CONSERVATION OF MASS AND BALANCE OF LINEAR MOMENTUM

- Summary of integral theorems
- Material time derivative
- Reynolds' transport theorem
- Principle of conservation of mass
- Principle of balance of linear momentum

INTEGRAL THEOREMS

Divergence Theorem

$$\oint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega$$

Gradient Theorem

$$\oint_{\Gamma} \hat{\mathbf{n}} \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \mathbf{F} \, d\Omega$$

Curl Theorem

$$\oint_{\Gamma} \hat{\mathbf{n}} \times \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \times \mathbf{F} \, d\Omega$$

General Theorem

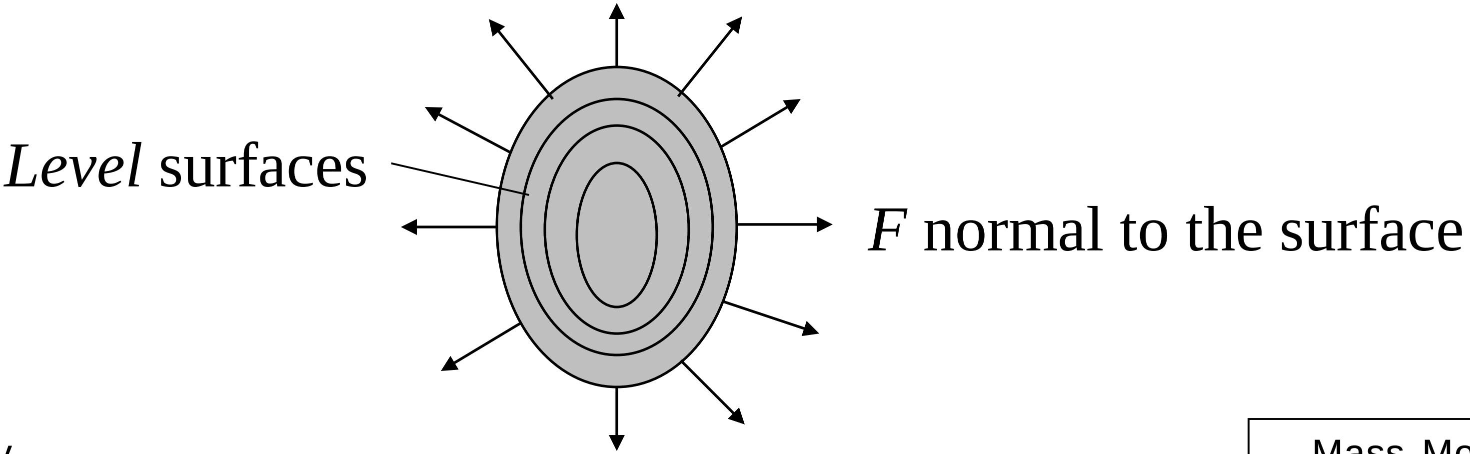
$$\boxed{\oint_{\Gamma} \hat{\mathbf{n}} * \mathbf{F} \, ds = \int_{\Omega} \nabla * \mathbf{F} \, d\Omega}$$

INTEGRAL THEOREMS

Gradient Theorem

$$\oint_{\Gamma} \hat{\mathbf{n}} F \, d\Gamma = \int_{\Omega} \nabla F \, d\Omega$$

The **gradient** of a function F represents the rate of change of F with respect to the coordinate directions. the partial derivative with respect to x, for example, gives the rate of change of F in the x direction.



Gradient of a Scalar Function

$$\nabla F = \text{a first-order tensor, that is, a } \textit{vector}$$

$$\left.\begin{aligned} \nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\ \nabla F &= \frac{\partial F}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial F}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial F}{\partial z}\hat{\mathbf{e}}_x \end{aligned}\right\}$$

in rectangular
Cartesian system

$$\left.\begin{aligned} \nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\ \nabla F &= \hat{\mathbf{e}}_r \frac{\partial F}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial F}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial F}{\partial z} \end{aligned}\right\}$$

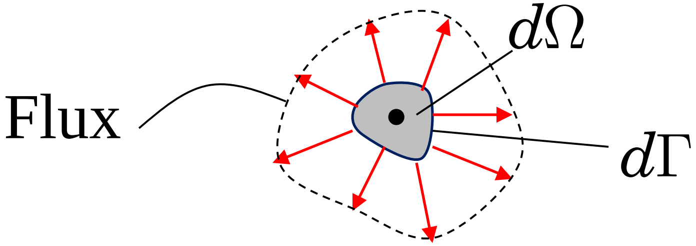
in cylindrical
coordinate system

INTEGRAL THEOREMS

Divergence Theorem

$$\oint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{F}\, d\Gamma = \int_{\Omega} \nabla \cdot \mathbf{F}\, d\Omega$$

The **divergence** represents the volume density of the outward flux of a **vector** field **F** from an infinitesimal volume $d\Omega$ around a given point. It is a local measure of its "outgoingness."



Divergence of First-Order Tensors

$$\nabla\cdot\mathbf{F} = \text{a zeroth tensor, that is, a scalar}$$

$$\left.\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla\cdot\mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\end{aligned}\right\} \begin{array}{c}\text{in rectangular}\\ \text{Cartesian system}\end{array}$$

$$\left.\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla\cdot\mathbf{F} &= \frac{1}{r}\left[\frac{\partial (rF_r)}{\partial r} + \frac{\partial F_\theta}{\partial \theta} + r\frac{\partial F_z}{\partial z}\right]
\end{aligned}\right\} \begin{array}{c}\text{in cylindrical}\\ \text{coordinate system}\end{array}$$

