KINEMATICS OF CONTINUA

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INTRODUCTION

- This part of the course is devoted to the study of geometric changes in a continuous medium that is in equilibrium.
- The study of geometric changes in a continuum without regard to the forces causing the changes is known as **kinematics**.

In a continuous medium, any property of the medium, for example density, can be defined at every point of the medium. This is possible only if the medium contains no gaps between points:

$$\rho(\mathbf{x}, t) \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$

DEFORMATION OF A CONTINUUM

- Consider a body of known geometry in a three-dimensional Euclidean space. The body of matter may be viewed as a set of particles, each particle representing a large collection of molecules with a continuous distribution of matter in space and time. Under external stimuli, the body will undergo macroscopic geometric changes, which are termed **deformations**.
- The geometric changes are accompanied by stresses that are induced in the body. If the applied loads are time dependent, the deformation of the body will be a function of time; that is, the geometry of the body of matter will change with time.

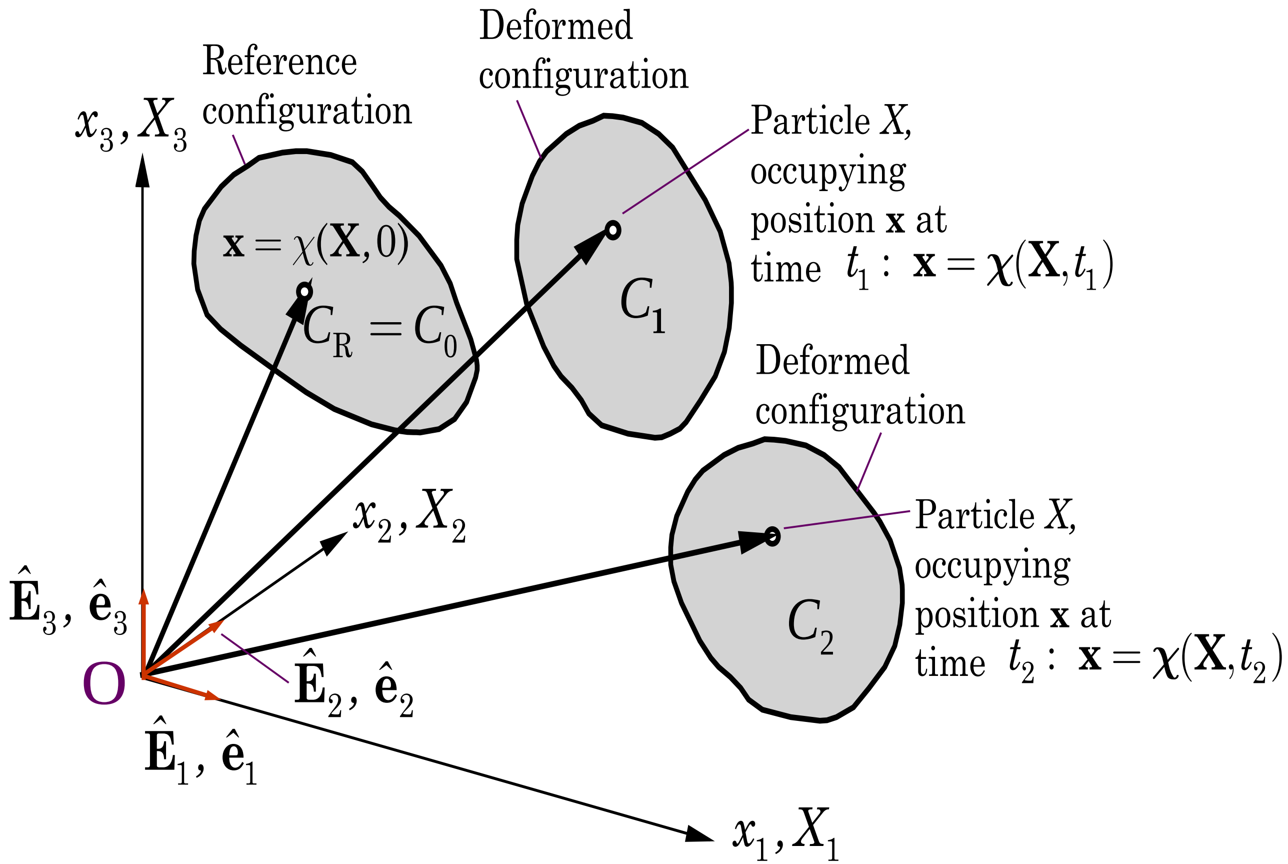
CONFIGURATIONS OF A CONTINUUM

Configurations of a Continuum

If the loads are applied slowly so that the deformation is dependent only on the loads, the body will occupy a sequence of geometrical regions. The region occupied by the continuum at a given time *t* is termed a **configuration**.

