STRESSES IN A CONTINUUM

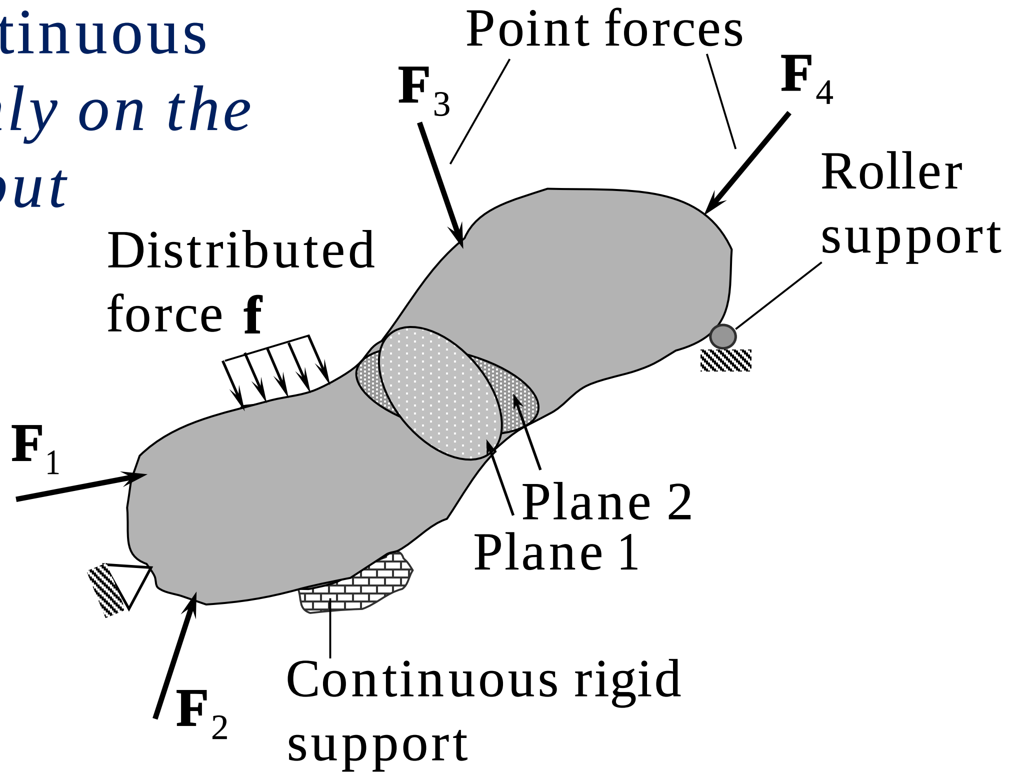
- Need for stresses
- Stress vector
- Cauchy’s formula-1
- Cauchy’s formula-2
- Derivation of 2-D Cauchy’s formula
- Cauchy stress tensor
- Principal values of stress
- Stress transformation relations
- Other measures of stress

NEED FOR STRESSES

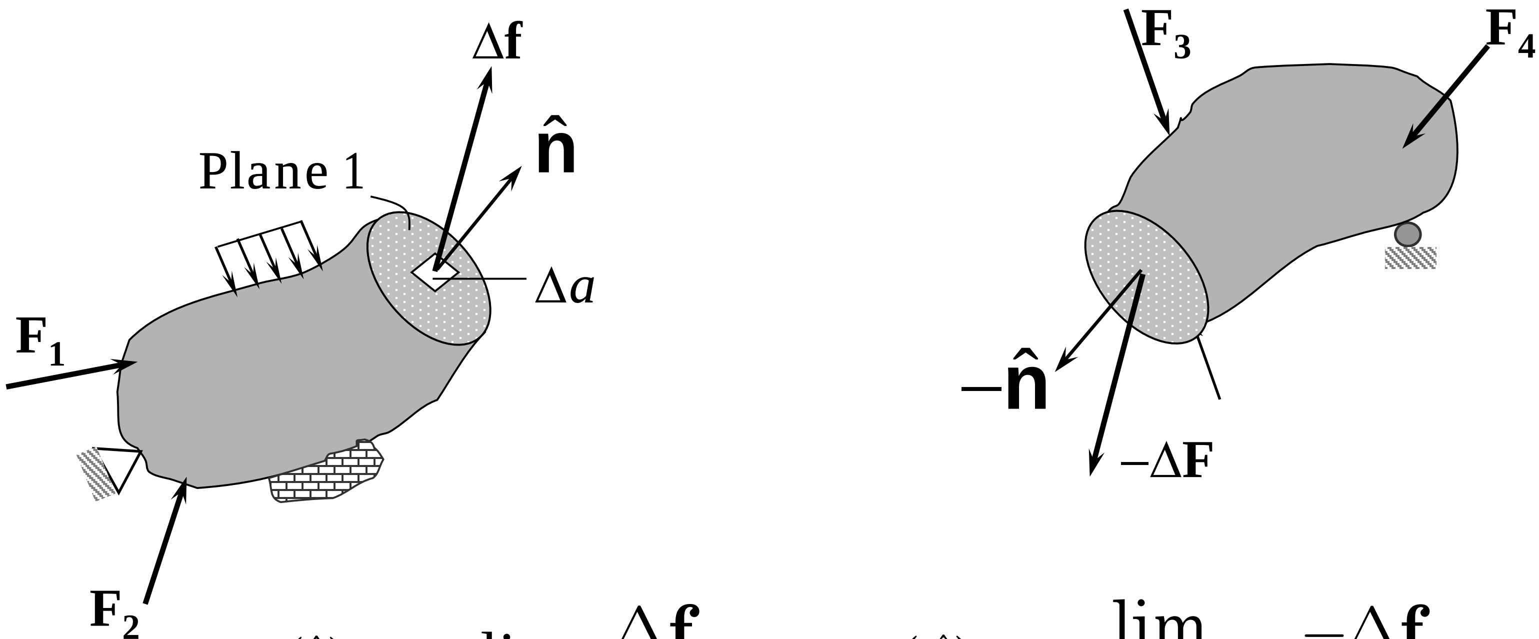
- All materials have certain thresholds to withstand forces, beyond which they “fail” to perform their intended function.
- The force per unit area, called *stress*, is a measure of the capacity of the material to carry loads, and all designs are based on the criterion that the materials used have the capacity to carry the working loads of the system.
- The stress at a point in a three-dimensional continuum can be measured in terms of nine quantities, three per plane, on three mutually perpendicular planes at the point. These nine quantities may be viewed as the components of a second-order tensor, called a *stress tensor*.

STRESSES IN A CONTINUUM

- The true stress is the force in the deformed configuration measured per unit area of the deformed configuration κ.
- The surface force acting on an element of area in a continuous medium *depends not only on the* *magnitude* *of the area but also on the* *orientation* *of the area*.
- The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the direction of travel along the boundary.



Stress Vector

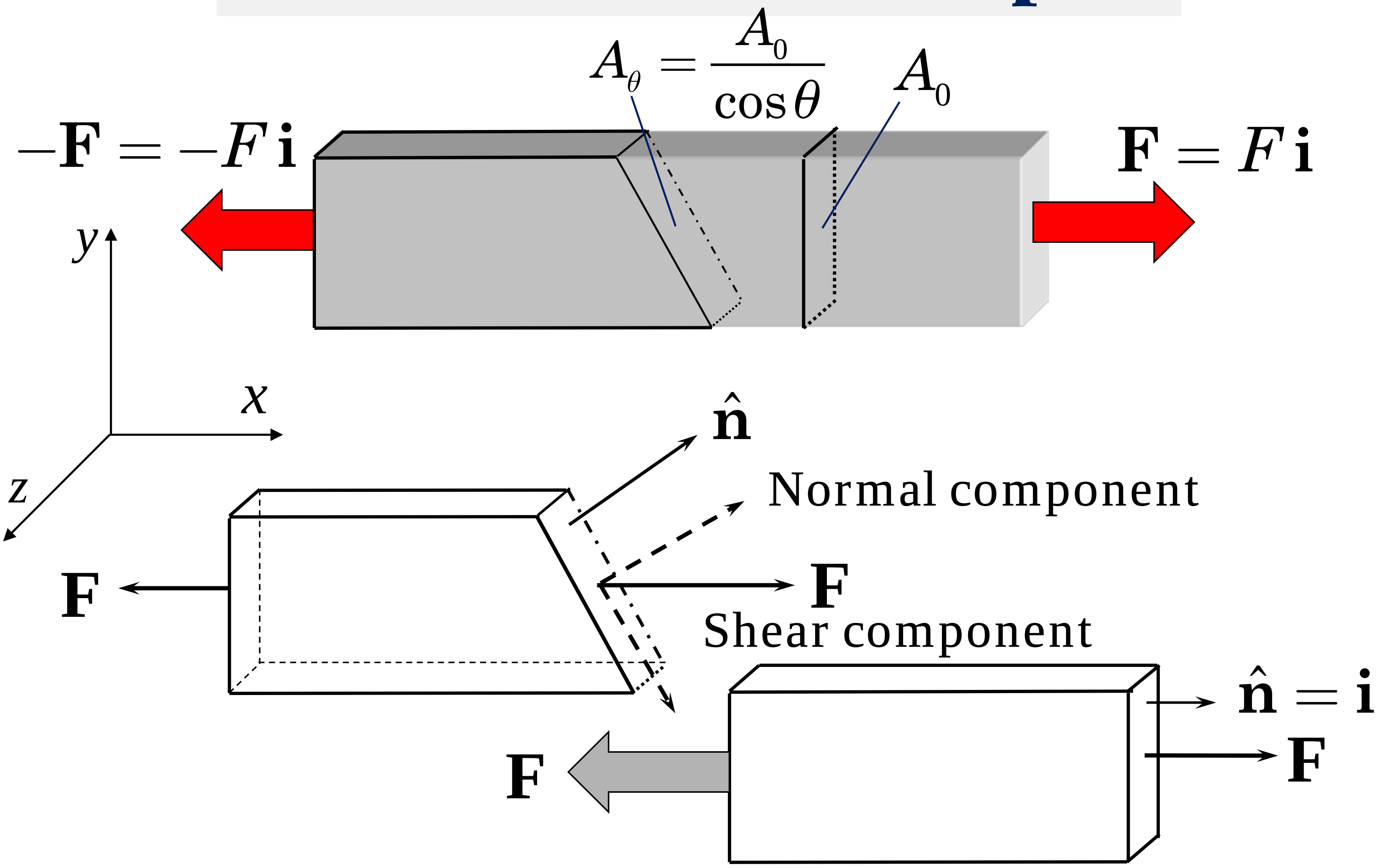


$$\mathbf{t}^{(\hat{\mathbf{n}})} = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}}{\Delta a} \qquad \mathbf{t}^{(-\hat{\mathbf{n}})} = \lim_{\Delta a \to 0} \frac{-\Delta \mathbf{f}}{\Delta a}$$