Divergence of Second-order Tensors

$$\nabla \cdot \mathbf{S} = \text{a first-order tensor, that is, a vector}$$

$$\nabla \cdot \mathbf{S} = \left(\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial F_{zx}}{\partial z} \right) \hat{\mathbf{e}}_x + \left(\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial F_{zy}}{\partial z} \right) \hat{\mathbf{e}}_y$$

$$+ \left(\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) \hat{\mathbf{e}}_z$$

in rectangular
Cartesian system

$$\nabla \cdot \mathbf{S} = \frac{1}{r} \left[\frac{\partial (r S_{rr})}{\partial r} + \frac{\partial S_{\theta r}}{\partial \theta} + r \frac{\partial S_{zr}}{\partial z} - S_{\theta\theta} \right] \hat{\mathbf{e}}_r$$

$$+ \frac{1}{r} \left[\frac{\partial (r S_{r\theta})}{\partial r} + \frac{\partial S_{\theta\theta}}{\partial \theta} + r \frac{\partial S_{z\theta}}{\partial z} + S_{\theta r} \right] \hat{\mathbf{e}}_\theta$$

$$+ \frac{1}{r} \left[\frac{\partial (r S_{rz})}{\partial r} + \frac{\partial S_{\theta z}}{\partial \theta} + r \frac{\partial S_{zr}}{\partial z} \right] \hat{\mathbf{e}}_z$$

in cylindrical
coordinate system

INTEGRAL THEOREMS

Curl Theorem (Stoke’s Theorem)

$$\oint_{\Gamma} \hat{\mathbf{n}} \times \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \times \mathbf{F} \, d\Omega$$

The **curl** of a vector **F** describes the infinitesimal rotation of **F**. A physical interpretation is as follows. Suppose the vector field describes the velocity field **F** = **v** of a fluid flow, say, in a large tank of liquid, and a small spherical ball is located within the fluid (the center of the ball being fixed at a certain point but free to rotate about an axis perpendicular to the plane of the flow). If the ball has a rough surface, the fluid flowing past the ball will make it rotate. The rotation axis (oriented according to the right hand rule) points in the direction of the curl of the field at the center of the ball, and the angular speed of the rotation is half the magnitude of the curl at this point.

Curl of a Vector Function in (x,y,z) system

$$\nabla\times\mathbf{A} = \text{a first-order tensor, that is, a vector}$$

$$\begin{aligned}\nabla\times\mathbf{A} &= \frac{\partial A_x}{\partial x}\overbrace{(\hat{\mathbf{e}}_x\times\hat{\mathbf{e}}_x)}^{0} + \frac{\partial A_y}{\partial x}\overbrace{(\hat{\mathbf{e}}_x\times\hat{\mathbf{e}}_y)}^{\hat{\mathbf{e}}_z} + \frac{\partial A_z}{\partial x}\overbrace{(\hat{\mathbf{e}}_x\times\hat{\mathbf{e}}_z)}^{-\hat{\mathbf{e}}_y}\\
&+ \frac{\partial A_x}{\partial y}\overbrace{(\hat{\mathbf{e}}_y\times\hat{\mathbf{e}}_x)}^{-\hat{\mathbf{e}}_z} + \frac{\partial A_y}{\partial y}\overbrace{(\hat{\mathbf{e}}_y\times\hat{\mathbf{e}}_y)}^{0} + \frac{\partial A_z}{\partial y}\overbrace{(\hat{\mathbf{e}}_y\times\hat{\mathbf{e}}_z)}^{\hat{\mathbf{e}}_x}\\
&+ \frac{\partial A_x}{\partial z}\overbrace{(\hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_x)}^{\hat{\mathbf{e}}_y} + \frac{\partial A_y}{\partial z}\overbrace{(\hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_y)}^{-\hat{\mathbf{e}}_x} + \frac{\partial A_z}{\partial z}\overbrace{(\hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_z)}^{0}\\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{e}}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{e}}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{e}}_z\end{aligned}$$

Material Time Derivative

Material time derivative is the time derivative of a function in spatial description with the material coordinate **X** held constant. We denote it with *D/Dt* (also called *substantive derivative*).

Material (or Lagrange) description

$$\varphi(\mathbf{X},t): \quad \left(\frac{D\varphi}{Dt}\right) = \left(\frac{\partial \varphi}{\partial t}\right)_{\mathbf{X}=\text{const}} = \frac{\partial \varphi}{\partial t}$$

Spatial (or Eulerian) description

$$\varphi(\mathbf{x},t): \quad \left(\frac{D\varphi}{Dt}\right) = \left(\frac{\partial \varphi}{\partial t}\right)_{\mathbf{X}=\text{const}} = \underbrace{\boxed{\left(\frac{\partial \varphi}{\partial t}\right)_{\mathbf{x}=\text{const}}}}_{\text{Local change}} + \overbrace{\boxed{\left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=\text{const}} \frac{\partial \varphi}{\partial x_i}}}^{\text{Translational change}}$$

$$\boxed{\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \mathbf{v}\cdot\nabla\varphi}$$

EXAMPLE 5-1

Problem statement:

Suppose that a motion is described by the one-dimensional mapping, $x = (1+t)X$ for $t > 0$. Determine (a) the velocities and accelerations in the spatial and material descriptions, and (b) the time derivative of a function $\varphi(X,t) = Xt^2$ in the spatial and material descriptions.

Solution:

The velocity $v \equiv Dx/Dt$ can be expressed in the material and spatial coordinates as

$$v(X,t) = \frac{Dx}{Dt} = \frac{\partial x}{\partial t} = X, \quad v(x,t) = X(x,t) = \frac{x}{1+t}$$

The acceleration $a \equiv Dv/Dt$ in the two descriptions is

$$a \equiv \frac{Dv(X,t)}{Dt} = \frac{\partial v}{\partial t} = 0, \;\; a \equiv \frac{Dv(x,t)}{Dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

EXAMPLE 5-1 (continued)

The material time derivative of $\varphi = \varphi(X,t)$ in the material description is simply

$$\frac{D\varphi(X,t)}{Dt} = \frac{\partial\varphi(X,t)}{\partial t} = 2Xt$$

The material time derivative of

$$\varphi = \varphi(x,t) = X(x,t)t^2 = xt^2\,/\,(1+t)$$

in the spatial description is

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} = \frac{2xt}{1+t} - \frac{xt^2}{(1+t)^2} + \left(\frac{x}{1+t}\right)\left(\frac{t^2}{1+t}\right) = \frac{2xt}{1+t}$$

which is the same as that calculated before, except that it is expressed in terms of the current coordinate, *x*.