Suppose that the continuum initially occupies a configuration C_0, in which a particle X occupies position **X**, referred to a **reference frame** of right-handed, rectangular Cartesian axes (X_1, X_2, X_3) at a fixed origin O with orthonormal basis vectors $(\hat{\mathbf{E}}_1, \hat{\mathbf{E}}_2, \hat{\mathbf{E}}_3)$, as shown in the figure.

CONFIGURATIONS OF A CONTINUUM

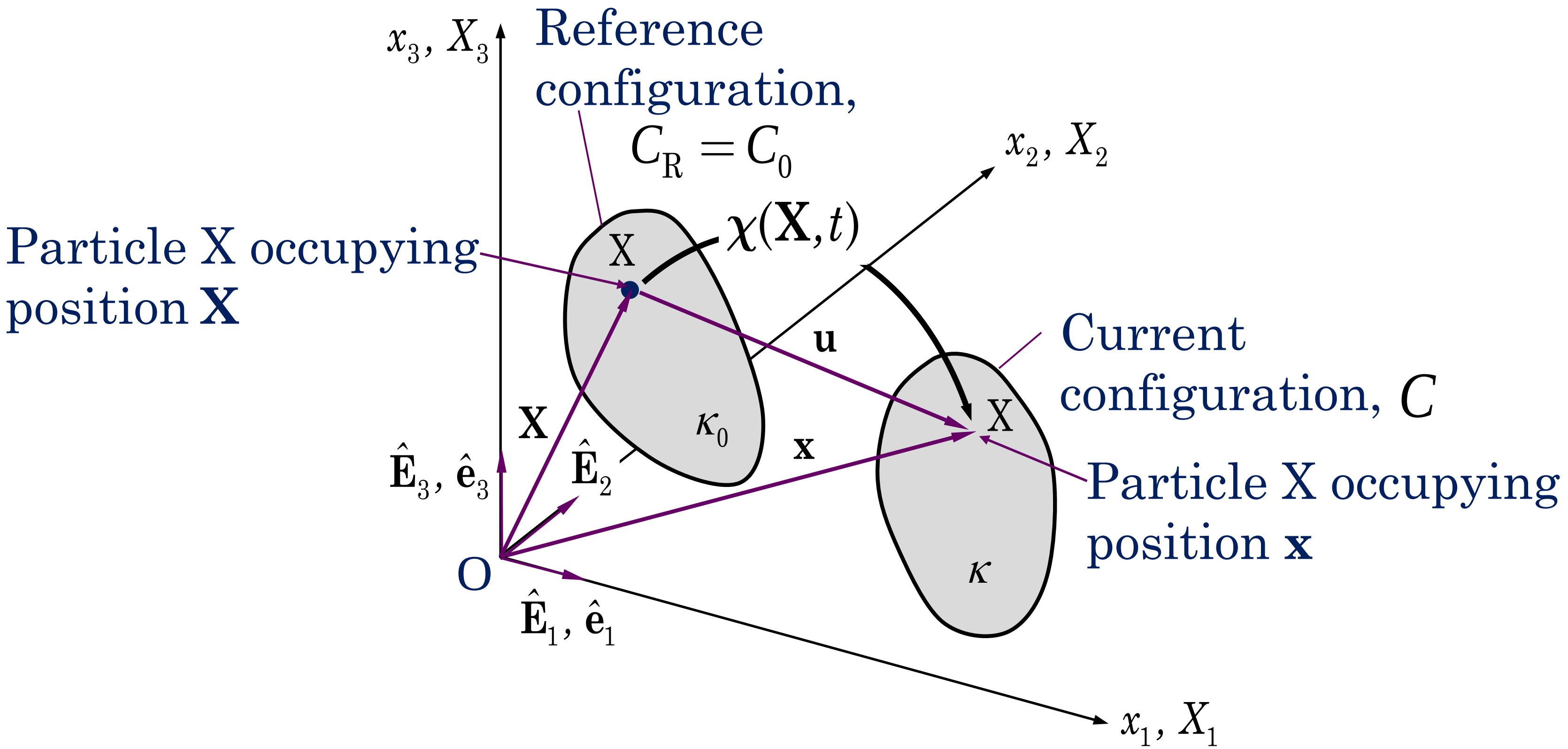


DEFORMATION MAPPING

Note that X (lightface roman letter) is the name of the particle that occupies location **X** (boldface letter) in configuration C_0 and therefore (X_1, X_2, X_3) are called the *material coordinates*. After the application of some external stimuli (e.g., loads), the continuum changes its geometric shape and thus assumes a new configuration C, called the *current* or *deformed configuration*. Particle X now occupies position **x** in the deformed configuration C.

The mapping $\boldsymbol{\chi} : C_0 \to C$ is called the *deformation mapping* of the body. The deformation mapping $\boldsymbol{\chi}(\mathbf{X}, t)$ takes the position vector **X** from the reference configuration and places the same point in the deformed configuration as $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$.

DEFORMATION MAPPING



DESCRIPTIONS OF MOTON

The inverse mapping $\boldsymbol{\chi}^{-1} : C \to C_0$ takes the position vector **x** from the deformed configuration back to the reference configuration $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t)$. It is not always possible to construct the inverse mapping from a known deformation mapping.

Material Description (Lagrangian description)

In the material description, the motion of the body is referred to a reference configuration C_R, which is often chosen to be the initial configuration. Thus, in the material description, also known as the *Lagrangian description*, the current coordinates **x** in C are expressed in terms of the reference coordinates $\mathbf{X} \in C_0$:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, 0)$$

DESCRIPTIONS OF MOTON

The variation of a typical variable $\phi(\mathbf{x},t)$ over the body is described with respect to the material coordinates $\mathbf{X}$ and time t:

