk} = \nabla \cdot \mathbf{v} = 0$)

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{D}, \quad \boldsymbol{\sigma}_{ij} = -p\boldsymbol{\delta}_{ij} + 2\mu D_{ij}$$

Fourier's Heat Conduction Law

For general solid

$$\mathbf{q} = -\mathbf{k} \cdot \nabla T$$

For an isotropic solid

$$\mathbf{q} = -k \nabla T, \;\; q_i = -k \frac{\partial T}{\partial x_i}$$

Energy equation for a solid

$$\rho c_v \frac{DT}{Dt} = \Phi - \nabla \cdot \mathbf{q} + \rho e, \quad \Phi = \boldsymbol{\tau} : \mathbf{D}$$