BALANCE OF ANGULAR MOMENTUM AND BALANCE OF ENERGY

- **Balance of angular momentum**
- **Balance of energy**

Balance of Angular Momentum

The principle of balance of angular momentum for the monopolar case can be stated as follows: *The time rate of change of the total moment of momentum for a continuum is equal to the vector sum of the moments of external forces acting on the continuum.*

Here we consider only the monopolar continuum mechanics, where the distributed couples present at the molecular level, are overlooked. A continuum said to have no body couples (that is, volume-dependent couples **M** if

$$\lim_{\Delta v \to 0} \frac{\Delta \mathbf{M}}{\Delta v} = \mathbf{0}$$

holds. For a continuum without body couples, the stress tensor can be shown to be symmetric.

Balance of Angular Momentum

$$\frac{D}{Dt}\int_{\Omega}(\mathbf{x}\times\rho\mathbf{v})d\Omega = \oint_{\Gamma}(\mathbf{x}\times\mathbf{t})\,d\Gamma + \int_{\Omega}(\mathbf{x}\times\rho\mathbf{f})\,d\Omega$$

$$\frac{D}{Dt}\int_{\Omega}\rho\varepsilon_{ijk}x_i v_j\,d\Omega = \oint_{\Gamma}\varepsilon_{ijk}x_i t_j\,d\Gamma + \int_{\Omega}\rho\varepsilon_{ijk}x_i f_j\,d\Omega$$

$$\int_{\Omega}\rho\varepsilon_{ijk}\frac{D}{Dt}\left(x_i v_j\right)d\Omega = \int_{\Omega}\varepsilon_{ijk}\left(x_i\sigma_{jp}\right)_{,p}d\Omega + \int_{\Omega}\rho\varepsilon_{ijk}x_i f_j\,d\Omega$$

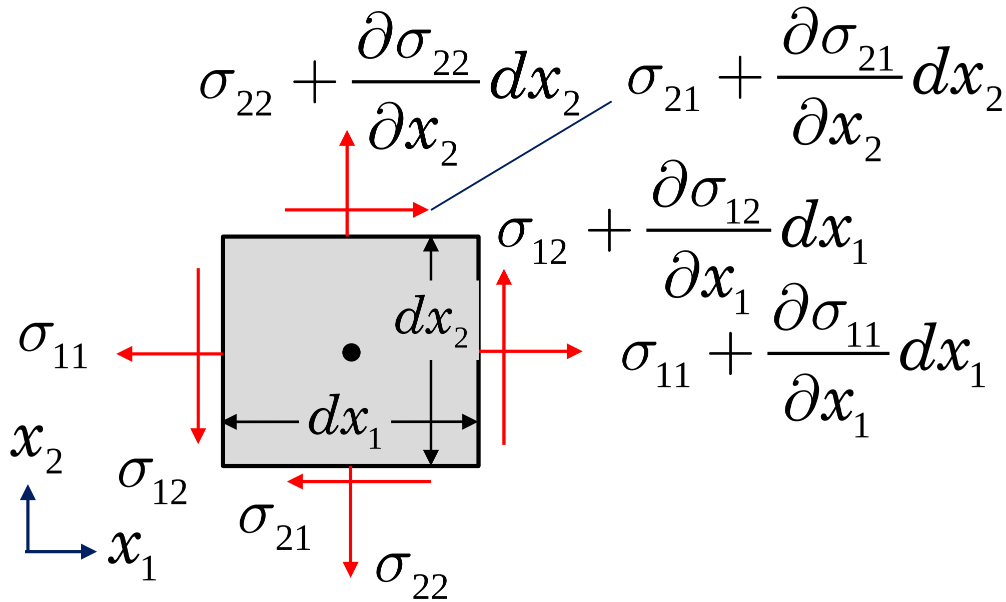
$$\int_{\Omega}\varepsilon_{ijk}\left(\rho v_j\frac{Dx_i}{Dt} + x_i\rho\frac{Dv_j}{Dt}\right)d\Omega = \int_{\Omega}\varepsilon_{ijk}\left(x_i\sigma_{jp,p} + \delta_{ip}\sigma_{jp}\right)d\Omega$$

$$+\int_{\Omega}\varepsilon_{ijk}x_i\rho f_j\,d\Omega$$

$$\varepsilon_{ijk}\sigma_{ji} = 0 \;\text{ or }\; \sigma_{ij} = \sigma_{ji}$$

Balance of Angular Momentum

Consider the moment of all forces acting on the parallelepiped about the x_3-axis, as shown in the figure. Using the right-handed screw rule for positive moment, we obtain