EXAMPLE 5-2

Problem statement:

Consider the motion of a body described by the mapping

$$x_1 = \frac{X_1}{1+tX_1}, \quad x_2 = X_2, \quad x_3 = X_3$$

Determine the density ρ as a function position **x** and time *t*.

Solution:

First, compute the velocity components

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}=fixed} \quad \text{or} \quad v_i = \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=fixed}$$

Thus, we have

$$v_1 = -\frac{X_1^2}{(1+tX_1)^2} = -x_1^2, \quad v_2 = 0, \quad v_3 = 0$$

EXAMPLE 5-2 (continued)

Next, compute

$$\frac{D\rho}{Dt} = -\rho\,\nabla\cdot\mathbf{v} = 2\rho x_1 = 2\rho\frac{X_1}{1+tX_1}.$$

Integrating this equation with respect to *t*, we obtain

$$\int\frac{1}{\rho}D\rho = \int 2\frac{X_1}{1+tX_1}Dt \Rightarrow \ln\rho = 2\ln\frac{X_1}{1+tX_1} + \ln c$$

where *c* is a constant of integration. If $\rho = \rho_0$ at time $t = 0$, then we have $\ln c = \ln c_0$, and the density becomes

$$\rho = \rho_0\left(1+tX_1\right)^2 = \frac{\rho_0}{(1-tx_1)^2}$$

REYNOLDS' TRANSPORT THEOREM

Let each element of mass in the volume $\Omega(t)$ with closed boundary Γ moves with the velocity $\mathbf{v}(\mathbf{x},t)$. Let $\phi(\mathbf{x},t)$ be any function. We are interested in the material time derivative of the integral

$$\frac{D}{Dt}\int_{\Omega}\phi(\mathbf{x},t)\,d\Omega$$

Since $\Omega(t)$ is changing with time, we cannot take the differentiation through the integral. However, if the integral were over the volume in the reference configuration (which is fixed), it is possible to interchange the integration and differentiation because D/Dt is differential with respect to time keeping $\mathbf{X}$ constant. The transformation $\mathbf{x}=\mathbf{x}(\mathbf{X},t)$ and $d\Omega = J\ d\Omega_0$ lets us to do exactly that.

REYNOLDS' TRANSPORT THEOREM

We have

$$\frac{D}{Dt}\int_{\Omega(t)}\phi(\mathbf{x},t)d\Omega=\frac{D}{Dt}\int_{\Omega_0}\phi[\mathbf{x}(\mathbf{X},t),\mathrm{t}]\,J\,d\Omega_0=\int_{\Omega_0}\left(\frac{D\phi}{Dt}J+\phi\frac{DJ}{Dt}\right)d\Omega_0$$

$$=\int_{\Omega_0}\left(\frac{D\phi}{Dt}+\phi\left(\nabla\cdot\mathbf{v}\right)\right)J\,d\Omega_0=\int_{\Omega(t)}\left(\frac{D\phi}{Dt}+\phi\left(\nabla\cdot\mathbf{v}\right)\right)d\Omega$$

$$=\int_{\Omega(t)}\left(\frac{\partial\phi}{\partial t}+\left(\mathbf{v}\cdot\nabla\phi\right)+\phi\left(\nabla\cdot\mathbf{v}\right)\right)d\Omega=\int_{\Omega(t)}\left(\frac{\partial\phi}{\partial t}+\nabla\cdot\left(\mathbf{v}\phi\right)\right)d\Omega$$

$$=\int_{\Omega(t)}\frac{\partial\phi}{\partial t}d\Omega+\oint_{\Gamma(t)}\hat{\mathbf{n}}\cdot\left(\mathbf{v}\phi\right)d\Gamma$$

Thus

$$\frac{D}{Dt}\int_{\Omega(t)}\phi(\mathbf{x},t)d\Omega=\int_{\Omega(t)}\frac{\partial\phi}{\partial t}d\Omega+\oint_{\Gamma(t)}\hat{\mathbf{n}}\cdot\left(\mathbf{v}\phi\right)d\Gamma$$

REYNOLDS' TRANSPORT THEOREM

$$\frac{D}{Dt}\int_{\Omega(t)}\phi(\mathbf{x},t)d\Omega=\int_{\Omega(t)}\frac{\partial\phi}{\partial t}d\Omega+\oint_{\Gamma(t)}\hat{\mathbf{n}}\cdot(\mathbf{v}\phi)d\Gamma$$

Thus the time rate of change of the integral of a function $\phi(\mathbf{x},t)$ over a moving volume is equal to the integral of the local time rate of $\phi(\mathbf{x},t)$ plus the net outflow of $\phi(\mathbf{x},t)$ over the surface Γ of the moving volume. Here $\phi(\mathbf{x},t)$ can be a scalar or a tensor of any order.

$$\frac{D}{Dt}\int_{\Omega(t)}\phi(\mathbf{x},t)d\Omega=\int_{\Omega(t)}\left[\frac{\partial\phi}{\partial t}+\nabla\cdot(\mathbf{v}\phi)\right]d\Omega$$

$$=\int_{\Omega(t)}\left(\frac{D\phi}{Dt}+\phi\nabla\cdot\mathbf{v}\right)d\Omega$$

The Principle of Conservation of Mass in Spatial Description

If a continuous medium of density ρ fills the volume $\Omega(t)$ at time t, the rate of increase of the total mass inside the volume is

$$\int_{\Omega} \frac{\partial \rho}{\partial t} d\Omega$$

The rate of mass outflow through the surface element $d\Gamma$ is $\rho v_n d\Gamma$ where $v_n = \mathbf{v} \cdot \hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ is the outward normal). Hence, the rate of inflow through the entire surface Γ is

$$-\oint_{\Gamma} \rho v_n \, d\Gamma = -\oint_{\Gamma} \rho \mathbf{v} \cdot \hat{\mathbf{n}} \, d\Gamma = -\int_{\Omega} \nabla \cdot \left(\rho \mathbf{v}\right) d\Omega$$

The Principle of Conservation of Mass

(continued)

If no mass is created or destroyed inside the volume $\Omega(t)$, this must be equal to rate of mass inflow through the surface. Equating the rate of mass inflow to the rate of increase of mass, we obtain

$$\int_{\Omega} \frac{\partial \rho}{\partial t}\, d\Omega = -\int_{\Omega} \nabla \cdot (\rho \mathbf{v})\, d\Omega \Rightarrow \int_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] d\Omega = 0$$

In we shrink the volume to a point, we obtain

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ \text{ or } \ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Conservation of Mass (continued)

This is the invariant form (i.e., valid in any coordinate system and dimension) of the statement of the principle of conservation of mass, also known as the **continuity equation** (this author does not prefer this name).