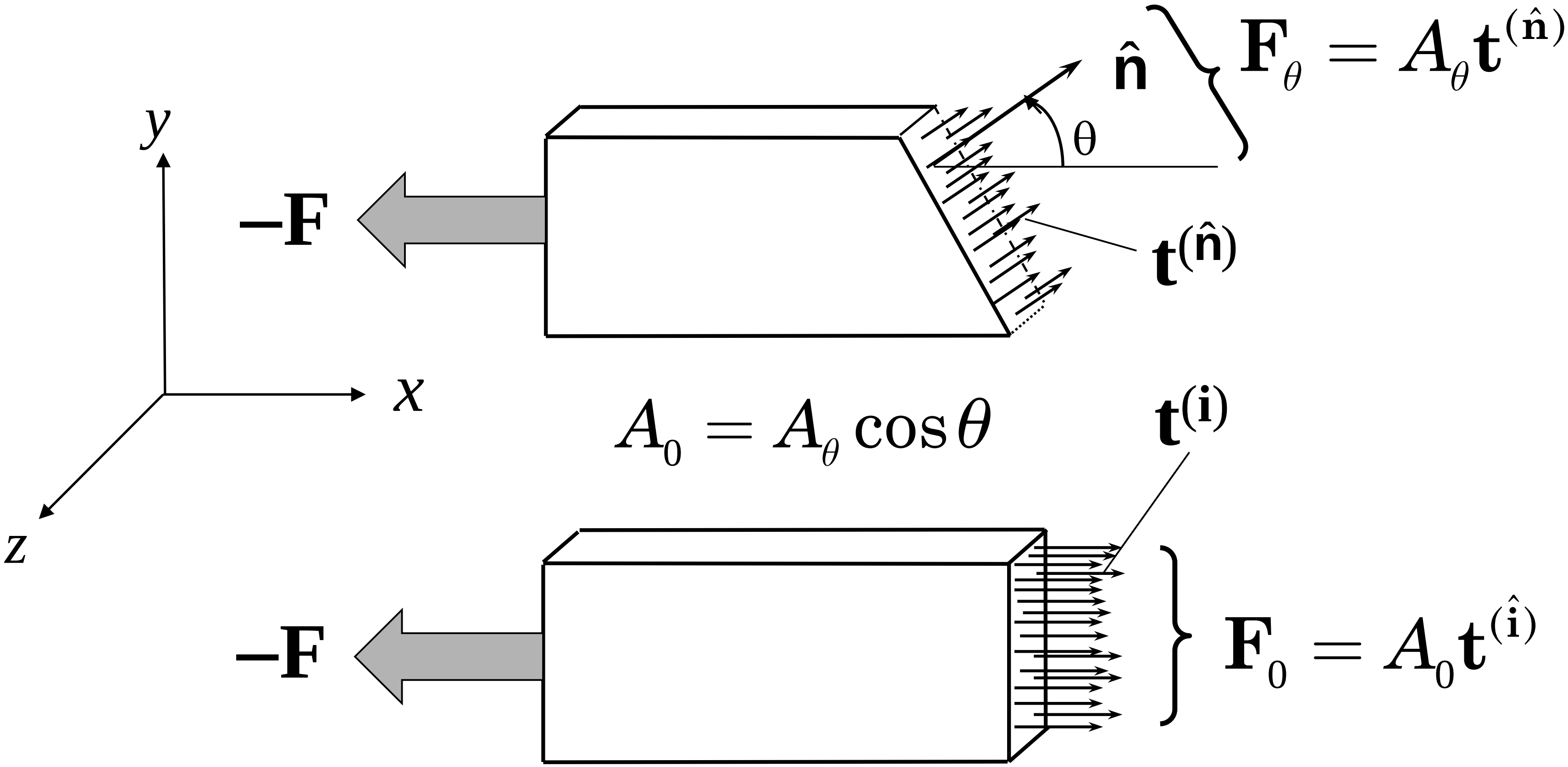
$$\mathbf{t}^{(\hat{\mathbf{n}})} = -\mathbf{t}^{(-\hat{\mathbf{n}})}$$ or $$\mathbf{t}^{(-\hat{\mathbf{n}})} = -\mathbf{t}^{(\hat{\mathbf{n}})}$$

Cauchy's Lemma

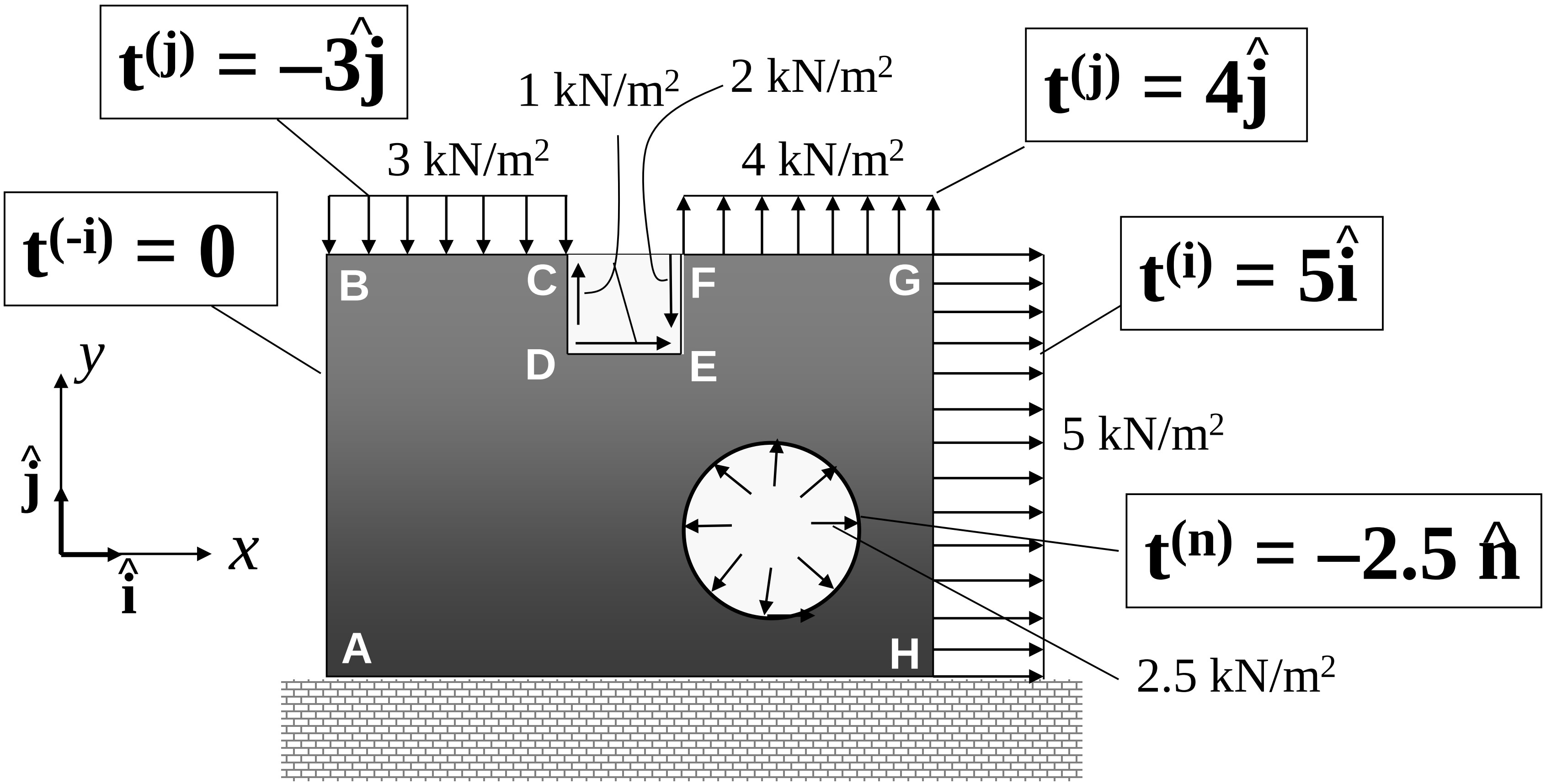
Traction Vector: Examples



Traction Vector: Examples



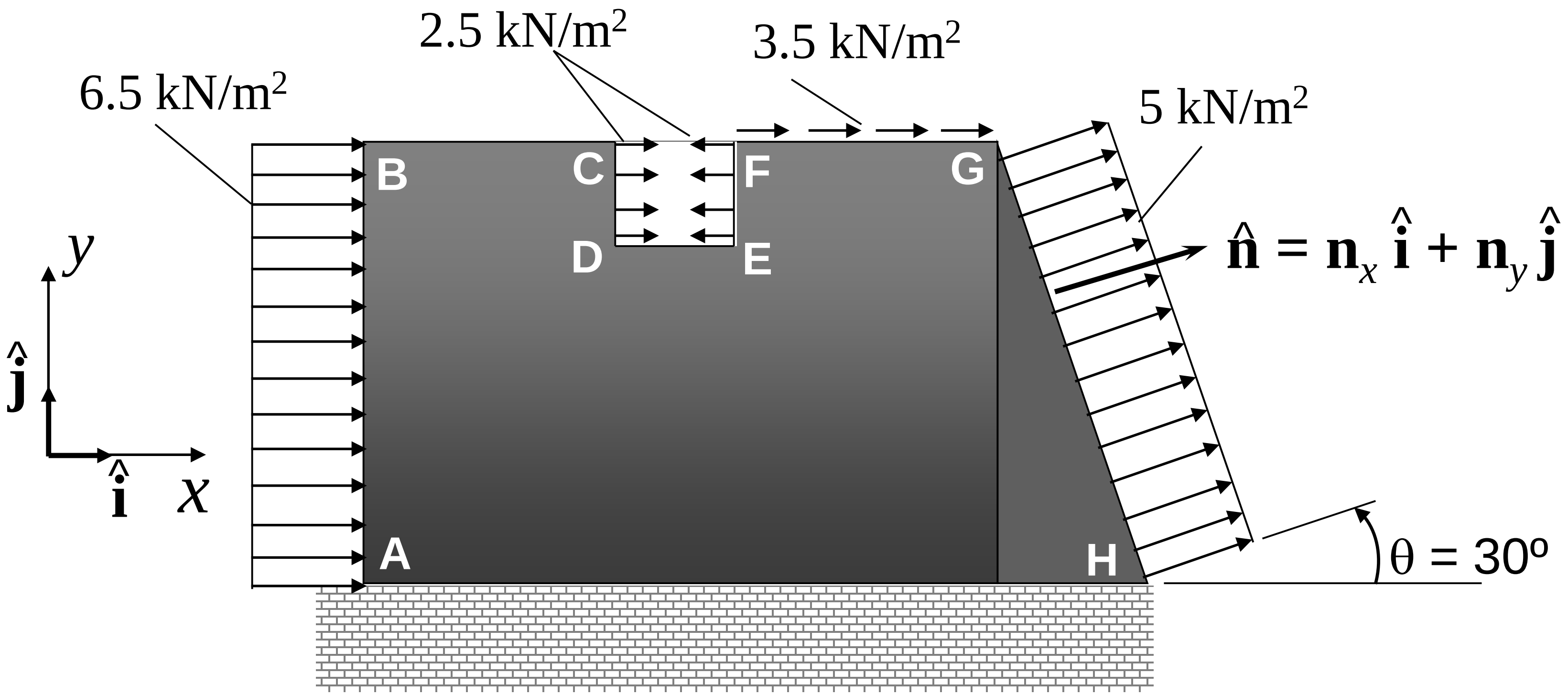
A TWO-DIMENSIONAL EXAMPLE



AN EXERCISE

Problem statement:

Write the traction vectors in terms of the magnitude of stress and the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ on each side of the block shown below.