$$\phi = \phi(\mathbf{x}(\mathbf{X}),t) = \phi(\mathbf{X},t)$$

Spatial Description (Eulerian description)

In the spatial description, also known as the *Eulerian description*, the motion is referred to the current configuration occupied by the body, and $\phi(\mathbf{x},t)$ is described with respect to the current position $\mathbf{x} \in C$ occupied by material particle X:

$$\phi = \phi(\mathbf{x},t), \quad \mathbf{X} = \mathbf{X}(\mathbf{x},t) = \boldsymbol{\chi}^{-1}(\mathbf{x},t)$$

The coordinates $\mathbf{x}$ are termed the *spatial coordinates*.

MATERIAL TIME DERIVTIVE

When a function $\phi(\mathbf{x},t)$ is known in the material description, $\phi = \phi(\mathbf{X},t)$, its total time derivative, D/Dt, is simply the partial derivative with respect to time because the material coordinates **X** do not change with time:

$$\frac{D}{Dt}[\phi(\mathbf{X},t)] \equiv \frac{\partial}{\partial t}[\phi(\mathbf{X},t)]\bigg|_{\mathbf{X}\ \text{fixed}} = \frac{\partial \phi(\mathbf{X},t)}{\partial t}$$

However, when $\phi(\mathbf{x},t)$ is known in the spatial description, its time derivative for a given particle, known as the **material derivative,** is

$$\frac{D}{Dt}[\phi(\mathbf{x},t)] = \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + \frac{Dx_i}{Dt}\frac{\partial}{\partial x_i}[\phi(\mathbf{x},t)]$$

$$= \frac{\partial \phi}{\partial t} + v_i \frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi(\mathbf{x},t)$$

VELOCITY AND ACCELERATION

The material time derivative of a function of current position and time is nothing but its total time derivative, which by the chain rule of differentiation is

$$\frac{d}{dt}\left[\phi(\mathbf{x},t)\right]=\frac{\partial}{\partial t}\left[\phi(\mathbf{x},t)\right]+\frac{dx_i}{dt}\frac{\partial}{\partial x_i}\left[\phi(\mathbf{x},t)\right]$$

where $\mathbf{x}=\mathbf{x}(t)$ and $\mathbf{v}$ is the velocity vector

$$\mathbf{v}=\frac{d\mathbf{x}}{dt}=\frac{D\mathbf{x}}{Dt}$$

The acceleration $\mathbf{a}$ of the particle $\mathbf{x}$ is defined by the total time derivative of the velocity $\mathbf{v}$:

$$\mathbf{a}(\mathbf{x},t)=\frac{D\mathbf{v}}{Dt}=\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$$