- **Steady-state flows** $\left(\dfrac{\partial \rho}{\partial t} = 0\right)$**:**

$$0 = \nabla \cdot (\rho \mathbf{v})$$

- **Incompressible fluids** $\left(\dfrac{D\rho}{Dt} = 0\right)$**:**

$$0 = \nabla \cdot \mathbf{v}$$

Conservation of Mass (continued)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

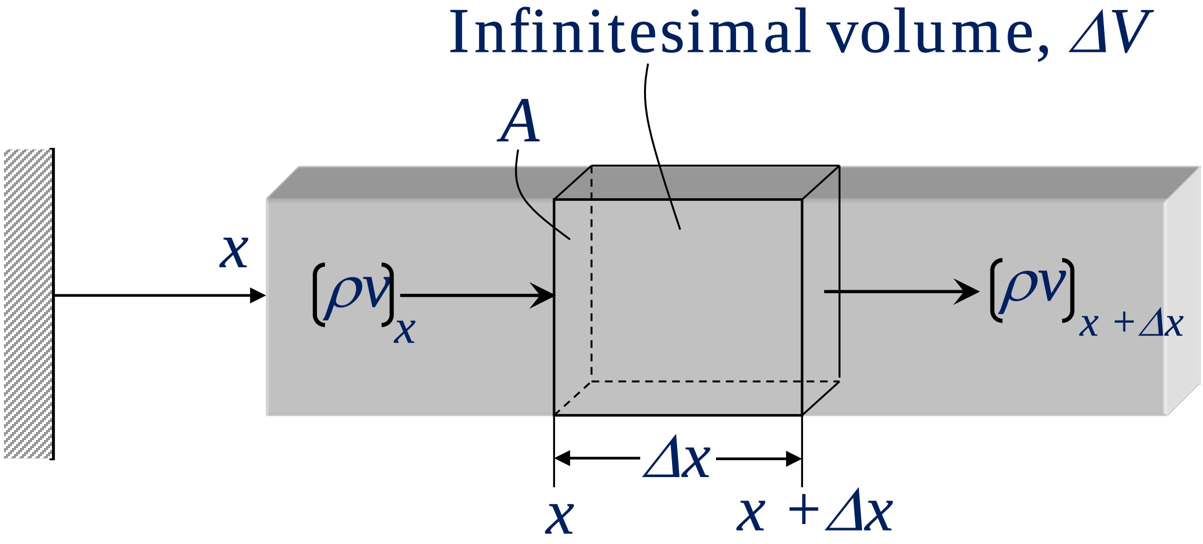
Cartesian component form

$$\nabla \cdot (\rho \mathbf{v}) = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \cdot \left(\rho v_x \hat{\mathbf{i}} + \rho v_y \hat{\mathbf{j}} + \rho v_z \hat{\mathbf{k}} \right)$$

$$= \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)$$

$$\boxed{\frac{\partial \rho}{\partial \mathrm{t}} + \frac{\partial \left(\rho v_x\right)}{\partial x} + \frac{\partial \left(\rho v_y\right)}{\partial y} + \frac{\partial \left(\rho v_z\right)}{\partial z} = 0}$$

The Principle of Conservation of Mass in one dimension



If no mass is created or destroyed inside ΔV, the time rate of change of mass must equal to the rate of inflow of mass through the surface.

The Principle of Conservation of Mass in one dimension (continued)

$$\frac{(\rho A)_{t+\Delta t}\,\Delta x-(\rho A)_t\,\Delta x}{\Delta t}=(\rho v)_x\,A-(\rho v)_{x+\Delta x}\,A$$

$$\frac{(\rho A)_{t+\Delta t}-(\rho A)_t}{\Delta t}=\frac{(\rho v)_x\,A-(\rho v)_{x+\Delta x}\,A}{\Delta x}$$

$$\frac{\partial\rho}{\partial t}=-\frac{\partial(\rho v)}{\partial x}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0$$

Conservation of Mass (continued)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

Cylindrical component form

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\left[\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + r\frac{\partial v_z}{\partial z}\right]$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial\left(\rho v_r r\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho v_\theta\right)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0}$$

Alternative form of
REYNOLDS' TRANSPORT THEOREM

Replace the function $\phi(\mathbf{x},t)$ with $\rho(\mathbf{x},t)\phi(\mathbf{x},t)$ in the Reynolds transport theorem and obtain

$$\frac{D}{Dt}\int_{\Omega(t)}\rho(\mathbf{x},t)\phi(\mathbf{x},t)d\Omega = \int_{\Omega(t)}\left(\frac{D\left(\rho\phi\right)}{Dt}+\rho\phi\nabla\cdot\mathbf{v}\right)d\Omega$$

$$= \int_{\Omega(t)}\left[\rho\frac{D\phi}{Dt}+\phi\left(\frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{v}\right)\right]d\Omega = \int_{\Omega(t)}\rho\frac{D\phi}{Dt}\,d\Omega$$

where the equation resulting from conservation of mass is used in arriving at the last step. We have

$$\frac{D}{Dt}\int_{\Omega}\rho(\mathbf{x},t)\phi(\mathbf{x},t)d\Omega = \int_{\Omega}\rho\frac{D\phi}{Dt}\,d\Omega$$

CONSERVATION OF MASS:
Material Description

Let Ω_0 be an arbitrary material volume occupied by *the body* in the reference configuration, and Ω be the volume occupied by the body in the current configuration.