AN EXERCISE

Solution (in kN/m²)

On AB: $\mathbf{t}^{(-\mathbf{i})} = 6.5\hat{\mathbf{i}}$

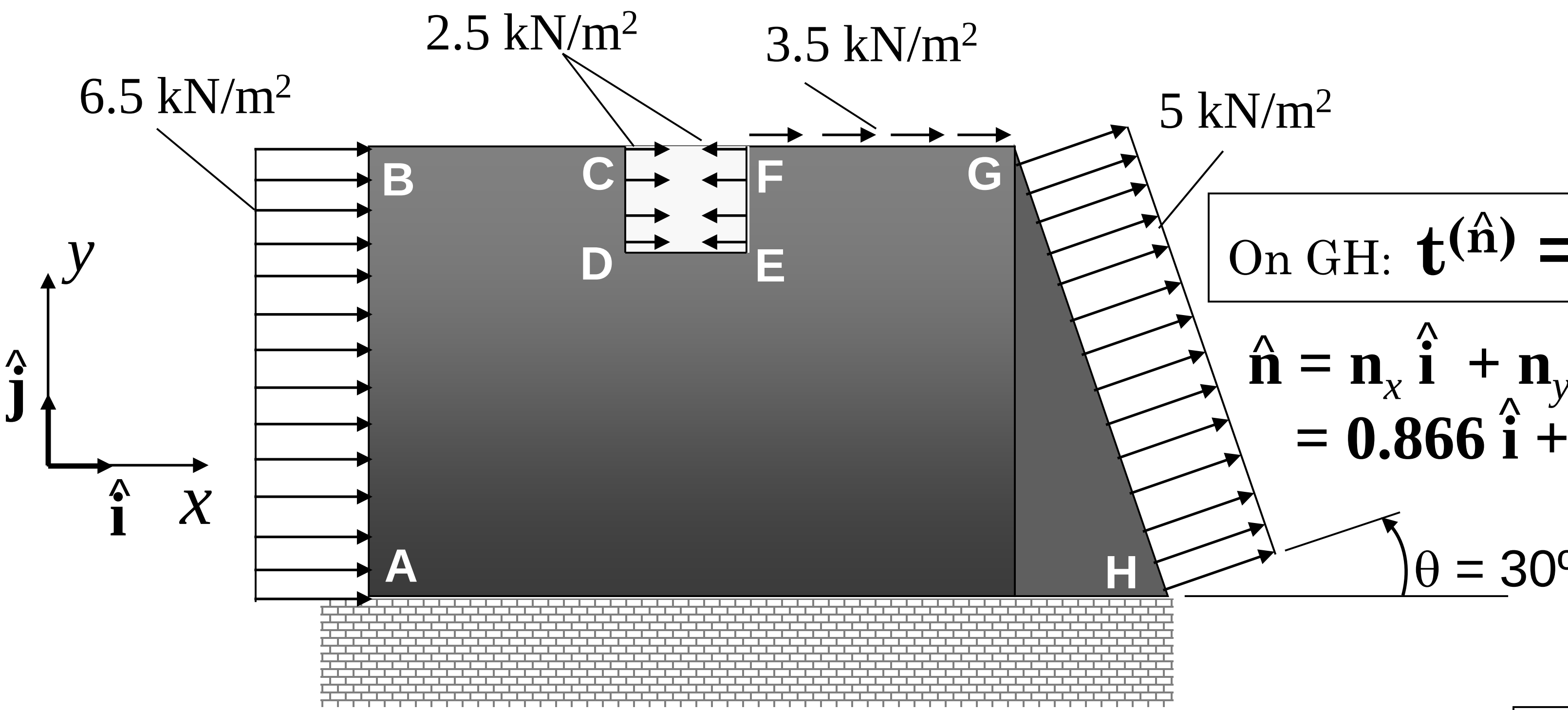
On BC: $\mathbf{t}^{(\mathbf{j})} = 0$

On CD: $\mathbf{t}^{(\mathbf{i})} = 2.5\,\hat{\mathbf{i}}$

On DE: $\mathbf{t}^{(\mathbf{j})} = 0$

On EF: $\mathbf{t}^{(-\mathbf{i})} = -2.5\,\hat{\mathbf{i}}$

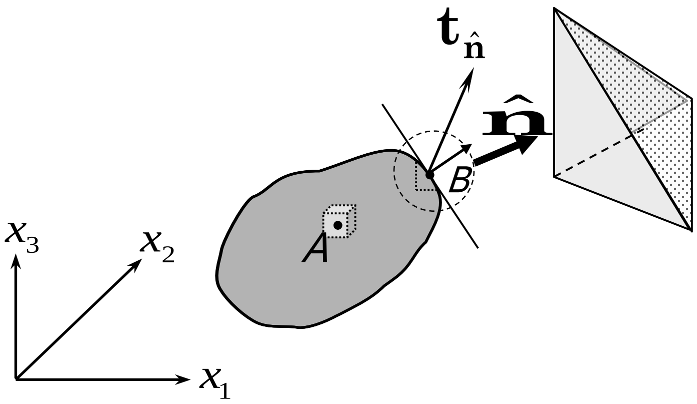
On FG: $\mathbf{t}^{(\mathbf{j})} = 3.5\hat{\mathbf{i}}$

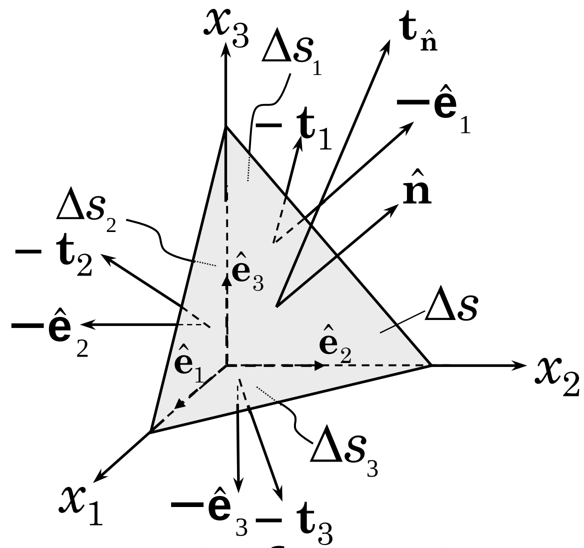


On GH: $\mathbf{t}^{(\hat{\mathbf{n}})} = 5\hat{\mathbf{n}}$

$$\hat{\mathbf{n}} = \mathbf{n}_x\,\hat{\mathbf{i}} + \mathbf{n}_y\,\hat{\mathbf{j}}$$
$$= 0.866\,\hat{\mathbf{i}} + 0.5\,\hat{\mathbf{j}}$$

CAUCHY'S FORMULA-1





$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3\cdot\hat{\mathbf{n}}) + \rho\frac{\Delta h}{3}(\mathbf{a}-\mathbf{f})$$

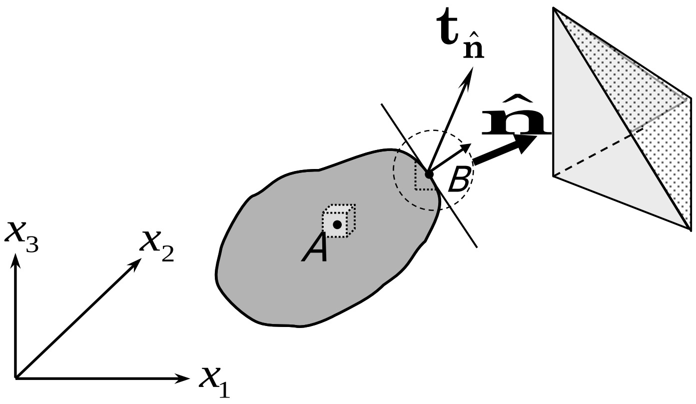
As $\Delta h \to 0$, we obtain

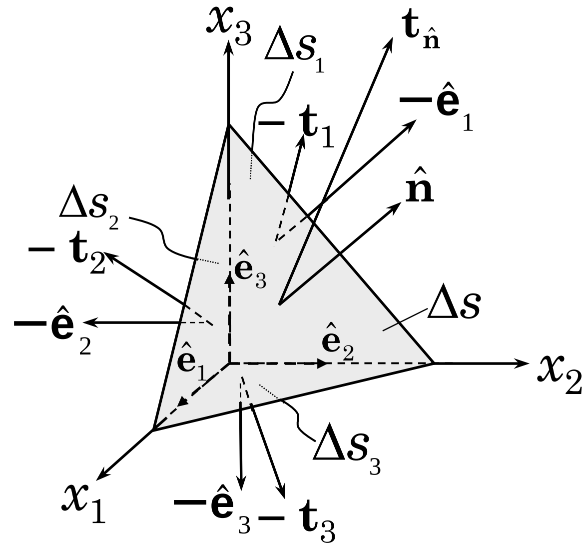
$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3\cdot\hat{\mathbf{n}}) = \mathbf{t}_j\hat{\mathbf{e}}_j\cdot\hat{\mathbf{n}} \equiv \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$$

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \quad \text{and} \quad \boldsymbol{\sigma} = \mathbf{t}_j\hat{\mathbf{e}}_j \quad \left[\; t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}n_j;\; \mathbf{t}_i = \sigma_{ji}\,\hat{\mathbf{e}}_j,\;\; \boldsymbol{\sigma} = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \;\right]$$

plane (i) direction (j)