AN EXAMPLE

Problem statement:

Suppose that the motion of a continuous medium is described by the mapping

$$\boldsymbol{\chi}(\mathbf{X},t) = \mathbf{x} = (X_1 + A\,t\,X_2)\hat{\mathbf{e}}_1 + (X_2 - A\,t\,X_1)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3$$

and that the temperature T in the continuum in the spatial description is given by

$$T(\mathbf{x},t) = c_1(x_1 + c_2 t x_2) = x_1 + t x_2$$

Determine (a) the inverse of the mapping $\boldsymbol{\chi}$, (b) the velocity components, and (c) the total time derivatives of T in the two descriptions.

AN EXAMPLE (continued)

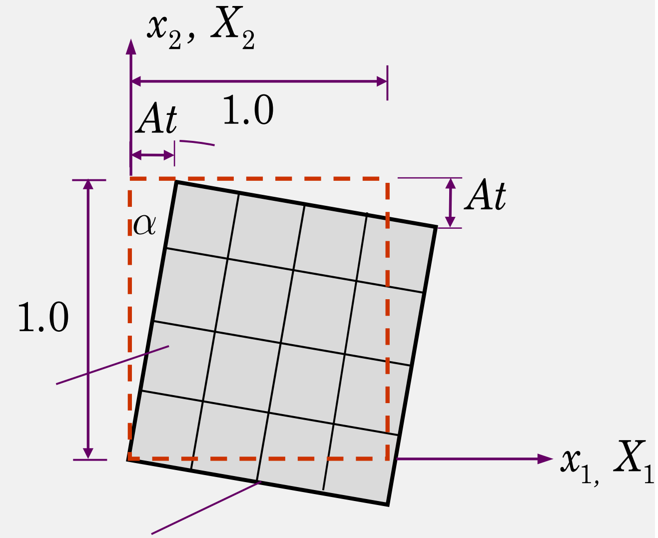
Problem solution: From the given mapping have

$$x_1 = X_1 + AtX_2,\ x_2 = X_2 - AtX_1,\ x_3 = X_3$$

or

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & At & 0 \\ -At & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad (1)$$

Clearly, the mapping is linear. Therefore, polygons are mapped into polygons. In particular, a unit square is mapped into a square that is rotated in a clockwise direction.



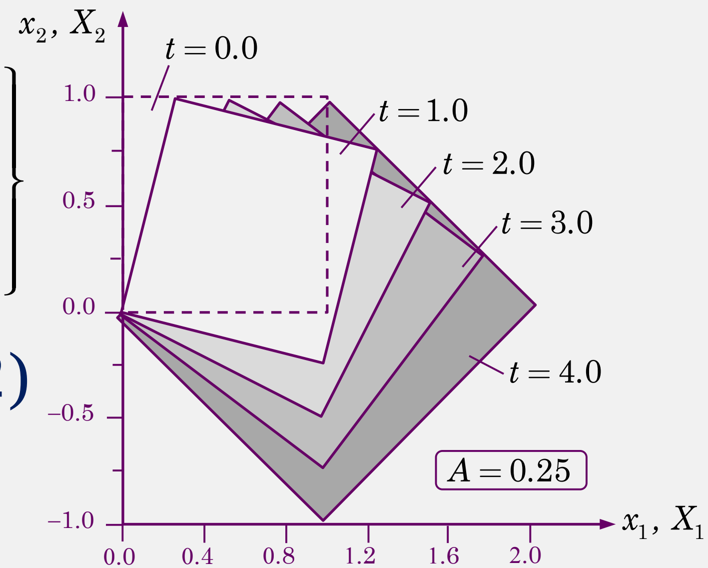
Deformed body

Initial (undeformed) body is shown in dotted lines

AN EXAMPLE (continued)

(a) The inverse mapping can be determined, when possible, by expressing (x_1, x_2, x_3) in terms of (X_1, X_2, X_3). In the present case, it is possible to invert the relations in Eq. (1) and obtain