Conservation of mass states that if the mass is neither created nor destroyed during the motion from Ω_0 to Ω, *the mass of the material volume is conserved:*

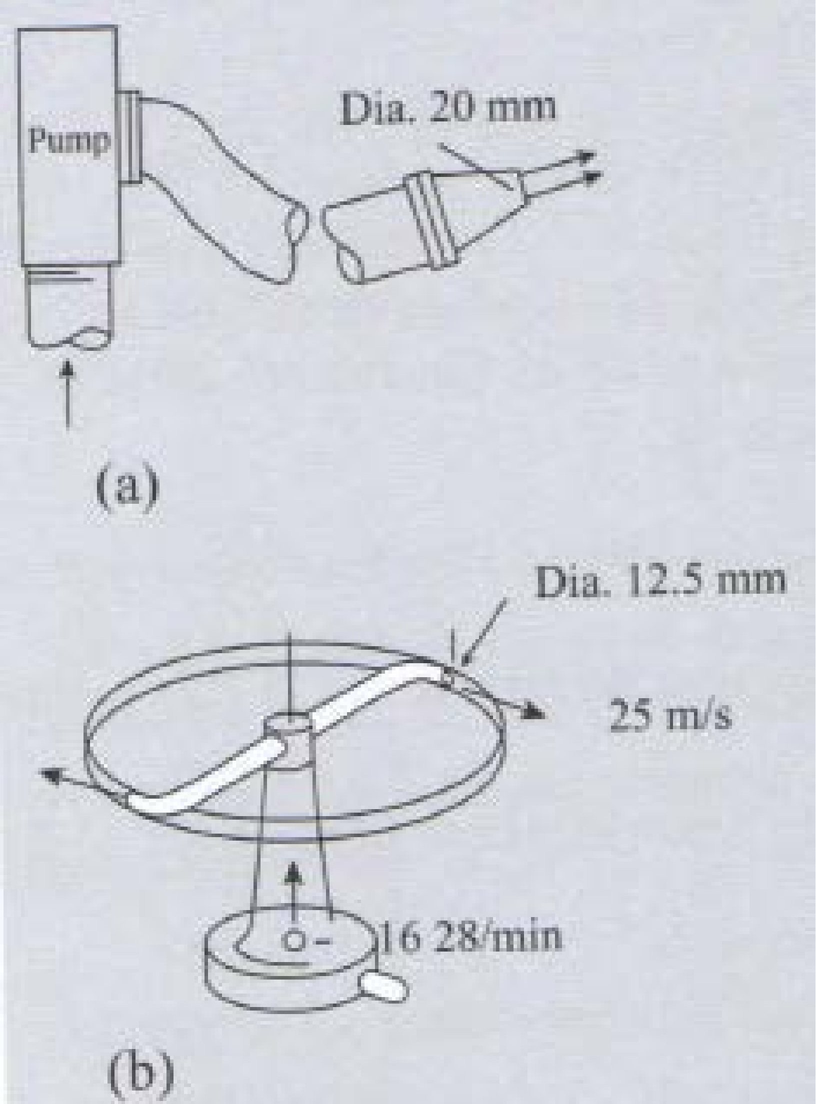
$$\int_{\Omega_0} \rho_0 \, d\Omega_0 = \int_{\Omega} \rho \, d\Omega$$

Since the volumes in the two configurations are related by $d\Omega = J \, d\Omega_0$, we obtain

$$\boxed{\rho_0 = J\rho}$$

AN EXAMPLE

Problem statement:
Consider a water hose with conical-shaped nozzle at its end, as shown in the figure.
(a) Determine the pumping capacity required in order The velocity of the water (assuming incompressible for the present case) exiting the nozzle be 25 m/s.
(b) If the hose is connected to a rotating sprinkler through its base, determine the average speed of the water Leaving the sprinkler nozzle.



AN EXAMPLE

Solution:

(a) The principle of conservation of mass for steady one-dimensional flow requires $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

If the exit of the nozzle is taken as the section 2, we can calculate the flow at section 1 as (for an incompressible fluid, $\rho_1 = \rho_2$)

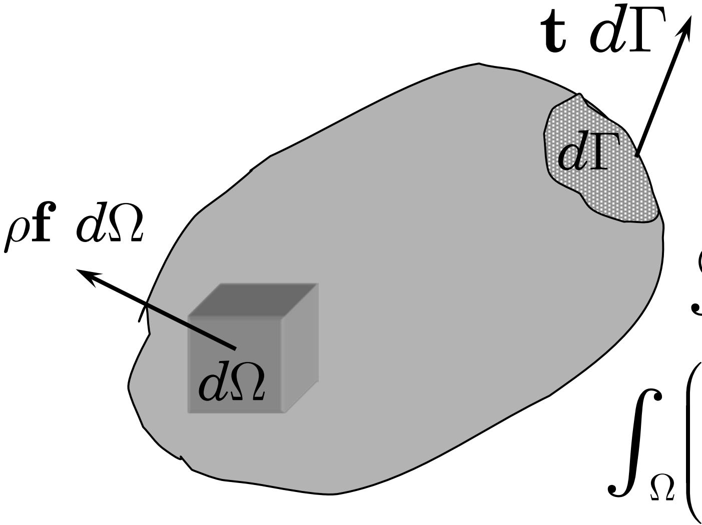
$$Q_1 = A_1 v_1 = A_2 v_2 = \frac{\pi(20\times 10^{-3})^2}{4} 25 = 0.0025\pi \text{ m}^3 / \text{s}.$$

(b) The average speed of the water leaving the sprinkler nozzle can be calculated using the principle of conservation of mass for steady one-dimensional flow. We obtain

$$Q_1 = 2A_2 v_2 \;\; \rightarrow \;\; v_2 = \frac{2Q_1}{\pi d^2} = \frac{0.005}{(12.5\times 10^{-3})^2} = 32 \text{ m/s}.$$

BALANCE OF LINEAR MOMENTUM

The time rate of change of total linear momentum of a given continuum equals the vector sum of all external forces acting on the continuum. This also known as **Newton’s Second Law**.



$$\oint_\Gamma \mathbf{t}\, d\Gamma + \int_\Omega \rho\mathbf{f}\, d\Omega = \frac{D}{Dt}\int_\Omega \rho\mathbf{v}\, d\Omega$$

$$\oint_\Gamma \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\, d\Gamma + \int_\Omega \rho\mathbf{f}\, d\Omega = \int_\Omega \rho\frac{D\mathbf{v}}{Dt} d\Omega$$

$$\int_\Omega \left(\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f} - \rho\frac{D\mathbf{v}}{Dt}\right) d\Omega = 0$$

Newton's First Law. Newton's First Law states that an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force.