CAUCHY'S FORMULA-2





$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a}-\mathbf{f})$$

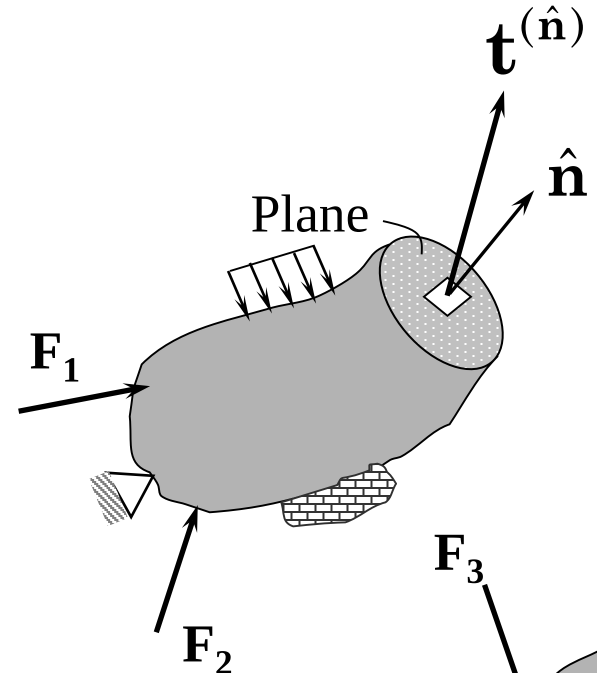
As $\Delta h \to 0$, we obtain

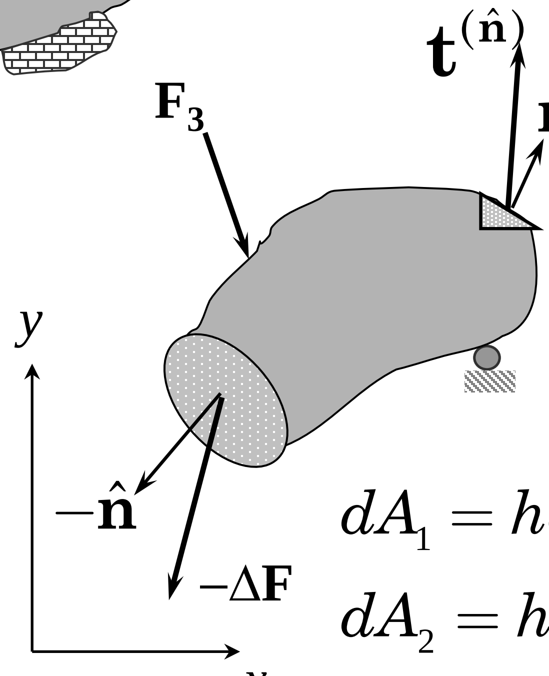
$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i)\mathbf{t}_i \equiv \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$$

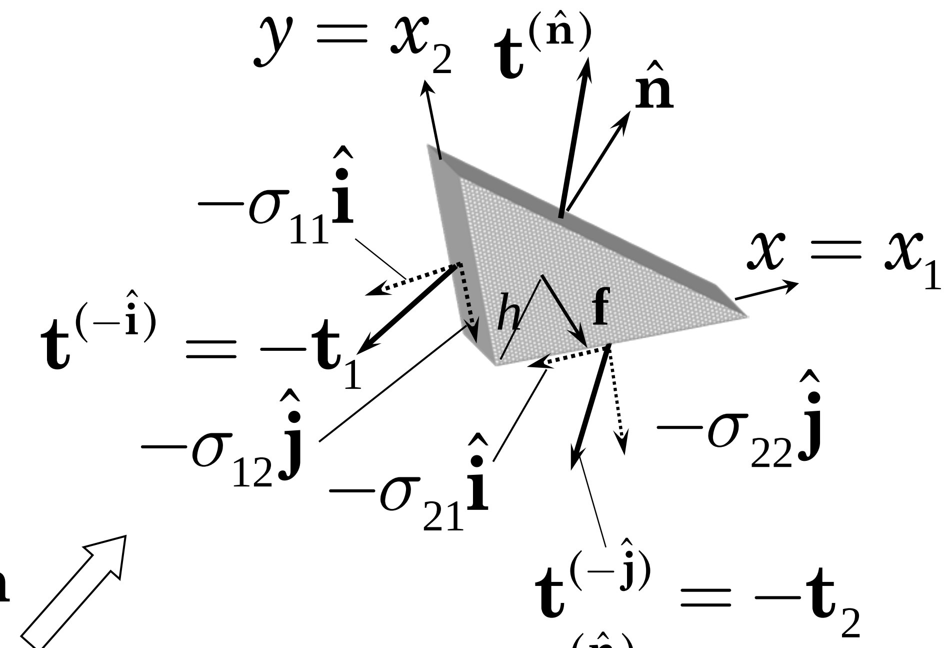
$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}}\cdot\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\sigma} = \hat{\mathbf{e}}_i\mathbf{t}_i \quad \left[t_i^{(\hat{\mathbf{n}})} = n_j\sigma_{ji},\ \mathbf{t}_i = \sigma_{ij}\,\hat{\mathbf{e}}_j\right]$$

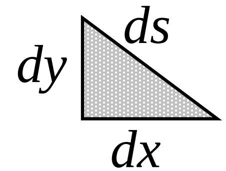
plane (index i) direction (index j)

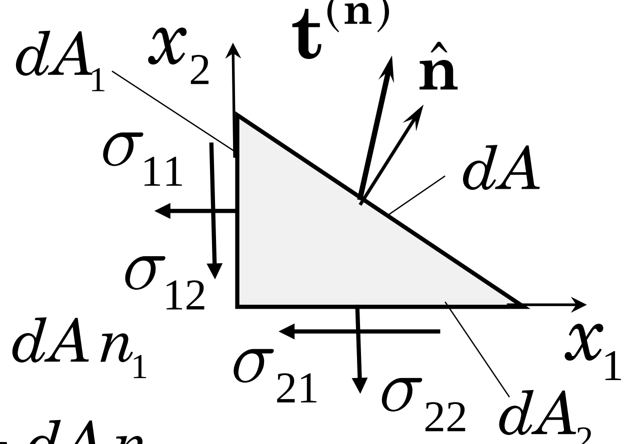
2-D CAUCHY'S FORMULA











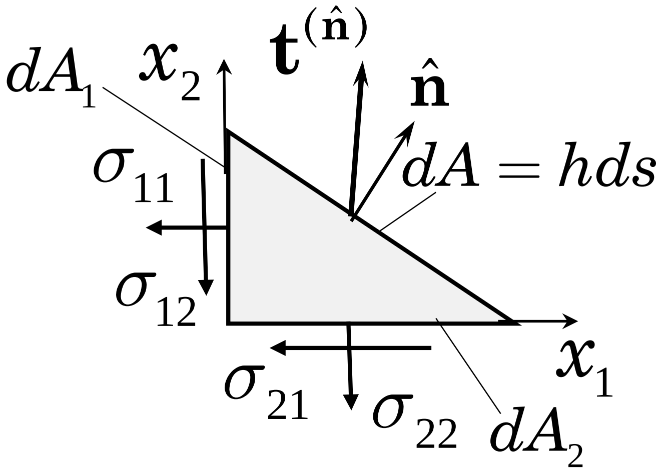
$$dA_1 = h\,ds\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{i}}) = dA\,n_1$$

$$dA_2 = h\,ds\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{j}}) = dA\,n_2$$

2-D CAUCHY'S FORMULA

Summing the forces in the horizontal and vertical directions, we obtain

$$-\left(\sigma_{11} dA_1 + \sigma_{21} dA_2\right) + t_1 dA = 0$$
$$-\left(\sigma_{12} dA_1 + \sigma_{22} dA_2\right) + t_2 dA = 0$$



or

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2$$
$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2$$

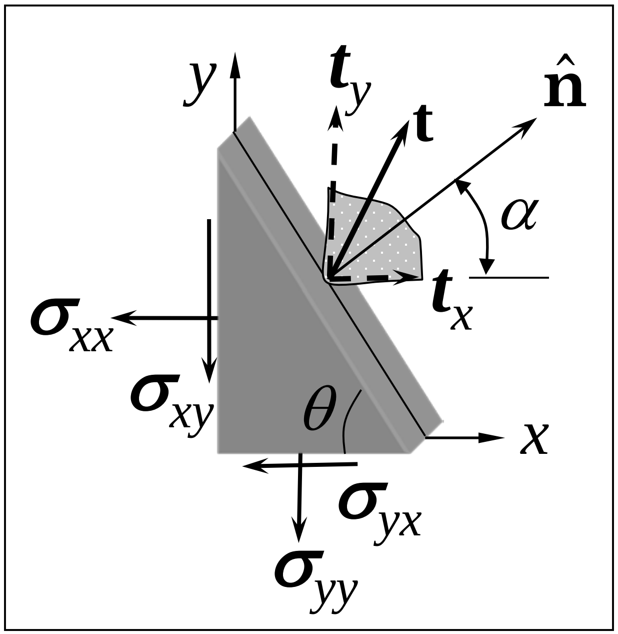
$$dA_1 = dA\, n_1, \quad dA_2 = dA\, n_2$$

3-D Cauchy's formula

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

$$t_i = \sigma_{ji} n_j = n_j \sigma_{ji}$$
$$\mathbf{t} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$$

CAUCHY'S FORMULA IN TWO DIMENSIONS



$$t_i = n_j \sigma_{ji}$$

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$$

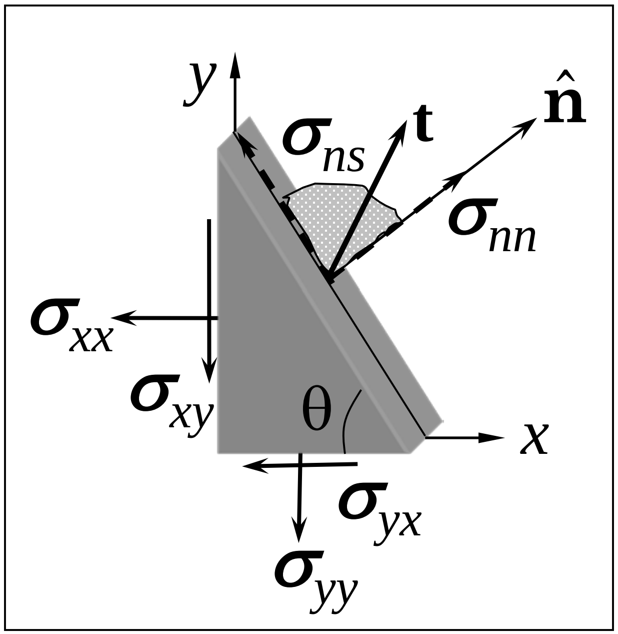
$$t_x = n_x \boldsymbol{\sigma}_{xx} + n_y \boldsymbol{\sigma}_{yx}$$