$$\left[\left(\sigma_{12}+\frac{\partial\sigma_{12}}{\partial x_1}dx_1\right)dx_2dx_3\right]\frac{dx_1}{2}+\left(\sigma_{12}dx_2dx_3\right)\frac{dx_1}{2}$$

$$-\left[\left(\sigma_{21}+\frac{\partial\sigma_{21}}{\partial x_2}dx_2\right)dx_1dx_3\right]\frac{dx_2}{2}-\left(\sigma_{21}dx_1dx_3\right)\frac{dx_2}{2}=0$$

$$\sigma_{21}-\sigma_{21}=0$$

BALANCE OF ENERGY

The principle of the balance of energy can be stated as: **the time rate of change of internal energy is equal to the sum of power due to external forces and the heat input to the system**. The internal energy may consist of strain energy, kinetic energy, chemical energy, and so on. Here we consider only the strain energy and kinetic energy. Then we have

$$\frac{D}{Dt}(K+U)=W+Q_h$$

K – Kinetic energy; U – Strain energy
W – Power due to external forces; Q_h – Heat input

BALANCE OF ENERGY

In spatial description these terms can be expressed as

$$K = \frac{1}{2}\int_\Omega \rho \mathbf{v}\cdot\mathbf{v}\, d\Omega, \;\; U = \int_\Omega \rho e\, d\Omega$$

$$W = \oint_\Gamma \mathbf{t}\cdot\mathbf{v} d\Gamma + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v} d\Omega$$

$$Q_h = -\oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\, d\Gamma + \int_\Omega \rho r_h\, d\Omega$$

Then we have

$$\frac{D}{Dt}\int_\Omega \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + e\right) d\Omega = \oint_\Gamma \mathbf{t}\cdot\mathbf{v} d\Gamma + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v} d\Omega - \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\, d\Gamma + \int_\Omega \rho r_h\, d\Omega$$

BALANCE OF ENERGY (continued)

Simplifying the expression

$$\int_\Omega \rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+e\right)d\Omega = \oint_\Gamma \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\cdot\mathbf{v}\,d\Gamma + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\,d\Omega - \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\,d\Gamma + \int_\Omega \rho r_h\,d\Omega$$

$$= \int_\Omega \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})d\Omega + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\,d\Omega - \int_\Omega \nabla\cdot\mathbf{q}\,d\Gamma + \int_\Omega \rho r_h\,d\Omega$$

$$\int_\Omega \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})d\Omega = \int_\Omega (\nabla\cdot\boldsymbol{\sigma}\cdot\mathbf{v} + \boldsymbol{\sigma}\cdot\cdot\nabla\mathbf{v})d\Omega$$

$$= \int_\Omega (\nabla\cdot\boldsymbol{\sigma}\cdot\mathbf{v} + \boldsymbol{\sigma}\cdot\cdot\mathbf{D})d\Omega$$

BALANCE OF ENERGY (continued)

Thus the principle of the balance of energy results in the final **global form**

$$\int_{\Omega}\left(\rho\frac{De}{Dt}-\boldsymbol{\sigma}:\mathbf{D}+\nabla\cdot\mathbf{q}-\rho r_h\right)d\Omega=0$$

The **local form** in the spatial description is given by

$$\rho\frac{De}{Dt}=\boldsymbol{\sigma}:\mathbf{D}-\nabla\cdot\mathbf{q}+\rho r_h$$

The local form in the material description is given by

$$\rho_0\frac{\partial e}{\partial t}=(\mathbf{S}\cdot\mathbf{F}^{\mathrm{T}}):\nabla_0\mathbf{v}-\nabla_0\cdot\mathbf{q}_0+\rho_0 r_h$$

BALANCE OF ENERGY (continued)

The energy equation for hear transfer in a solid medium becomes ($\rho r_h = Q$)

$$\rho c_v \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + Q$$

Rectangular Cartesian system (*x*, *y*, *z*)

$$\rho c_v \frac{\partial T}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + Q$$

Cylindrical coordinate system (r, θ, z)

$$\rho c_v \frac{\partial T}{\partial t} = -\frac{1}{r}\left[\frac{\partial (r q_r)}{\partial r} + \frac{\partial q_\theta}{\partial \theta} + r \frac{\partial q_z}{\partial z} \right] + Q$$

EXERCISES

1. Show that the energy equation can be expressed in the alternative form

$$\rho \frac{D}{Dt}\left(e + \frac{v^2}{2}\right) = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v} + \rho r_h - \nabla \cdot \mathbf{q}$$

2. Establish the following *thermodynamic form* of the energy equation

$$\rho \frac{De}{Dt} = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma} + \rho r_h - \nabla \cdot \mathbf{q}$$