Conservation of Linear Momentum
(continued)

Vector form of the equation of motion

$$\boxed{\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}}$$

Cartesian Component Form

$$\left(\hat{\mathbf{e}}_k \frac{\partial}{\partial x_k}\right) \cdot \left(\boldsymbol{\sigma}_{ji} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i\right) + \rho f_i \hat{\mathbf{e}}_i = \rho \frac{D(v_i \hat{\mathbf{e}}_i)}{Dt}$$

$$\frac{\partial \sigma_{ji}}{\partial x_j} \hat{\mathbf{e}}_i + \rho f_i \hat{\mathbf{e}}_i = \rho \frac{Dv_i}{Dt} \hat{\mathbf{e}}_i \ \Rightarrow \ \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{Dv_i}{Dt}$$

Conservation of Linear Momentum

(continued)

Cartesian Component Form (expanded form)

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{Dv_i}{Dt}$$

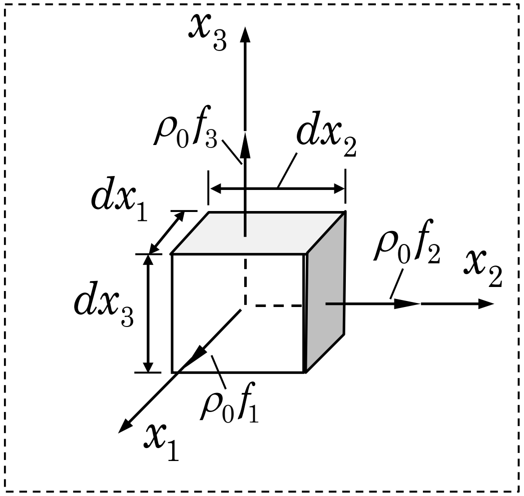
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1 = \rho \frac{Dv_1}{Dt}$$

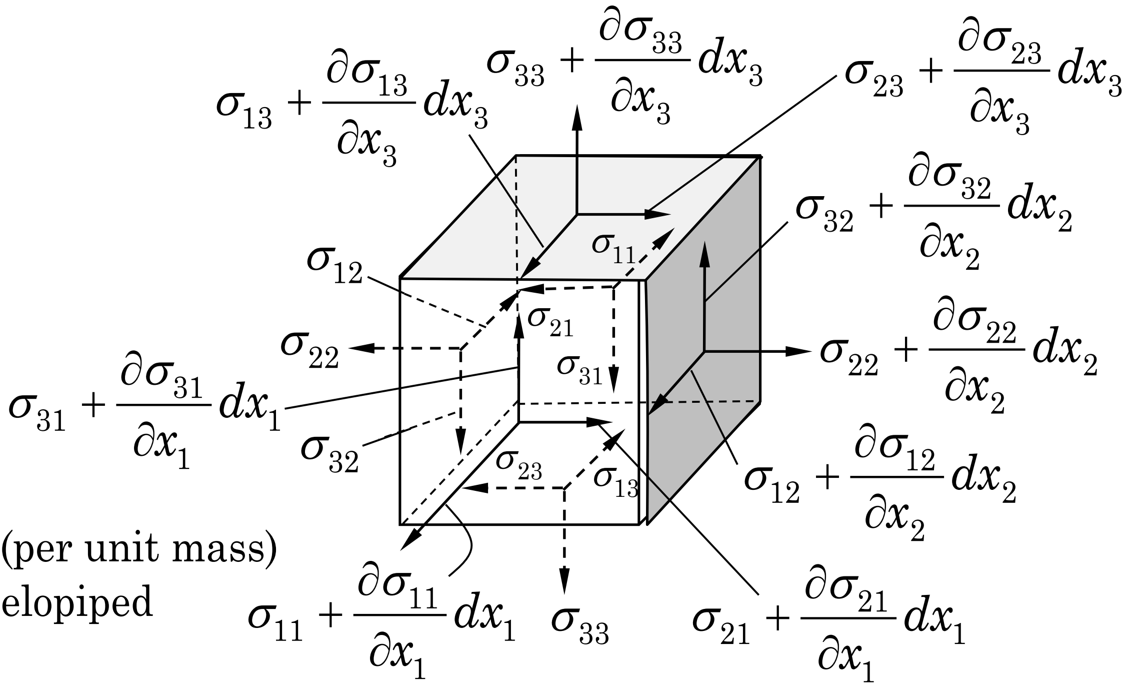
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho f_2 = \rho \frac{Dv_2}{Dt}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_3 = \rho \frac{Dv_3}{Dt}$$

EQUATIONS OF MOTION
in Rectangular Cartesian System

Alternative approach to the derivation: Sum all the forces on the infinitesimal block of dimensions $dx_1,\ dx_2$, and dx_3





$\rho_0 f_1,\ \rho_0 f_2,\ \rho_0 f_3$ = body force components (per unit mass)

Origin is at the center of the parallelopiped

Conservation of Linear Momentum

Special Cases

0, for steady state

$$\rho\mathbf{f}+\nabla\cdot\boldsymbol{\sigma}=\rho\left(\cancel{\frac{\partial\mathbf{v}}{\partial t}}+\cancel{\mathbf{v}}\cdot\nabla\mathbf{v}\right)$$

0, for solid bodies

Fluid Mechanics

Viscous stress tensor

$$\rho\mathbf{f}+\nabla\cdot\boldsymbol{\sigma}=\rho\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right),\qquad \boldsymbol{\sigma}=\boldsymbol{\tau}-P\mathbf{I}$$

Hydrostatic pressure

Solid Mechanics

$$\rho\mathbf{f}+\nabla\cdot\boldsymbol{\sigma}=\rho\frac{\partial\mathbf{v}}{\partial t}$$