$$t_y = n_x \boldsymbol{\sigma}_{xy} + n_y \boldsymbol{\sigma}_{yy}$$

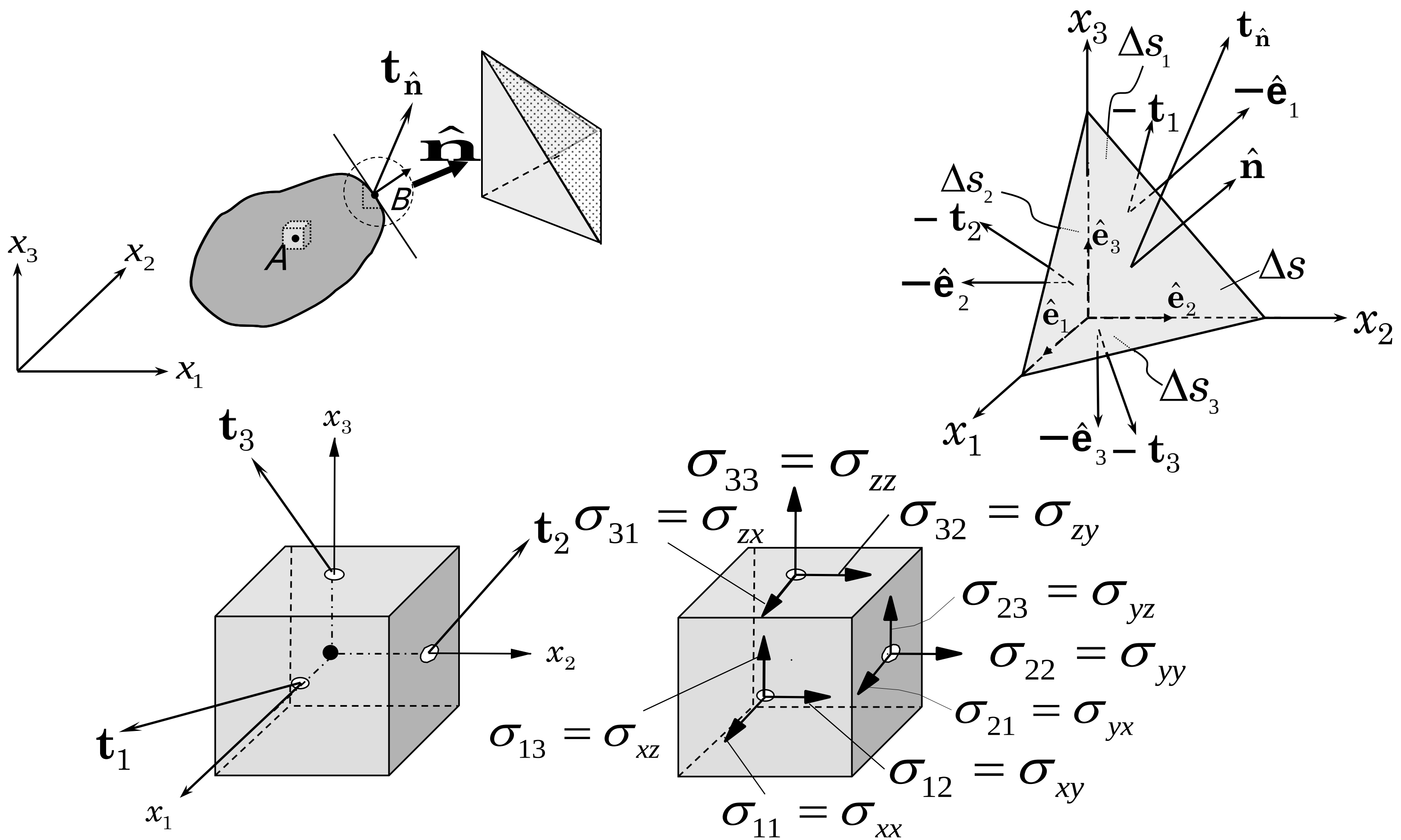
$$\hat{\mathbf{n}} = n_x \hat{\mathbf{i}} + n_x \hat{\mathbf{j}}$$

$$n_x = \cos\alpha = \sin\theta,$$

$$n_y = \sin\alpha = \cos\theta$$



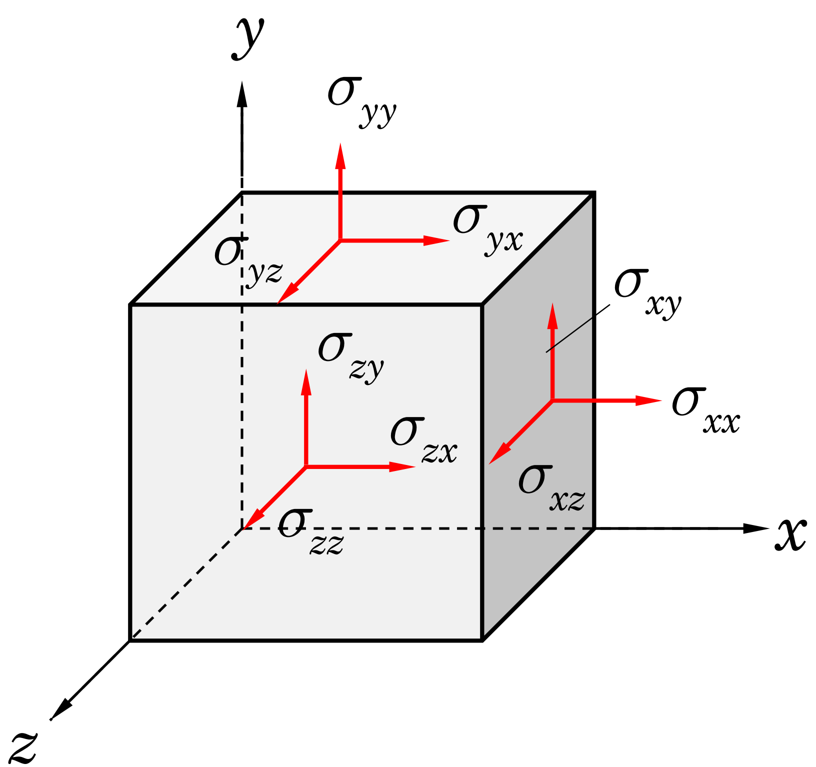
CAUCHY STRESS TENSOR



$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \sigma \quad \text{or} \quad t_i^{(\hat{\mathbf{n}})} = n_j \sigma_{ji}; \quad \mathbf{t}_i = \sigma_{ij} \hat{\mathbf{e}}_j, \quad \sigma = \hat{\mathbf{e}}_i \mathbf{t}_i = \sigma_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$$

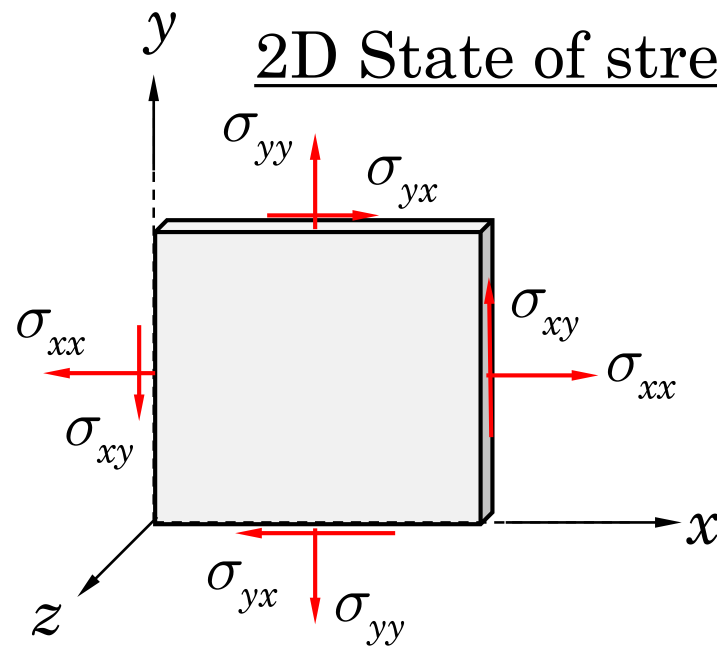
Notation of Stress Components

3D State of stress



The first subscript denotes the plane on which it is acting, and the second subscript denotes the direction of the stress component

2D State of stress



Matrix Form of Cauchy's Formula

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \mathbf{n}$$

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \\ \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z \\ \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z \end{Bmatrix}$$

$$\begin{aligned} \mathbf{t} &= t_x \hat{\mathbf{e}}_x + t_y \hat{\mathbf{e}}_y + t_z \hat{\mathbf{e}}_z \\ &= (\sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z)\hat{\mathbf{e}}_x + (\sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z)\hat{\mathbf{e}}_y \\ &\quad + (\sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z)\hat{\mathbf{e}}_z \end{aligned}$$

AN EXAMPLE

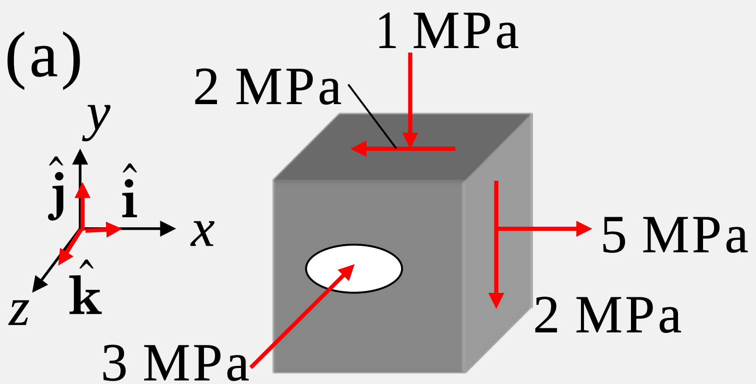
Problem: Given the following stress tensor components in Cartesian coordinates

$$[\sigma] = \begin{bmatrix} 5 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} (\text{MPa})$$

(a) Show the stress components on the stress cube.
(b) Determine the traction vectors $\mathbf{t}^{(\hat{\mathbf{i}})}$, $\mathbf{t}^{(\hat{\mathbf{j}})}$, and $\mathbf{t}^{(\hat{\mathbf{k}})}$
(c) Sketch the traction vectors on the stress cube.

Solution: We have

(a)



Stress cube

(b)

$$\mathbf{t}^{(\hat{\mathbf{i}})} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

$$\mathbf{t}^{(\hat{\mathbf{j}})} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}}$$

$$\mathbf{t}^{(\hat{\mathbf{k}})} = -3\hat{\mathbf{k}}$$

Example (continued)

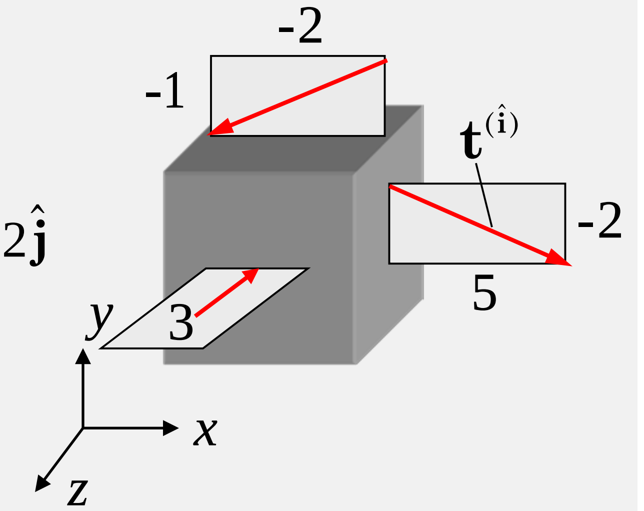
(b) Solution by use of Cauchy's formula

$$[\sigma]=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}(\text{MPa}),\qquad \begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}n_x\\ n_y\\ n_z\end{Bmatrix}$$

(c)

(i) When $\hat{\mathbf{n}}=\hat{\mathbf{i}}\ (n_x=1,\ n_y=0,\ n_z=0)$

$$\begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}^{(\hat{\mathbf{i}})}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}1\\ 0\\ 0\end{Bmatrix}=\begin{Bmatrix}5\\ -2\\ 0\end{Bmatrix}\Rightarrow \mathbf{t}^{(\hat{\mathbf{i}})}=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}$$



(ii) When $\hat{\mathbf{n}}=\hat{\mathbf{j}}\ (n_x=0,\ n_y=1,\ n_z=0)$

$$\begin{Bmatrix}t_x\\ t_y\\ t_z\end{Bmatrix}^{(\hat{\mathbf{j}})}=\begin{bmatrix}5 & -2 & 0\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}0\\ 1\\ 0\end{Bmatrix}=\begin{Bmatrix}-2\\ -1\\ 0\end{Bmatrix}\Rightarrow \mathbf{t}^{(\hat{\mathbf{j}})}=-2\hat{\mathbf{i}}-\hat{\mathbf{j}}$$

(iii) When $\hat{\mathbf{n}}=\hat{\mathbf{k}}\ (n_x=0,\ n_y=0,\ n_z=1)\ \Rightarrow \mathbf{t}^{(\hat{\mathbf{k}})}=-3\hat{\mathbf{k}}$

AN EXAMPLE

Problem statement:

With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ psi.}$$

Determine stress vector and its normal and tangential components at the point on the plane

$$\phi(x_1, x_2, x_3) \equiv x_1 + 2x_2 + 2x_3 = \text{constant}$$

which is planning through the point.