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{1}{(1+A^2t^2)} \begin{bmatrix} 1 & -At & 0 \\ At & 1 & 0 \\ 0 & 0 & 1+A^2t^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad (2)$$



(b) The velocity vector is given by

$$\mathbf{v} = v_1 \hat{\mathbf{E}}_1 + v_2 \hat{\mathbf{E}}_2, \quad v_1 = \frac{Dx_1}{Dt} = AX_2, \quad v_2 = \frac{Dx_2}{Dt} = -AX_1 \quad (3)$$

AN EXAMPLE (continued)

(c) The time rate of change of temperature of a material particle in the body is simply

$$\frac{D}{Dt}\left[T(\mathbf{X},t)\right] = \frac{\partial}{\partial t}\left[T(\mathbf{X},t)\right]\Big|_{\mathbf{X}\ \text{fixed}} = -2AtX_1 + (1+A)X_2 \quad (4)$$

On the other hand, the time rate of change of temperature at point **x**, which is now occupied by particle X in the spatial description, is

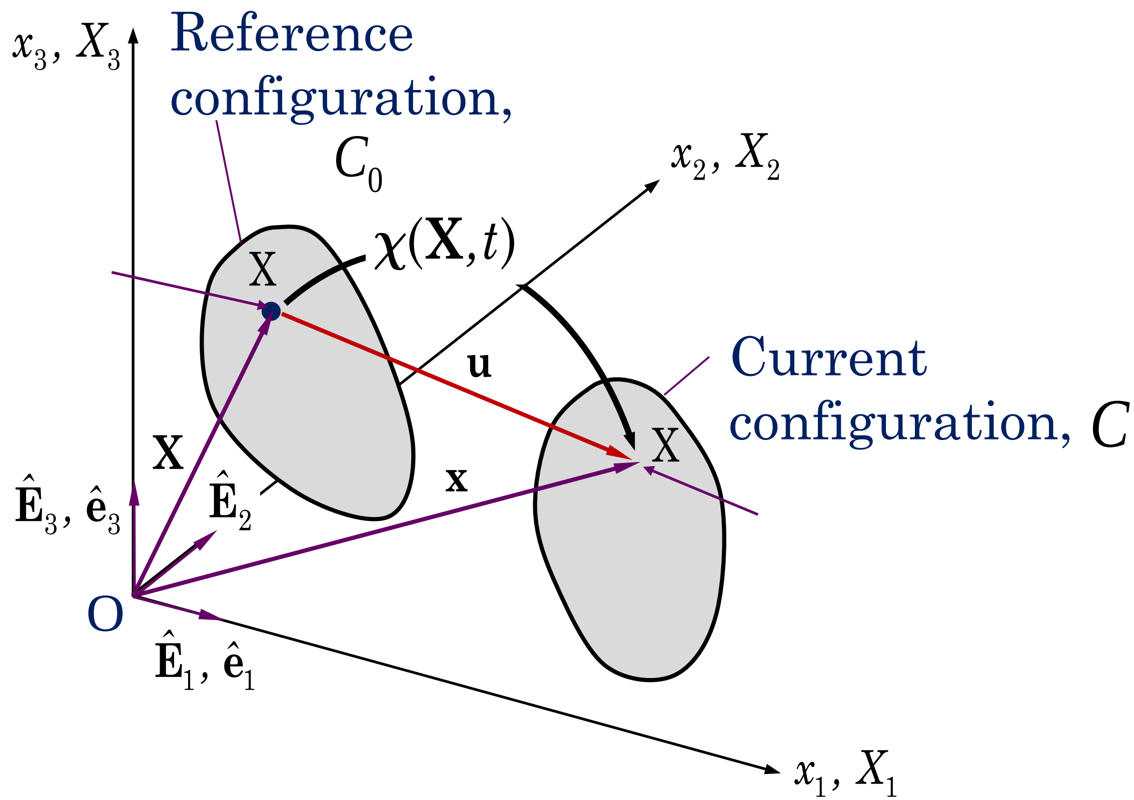
$$\frac{D}{Dt}\left[T(\mathbf{x},t)\right] = \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = x_2 + v_1 \cdot 1 + v_2 \cdot t$$

$$= -2AtX_1 + (1+A)X_2 \quad (5)$$

KINEMATICS OF DEFORMATION OF SOLID CONTINUA

Displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X} \quad (u_i = x_i - X_i)$