EQUATIONS OF MOTION

Fluid Mechanics

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1 = \rho \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho f_2 = \rho \left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right)$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_3 = \rho \left(\frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho f_x = \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho f_y = \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right)$$

EQUATIONS OF MOTION
in cylindrical coordinates

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + \rho f_r$$

$$= \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho f_\theta$$

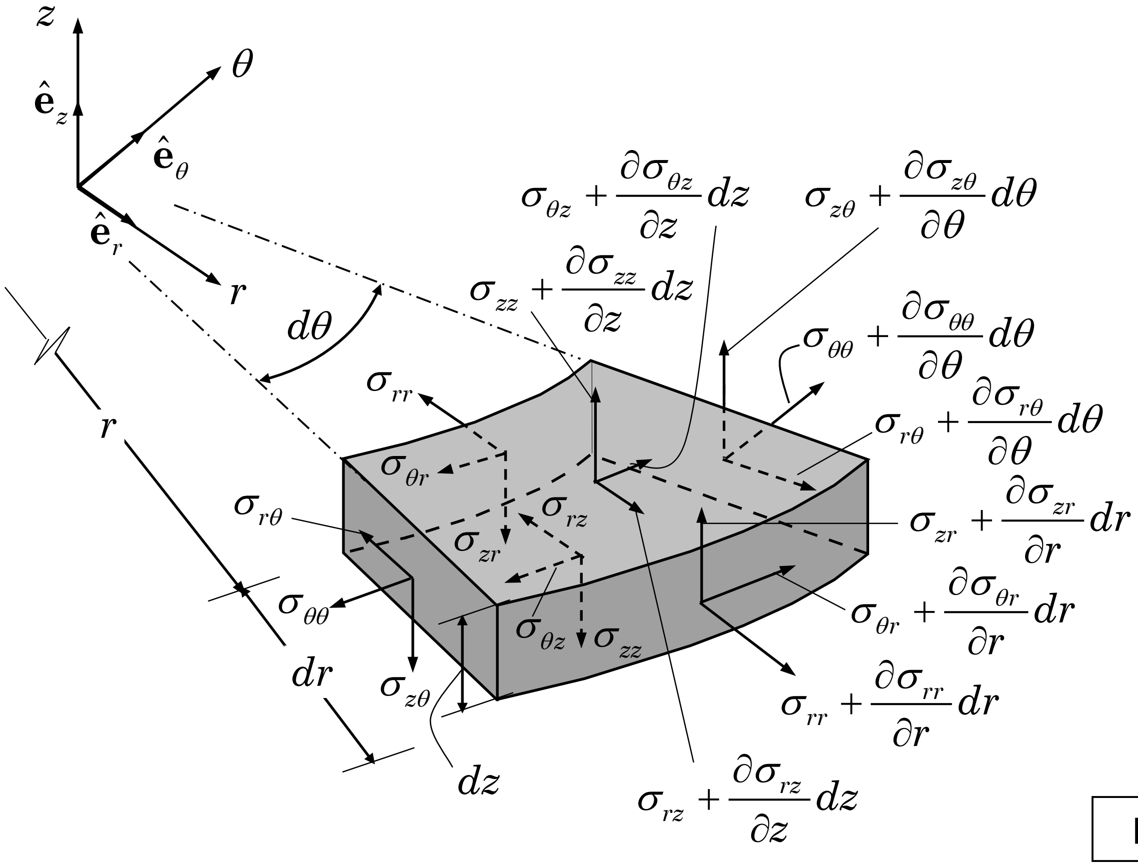
$$= \rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho f_z$$

$$= \rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

EQUATIONS OF MOTION
in cylindrical coordinates

Alternative approach to the derivation: Sum all the forces on the infinitesimal block of dimensions $dr,\ rd\theta$, and dz



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EQUATIONS OF MOTION

in material description

$$\int_{\Omega} \rho \mathbf{f}(\mathbf{x}) d\Omega = \int_{\Omega_0} \rho_0 \mathbf{f}(\mathbf{X}) d\Omega_0,$$

$$\oint_{\Gamma} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, d\Gamma = \oint_{\Gamma_0} \mathbf{P} \cdot \widehat{\mathbf{N}} d\Gamma_0 = \int_{\Omega_0} \nabla_0 \cdot \mathbf{P}^{\mathrm{T}} d\Omega_0,$$

$$\frac{D}{Dt} \int_{\Omega} \rho \mathbf{v} d\Omega = \frac{\partial}{\partial t} \int_{\Omega_0} \rho_0 \frac{\partial \mathbf{u}}{\partial t} d\Omega_0 = \int_{\Omega_0} \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} d\Omega_0$$

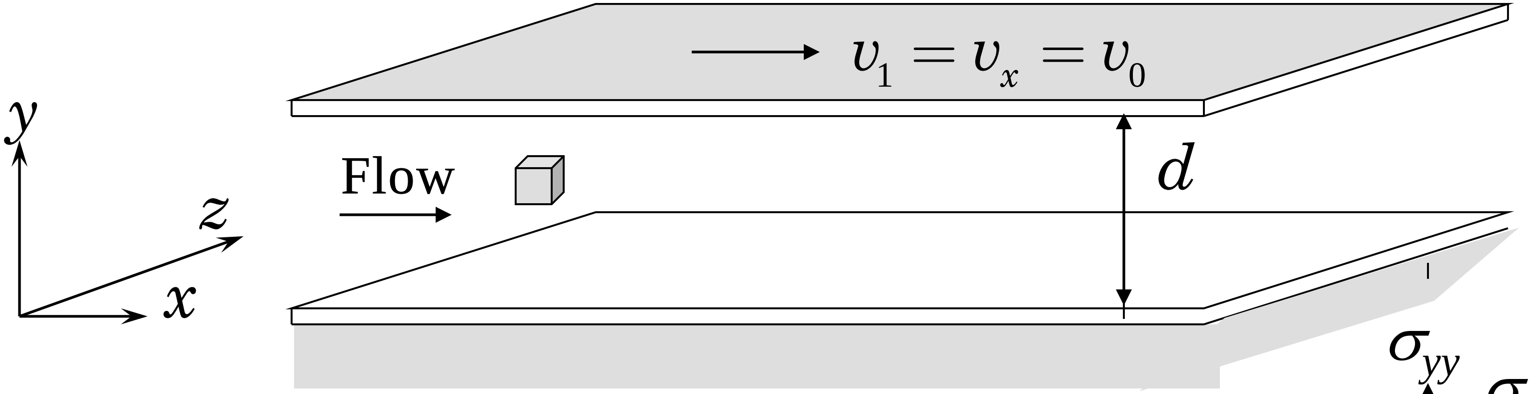
$$\boldsymbol{\sigma} \cdot d\mathbf{a} = \mathbf{P} \cdot d\mathbf{A}, \quad dv = J\, dV \quad (\text{or } d\Omega = J\, d\Omega_0).$$