AN EXAMPLE

Solution:

First, we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal to the plane is

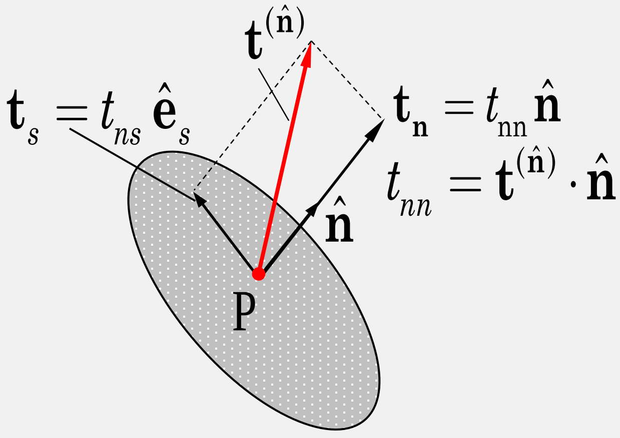
$$\hat{\mathbf{n}} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)$$

The components of the stress vector are

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi}$$

Solution (continued):

The traction vector normal to the plane is given by



$$\mathbf{t}_{nn} = (\mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \frac{2600}{9}\hat{\mathbf{n}}$$

$$= \frac{2600}{27}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \text{ psi}$$

and the traction vector projected onto the plane (i.e., shear traction) is given by

$$\mathbf{t}_{ns} = \mathbf{t}(\hat{\mathbf{n}}) - \mathbf{t}_{nn} = \tfrac{100}{27}(118\hat{\mathbf{e}}_1 - 16\hat{\mathbf{e}}_2 - 43\hat{\mathbf{e}}_3) \text{ psi}.$$

The magnitudes are

$$|\mathbf{t}_{nn}| = t_{nn} = \frac{2600}{9} = 288.89 \text{ psi}, \quad |\mathbf{t}_{ns}| = t_{ns} = 468.91 \text{ psi}.$$

PRINCIPAL VALUES OF STRESS

For a given state of stress, the determination of maximum normal stresses and shear stresses at a point is of considerable interest in the design of structures because failures occur when the magnitudes of stresses exceed the allowable (normal or shear) stress values, called strengths, of the material. In this regard it is of interest to determine the values and the planes on which the stresses are the maximum. Thus, we must determine the eigenvalues and eigenvectors associated with the stress tensor. This amounts to finding the plane on which $\mathbf{t}_n$ is largest $\lambda\hat{\mathbf{n}}$. It turns out there are three such planes on which the normal stress is the largest:

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \quad \text{or} \quad \left(\boldsymbol{\sigma} - \lambda\mathbf{I}\right)\cdot\hat{\mathbf{n}} = \mathbf{0} \qquad (1)$$

PRINCIPAL VALUES OF STRESS

The vanishing of the determinant, $|\boldsymbol{\sigma} - \lambda \mathbf{I}| = 0$, yields a cubic equation for λ, called the *characteristic equation:*

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where $I_1, I_2, \text{and}\, I_3$ are the invariants of $\boldsymbol{\sigma}$ as defined

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}, \quad I_2 = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}), \quad I_3 = |\sigma|,$$

$$I_2 = \tfrac{1}{2}\left(I_1^2 - \sigma_{11}^2 - \sigma_{22}^2 - \sigma_{33}^2 - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 - \sigma_{21}^2 - \sigma_{31}^2 - \sigma_{32}^2\right)$$

The eigenvector $\hat{\mathbf{n}}^{(i)}$ associated with any particular eigenvalue λ_i is calculated using Eq. (1), which gives only two independent relations among the three components $n_1^{(i)}, n_2^{(i)}, \text{and}\, n_3^{(i)}$. The third equation is provided by

$$(n_1^{(i)})^2 + (n_2^{(i)})^2 + (n_3^{(i)})^2 = 1$$

AN EXAMPLE

Problem statement:

The components of a stress dyadic at a point, referred to the system (x_1, x_2, x_3), are:

$$[\sigma] = \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{MPa}$$

Find the principal stresses and the principal plane associated with the maximum stress.

Solution: Clearly, $\lambda = 6$ is an eigenvalue. Expanding the determinant $|\sigma - \lambda I|$ with the last row or column, we obtain

$$(6-\lambda)[(12-\lambda)(-12-\lambda)-81] = 0 \;\Rightarrow\; (\lambda^2 - 225)(6-\lambda) = 0$$

AN EXAMPLE (continued)

Thus, the three principal stresses are

$$\sigma_1 = \lambda_1 = 15\,\text{MPa},\;\; \sigma_2 = \lambda_2 = 6\text{ MPa},\;\; \sigma_3 = \lambda_3 = -15\,\text{MPa}$$

The plane associated with the maximum principal stress $\lambda_1 = 15$ MPa can be calculated from

$$\begin{bmatrix} 12-15 & 9 & 0 \\ 9 & -12-15 & 0 \\ 0 & 0 & 6-15 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

which gives

$$-3n_1 + 9n_2 = 0,\;\; 9n_1 - 27n_2 = 0,\;\; -9n_3 = 0 \quad \rightarrow \quad n_3 = 0,\;\; n_1 = 3n_2$$

$$\hat{\mathbf{n}}^{(1)} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \;\; \text{or} \;\; \hat{\mathbf{n}}^{(1)} = \frac{1}{\sqrt{10}}\left(3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\right)$$

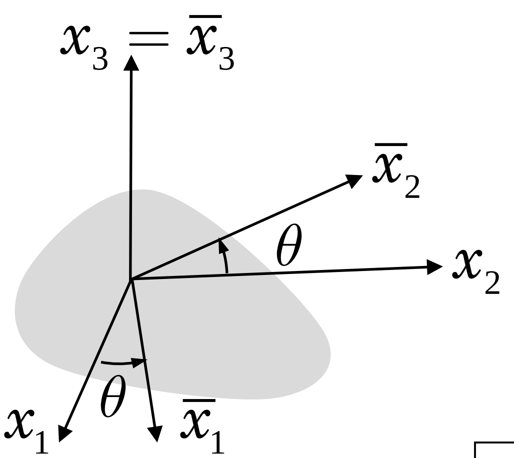
STRESS TRANSFORMATION RELATIONS

The transformations for stress tensor components follow the same relations as for a second-order tensor:

$$\bar{\sigma}_{ij} = \ell_{im}\ell_{jn}\sigma_{mn} \Rightarrow [\bar{\sigma}] = [L][\sigma][L]^{\mathrm{T}}$$

As an example, consider the case in which the barred coordinates are obtained by rotating the x_1x_2- plane counterclockwise about the x_3- axis by an angle θ. Thus we have

$$[L] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & l_{33} \end{bmatrix}$$



STRESS TRANSFORMATION RELATIONS (continued)

The stress components in the barred coordinate system are thus related to those in the unbarred system by

$$\begin{bmatrix} \bar\sigma_{11} & \bar\sigma_{12} & \bar\sigma_{13} \\ \bar\sigma_{21} & \bar\sigma_{22} & \bar\sigma_{23} \\ \bar\sigma_{31} & \bar\sigma_{32} & \bar\sigma_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & 0 \\ l_{12} & l_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_{11}\sigma_{11} + l_{12}\sigma_{12} & l_{21}\sigma_{11} + l_{22}\sigma_{12} & \sigma_{13} \\ l_{11}\sigma_{21} + l_{12}\sigma_{22} & l_{21}\sigma_{21} + l_{22}\sigma_{22} & \sigma_{23} \\ l_{11}\sigma_{31} + l_{12}\sigma_{32} & l_{21}\sigma_{31} + l_{22}\sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}\left(l_{11}\sigma_{11} + l_{12}\sigma_{12}\right) + l_{12}\left(l_{11}\sigma_{21} + l_{12}\sigma_{22}\right) & l_{11}\left(l_{21}\sigma_{11} + l_{22}\sigma_{12}\right) + l_{12}\left(l_{21}\sigma_{21} + l_{22}\sigma_{22}\right) & l_{11}\sigma_{13} + l_{12}\sigma_{23} \\ l_{21}\left(l_{11}\sigma_{11} + l_{12}\sigma_{12}\right) + l_{22}\left(l_{11}\sigma_{21} + l_{12}\sigma_{22}\right) & l_{21}\left(l_{21}\sigma_{11} + l_{22}\sigma_{12}\right) + l_{22}\left(l_{21}\sigma_{21} + l_{22}\sigma_{22}\right) & l_{21}\sigma_{13} + l_{22}\sigma_{23} \\ l_{11}\sigma_{31} + l_{12}\sigma_{32} & l_{21}\sigma_{31} + l_{22}\sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\bar\sigma_{11} = \sigma_{11}\cos^2\theta + (\sigma_{12} + \sigma_{21})\cos\theta\sin\theta + \sigma_{22}\sin^2\theta;$$

$$\bar\sigma_{12} = (\sigma_{22} - \sigma_{11})\cos\theta\sin\theta + \sigma_{12}\cos^2\theta - \sigma_{21}\sin^2\theta$$