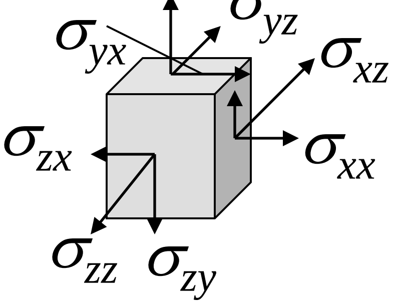
$$\nabla_0 \cdot \mathbf{P}^{\mathrm{T}} + \rho_0 \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

$$\nabla_0 \cdot \left(\mathbf{S}^{\mathrm{T}} \cdot \mathbf{F}^{\mathrm{T}} \right) + \rho_0 \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

EXAMPLE: COUETTE FLOW

Flow of a viscous fluid between parallel plates (**Couette Flow**)





Assume: $v_2 = 0,\ v_3 = 0$

steady state: $\dfrac{\partial(\cdot)}{\partial t} = 0$

incompressible: $\nabla.\mathbf{v} = 0 \longrightarrow \dfrac{\partial v_1}{\partial x_1} = 0 \longrightarrow v_1 = f(x_2)$

$$v_x = f(y)$$

no body forces: $f_1 = f_2 = f_3 = 0$

COUETTE FLOW (continued)

For viscous fluids, the total stresses are given by

$$\sigma_{xx} = \tau_{xx} - P, \ \ \sigma_{yy} = \tau_{yy} - P, \ \ \sigma_{zz} = \tau_{zz} - P$$

$$\sigma_{xy} = \tau_{xy}, \quad \sigma_{xz} = \tau_{xz}, \quad \sigma_{yz} = \tau_{yz}$$

The viscous stresses σ_{ij} are proportional to the gradient of the velocity field, and they are independent of *x* and *z*.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho f_x = \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho f_y = \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right)$$

COUETTE FLOW (continued)

Thus the linear momentum equations become

$$-\frac{\partial P}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}=0, \quad -\frac{\partial P}{\partial y}=0, \quad -\frac{\partial P}{\partial z}=0$$

This implies that the pressure *P* is only function of *x*. Thus, we have one equation

$$\boxed{\frac{\partial P}{\partial x}=\frac{\partial \sigma_{yx}}{\partial y}}$$

Thus we have

$$\boxed{\frac{\partial \sigma_{yx}}{\partial y}=\mu\frac{\partial^2 v_x}{\partial y^2} \quad \Rightarrow \quad \frac{\partial P}{\partial x}=\mu\frac{\partial^2 v_x}{\partial y^2}}$$

AN EXAMPLE

Problem statement:

Given the following state of stress in a kinematically infinitesimal deformation $(\sigma_{ij} = \sigma_{ji})$,

$$\sigma_{11} = -2x_1^2, \quad \sigma_{12} = -7 + 4x_1x_2 + x_3, \quad \sigma_{13} = 1 + x_1 - 3x_2,$$

$$\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \quad \sigma_{23} = 0, \quad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3$$

determine the body force components for which the stress field describes a state of equilibrium.

Solution: The body force components are

$$\rho f_1 = -\left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3}\right) = -[(-4x_1) + (4x_1) + 0] = 0,$$

$$\rho f_2 = -\left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right) = -[(4x_2) + (-4x_2) + 0] = 0,$$

$$\rho f_3 = -\left(\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}\right) = -[1 + 0 + 3] = -4.$$

EXERCISES

1. Determine if the following velocity fields for an incompressible flow satisfies the continuity equation:

$$v_1(x_1,x_2) = -\frac{x_1}{r^2}, \quad v_2(x_1,x_2) = -\frac{x_2}{r^2} \quad \text{where} \quad r^2 = x_1^2 + x_2^2$$

2. The velocity distribution between two parallel plates separated by distance b is

$$v_x(y) = \frac{y}{b}v_0 - c\frac{y}{b}\left(1 - \frac{y}{b}\right), \; v_y = 0, \; v_z = 0, \; 0 < y < b,$$

where y is measured from and normal to the bottom plate, x is taken along the plates, v_x is the velocity component parallel to the plates, v_0 is the velocity of the top plate in the x direction, and c is a constant. Determine if the velocity field satisfies the continuity equation and find the volume rate of flow and the average velocity.

EXERCISES

3. If the stress field in a body has the following components in a rectangular Cartesian coordinate system

$$[\sigma] = a\begin{bmatrix} x_1^2 x_2 & (b^2 - x_2^2)x_1 & 0 \\ (b^2 - x_2^2)x_1 & \frac{1}{3}(x_2^2 - 3b^2)x_2 & 0 \\ 0 & 0 & 2bx_3^2 \end{bmatrix}$$

where a and b constants, determine the body force components necessary for the body to be in equilibrium.

4. For a cantilevered beam bent by a point load at the free end, the bending moment M about the y-axis is given by $M = -Px$. The axial stress is given by

$$\sigma_{xx} = \frac{Mz}{I} = -\frac{Pxz}{I},$$

where I is the moment of inertia of the cross section about the y-axis. Starting with this equation, use the two-dimensional equilibrium equations to determine stresses and σ_{zz} and σ_{xz} as functions of x and z.