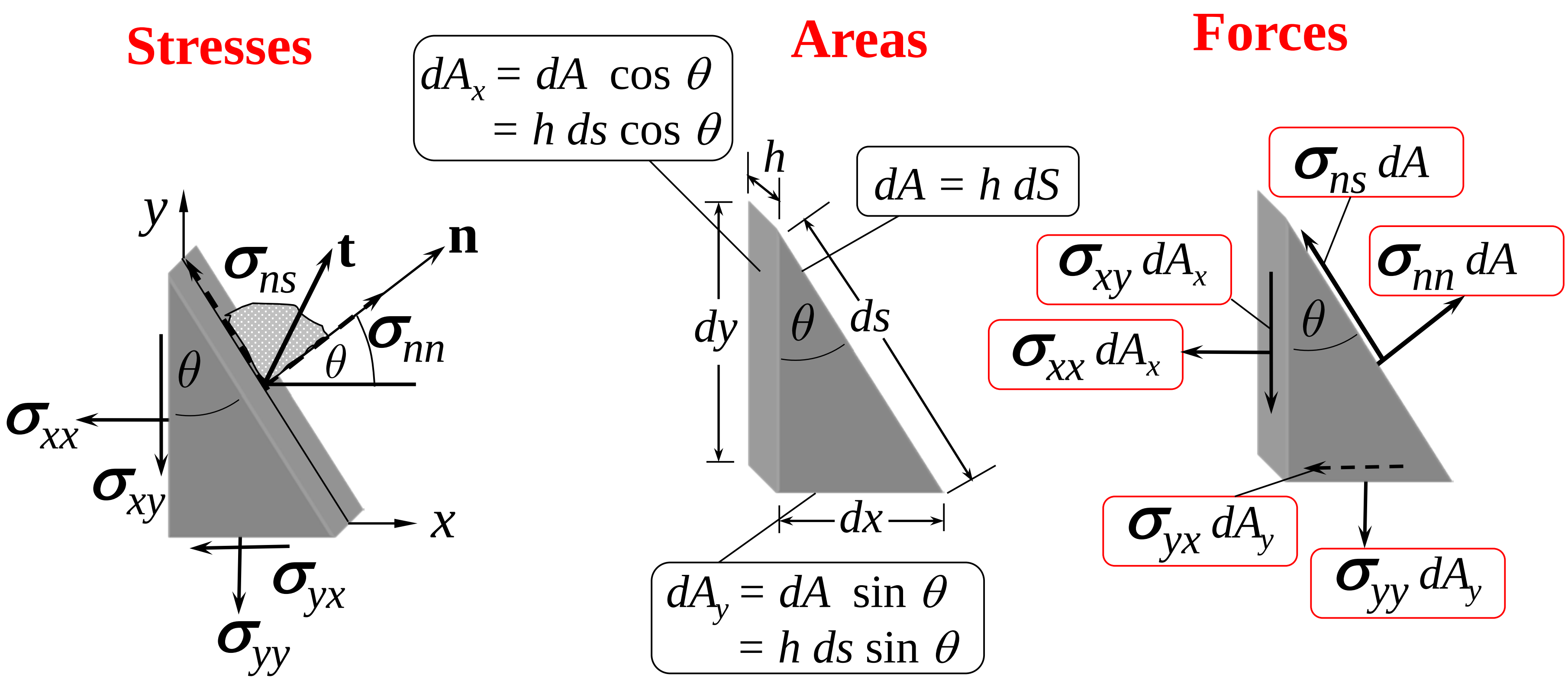
$$\bar\sigma_{13} = \sigma_{13}\cos\theta + \sigma_{23}\sin\theta; \quad \bar\sigma_{31} = \sigma_{31}\cos\theta + \sigma_{32}\sin\theta;$$

$$\bar\sigma_{21} = (\sigma_{22} - \sigma_{11})\cos\theta\sin\theta + \sigma_{21}\cos^2\theta - \sigma_{12}\sin^2\theta$$

$$\bar\sigma_{22} = \sigma_{11}\sin^2\theta - (\sigma_{12} + \sigma_{21})\cos\theta\sin\theta + \sigma_{22}\cos^2\theta;$$

$$\bar\sigma_{23} = -\sigma_{13}\sin\theta + \sigma_{23}\cos\theta; \; \bar\sigma_{32} = -\sigma_{31}\sin\theta + \sigma_{32}\cos\theta; \; \bar\sigma_{33} = \sigma_{33}$$

STRESSES ON A WEDGE (2-D)



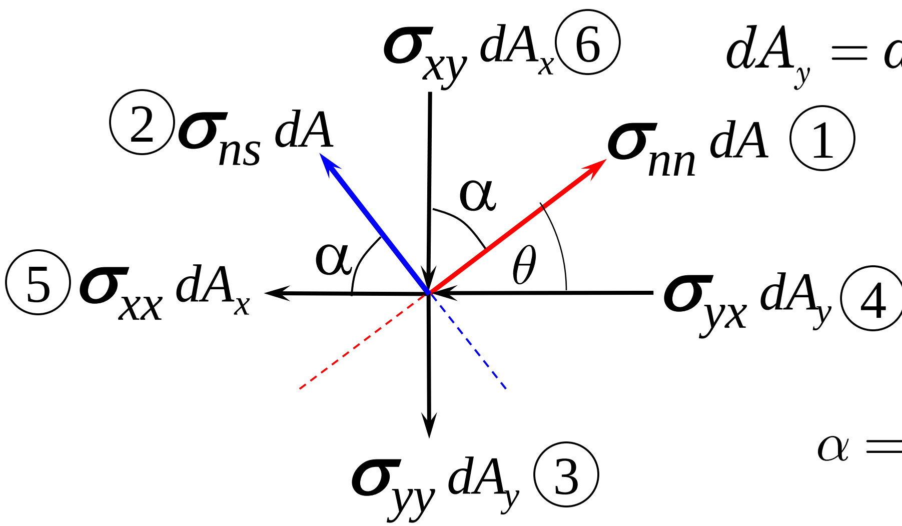
Stresses on a Wedge (2-D)
(Continued)

Forces placed at a common point

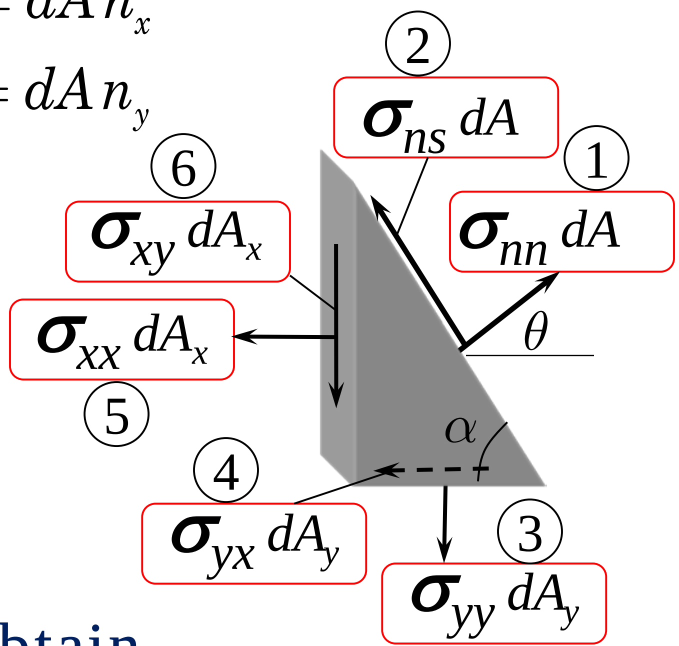
Forces on the wedge

$$dA_x = dA\cos\theta = dA\,n_x$$

$$dA_y = dA\sin\theta = dA\,n_y$$



$$\alpha = 90^\circ - \theta$$



Summing forces along the red line, we obtain

$$\sigma_{nn}dA - \left(\sigma_{xy}dA_x\right)\sin\theta - \left(\sigma_{yx}dA_y\right)\cos\theta - \left(\sigma_{xx}dA_x\right)\cos\theta$$

$$-\left(\sigma_{yy}dA_y\right)\sin\theta = 0$$

Stress Transformation Relations (Continued)

Similarly, by summing the forces along s-direction, we obtain

$$\sigma_{ns} dA + \left(\sigma_{xx} dA_x\right)\sin\theta - \left(\sigma_{xy} dA_x\right)\cos\theta + \left(\sigma_{yx} dA_y\right)\sin\theta - \left(\sigma_{yy} dA_y\right)\cos\theta = 0$$

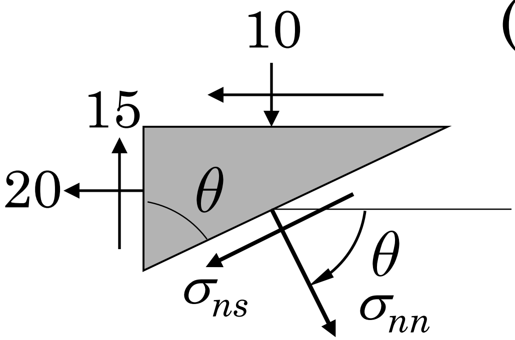
Substitute for $dA_x = dA\cos\theta$ and $dA_y = dA\sin\theta$ into the equations and divide throughout by dA and obtain

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + (\sigma_{xy} + \sigma_{yx})\cos\theta\sin\theta$$

$$\sigma_{ns} = (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}\cos^2\theta - \sigma_{yx}\sin^2\theta$$

which are the same as those derived earlier.

AN EXAMPLE



(all stresses in MPa)

$$\sigma_{xx} = 20,\ \sigma_{xy} = -15,\ \ \sigma_{yy} = -10,\ \ \theta = -60^\circ$$

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + (\sigma_{xy} + \sigma_{yx})\cos\theta\sin\theta$$

$$= 20 \times \frac{1}{4} - 10 \times \frac{3}{4} - (15 + 15)\frac{1}{2} \times \left(-\frac{\sqrt{3}}{2}\right) = 10.49\ \text{ MPa}$$

$$\sigma_{ns} = (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}\cos^2\theta - \sigma_{yx}\sin^2\theta$$

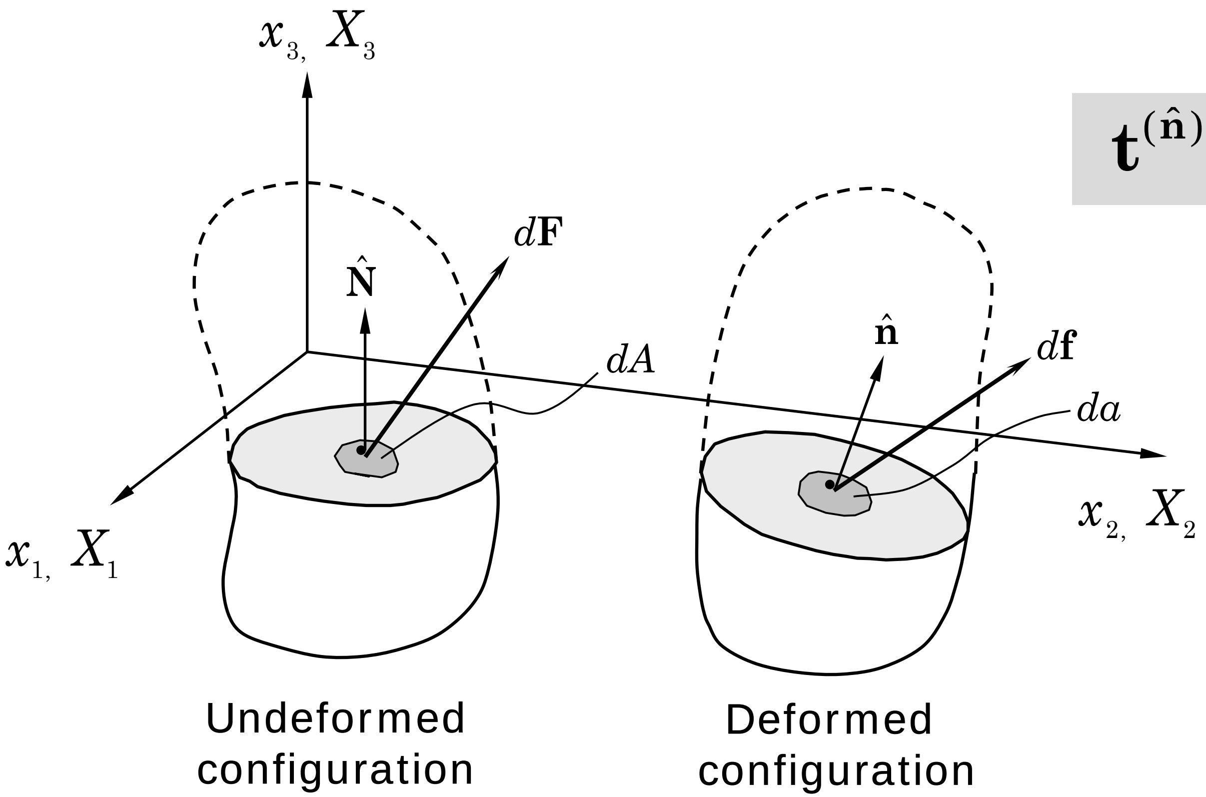
$$= (-10 - 20)\frac{1}{2} \times \left(-\frac{\sqrt{3}}{2}\right) - 15\left(\frac{1}{4} - \frac{3}{4}\right) = -20.49\ \text{MPa}$$

OTHER MEASURES OF STRESS

Stress vector, $\mathbf{t}(\hat{\mathbf{n}}) = \dfrac{d\mathbf{f}}{da}$ Cauchy stress tensor, $\boldsymbol{\sigma}$

$$d\mathbf{f} = \mathbf{t}\, da = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}\, da = d\mathbf{a} \cdot \sigma \text{ where } d\mathbf{a} = da\, \hat{\mathbf{n}}$$

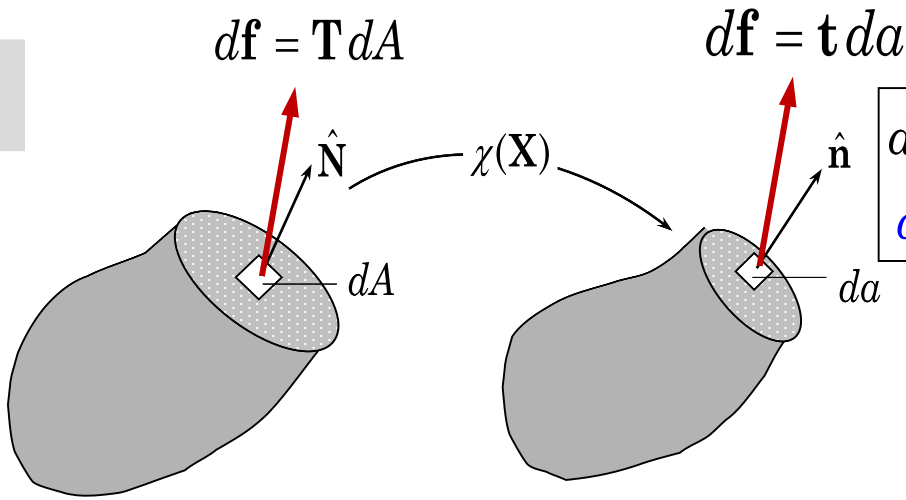
$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \sigma$$



FIRST PIOLA-KIRCHHOFF STRESS TENSOR

P ***First*** Pioloa-Kirchhoff stress tensor

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$$



$$d\mathbf{A} = \hat{\mathbf{N}}\,dA, \quad d\mathbf{a} = \hat{\mathbf{n}}\,da$$

$$d\mathbf{a} = J\,\mathbf{F}^{-\mathrm{T}} \cdot d\mathbf{A} = J\,d\mathbf{A} \cdot \mathbf{F}^{-1}$$

Reference configuration | Current configuration

$$d\mathbf{f} = \mathbf{T}\,dA = \hat{\mathbf{N}} \cdot \mathbf{P}\,dA = d\mathbf{A} \cdot \mathbf{P} \qquad d\mathbf{f} = \mathbf{t}\,da = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}\,da = d\mathbf{a} \cdot \boldsymbol{\sigma}$$

$$d\mathbf{f} = d\mathbf{A} \cdot \mathbf{P} = d\mathbf{a} \cdot \boldsymbol{\sigma}$$

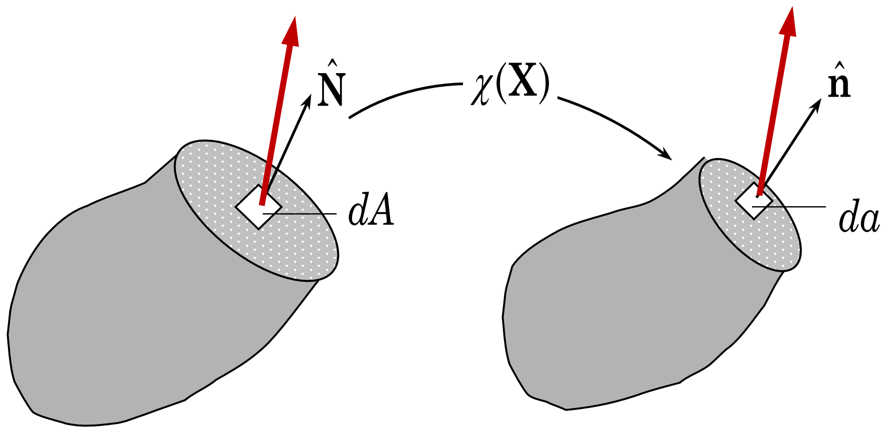
$$d\mathbf{A} \cdot \mathbf{P} = d\mathbf{a} \cdot \boldsymbol{\sigma} = J\,d\mathbf{A} \cdot \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \quad \Rightarrow \quad \mathbf{P} = J\,\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$$

SECOND PIOLA-KIRCHHOFF STRESS TENSOR

S ***Second*** Pioloa-Kirchhoff stress tensor

$$d\mathfrak{I} = \mathbf{F}^{-1} \cdot d\mathbf{f}$$

$d\mathbf{f}$



$$d\mathbf{A} = \hat{\mathbf{N}}\, dA, \quad d\mathbf{a} = \hat{\mathbf{n}}\, da$$

$$d\mathbf{a} = J\ d\mathbf{A} \cdot \mathbf{F}^{-1}$$

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$$

Reference configuration

$$d\mathfrak{I} = d\mathbf{A} \cdot \mathbf{S}$$

Current configuration

$$d\mathbf{f} = d\mathbf{a} \cdot \boldsymbol{\sigma}$$

$$d\mathfrak{I} = \mathbf{F}^{-1} \cdot d\mathbf{f}$$

$$d\mathbf{A} \cdot \mathbf{S} = \mathbf{F}^{-1} \cdot \left(d\mathbf{a} \cdot \boldsymbol{\sigma}\right) = \left(d\mathbf{a} \cdot \boldsymbol{\sigma}\right) \cdot \mathbf{F}^{-\mathrm{T}} = J d\mathbf{A} \cdot \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}}$$

$$\mathbf{S} = J\ \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} = \mathbf{P} \cdot \mathbf{F}^{-\mathrm{T}}$$

RELATIONS AMONG VARIOUS STRESS TENSORS

Component forms $\mathbf{P} = P_{Ii}\hat{\mathbf{E}}_I\hat{\mathbf{e}}_i, \qquad \mathbf{S} = S_{IJ}\hat{\mathbf{E}}_I\hat{\mathbf{E}}_J$

$$\text{Nanson's formula}$$

$$d\mathbf{a} = \hat{\mathbf{n}}\,da = \mathbf{F}^{-\mathrm{T}}\cdot\hat{\mathbf{N}}\,J\,dA = J d\mathbf{A}\cdot\mathbf{F}^{-1}$$

Relations among the stress tensors

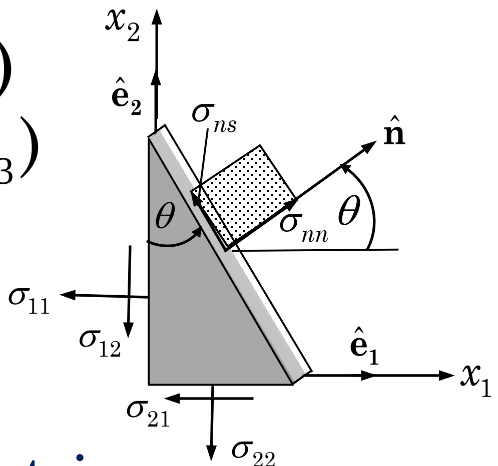
$$d\mathbf{A}\cdot\mathbf{P} = d\mathbf{a}\cdot\sigma = J d\mathbf{A}\cdot\mathbf{F}^{-1}\cdot\sigma \Rightarrow \boxed{\mathbf{P} = J\mathbf{F}^{-1}\cdot\sigma}$$

$$\mathbf{P}\cdot\mathbf{F}^{-\mathrm{T}} = \mathbf{S} \quad \text{or} \quad \boxed{\mathbf{P} = \mathbf{S}\cdot\mathbf{F}^{\mathrm{T}}}$$

$$\boxed{\mathbf{S} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}}}$$

EXERCISE PROBLEMS

(1) Determine the transformation relations between stress components $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ referred to the coordinate system (x_1, x_2, x_3) and those $(\sigma_{nn}, \sigma_{ns})$ on a plane oriented at an angle θ from the x_1- axis.



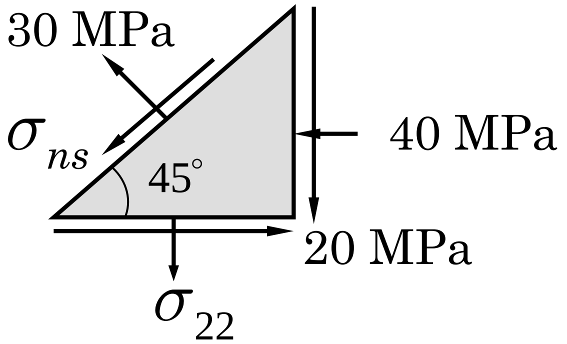
(2) Determine the rotation transformation matrix such that the new base vector $\hat{\bar{\mathbf{e}}}_1$ is along $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$, and $\hat{\bar{\mathbf{e}}}_2$ is along the normal to the plane $2x_1 + 3x_2 + x_3 = 5$. If $\mathbf{S}$ is the tensor whose components in the unbarred system are given by

$$s_{11} = 1, s_{12} = s_{21} = 0, s_{13} = s_{31} = -1, s_{22} = 3, s_{23} = s_{32} = -2, s_{33} = 0$$

find the components in the barred coordinates.

EXERCISE PROBLEMS

(3) Find the values of σ_{ns} and σ_{22} for the state of stress shown in the figure.



(4) Find the maximum principal stress, maximum shear stress and their orientations for the state of stress given.

$$[\sigma] = \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{MPa}$$

(5) Find the maximum principal stress, maximum shear stress and their orientations for the state of stress given.

$$[\sigma] = \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \text{MPa}$$