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}, \quad \mathbf{v}(\mathbf{X},t) = \frac{D\mathbf{x}(\mathbf{X},t)}{Dt} = \frac{\partial \mathbf{u}(\mathbf{X},t)}{\partial t}$$



DEFORMATION GRADIENT

One of the key quantities in deformation analysis is the **deformation gradient**, denoted **F,** which provides the relationship between a material line $d\mathbf{X}$ before deformation and the line $d\mathbf{x}$ (consisting of the same material as $d\mathbf{X}$) after deformation. It is defined as follows:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}, \quad \mathbf{F} = \left(\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}\right)^{\mathrm{T}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^{\mathrm{T}} \equiv \mathbf{x}\overleftarrow{\nabla}_0 = \left(\nabla_0 \mathbf{x}\right)^{\mathrm{T}}$$

We note that **F** is a mapping from the undeformed body to deformed body; it is not a tensor. The rectangular Cartesian component form is

$$\mathbf{F} = F_{iJ}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{E}}_J, \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}$$

$$\boxed{d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}}, \quad \mathbf{F}^{-\mathrm{T}} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \equiv \nabla \mathbf{X}}$$

AN EXAMPLE

Problem statement:

Consider the uniform deformation of a square block of side 2 units and initially centered at ${\bf X}=(0,0)$. If the deformation is defined by the mapping

$$\chi(\mathbf{X}) = (3.5 + X_1 + 0.5X_2)\hat{\mathbf{e}}_1 + (4 + X_2)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3$$

(a) sketch the deformation, (b) determine the deformation gradient $\mathbf{F}$, and (c) compute the displacements.

Solution:

(a) From the given mapping, we have in matrix form, we have

$$x_1 = 3.5 + X_1 + 0.5X_2,\ x_2 = 4 + X_2,\ x_3 = X_3$$

AN EXAMPLE (continued)

(a)

$$x_1 = 3.5 + X_1 + 0.5X_2,$$

$$x_2 = 4 + X_2, \;\; x_3 = X_3.$$

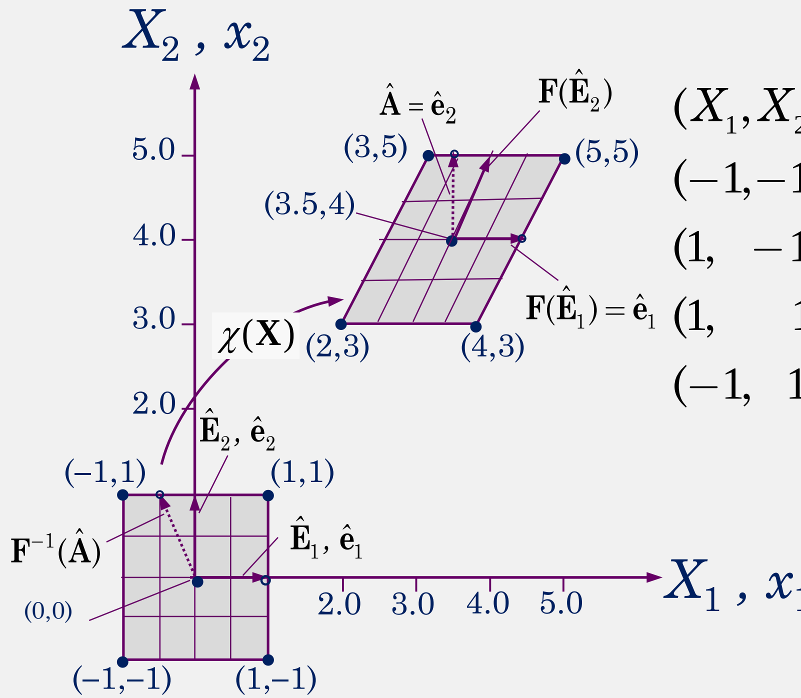
$$X_1 = -1.5 + x_1 - 0.5x_2,$$

$$X_2 = -4 + x_2, \; X_3 = x_3.$$

(b)

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix},$$

$$[F]^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

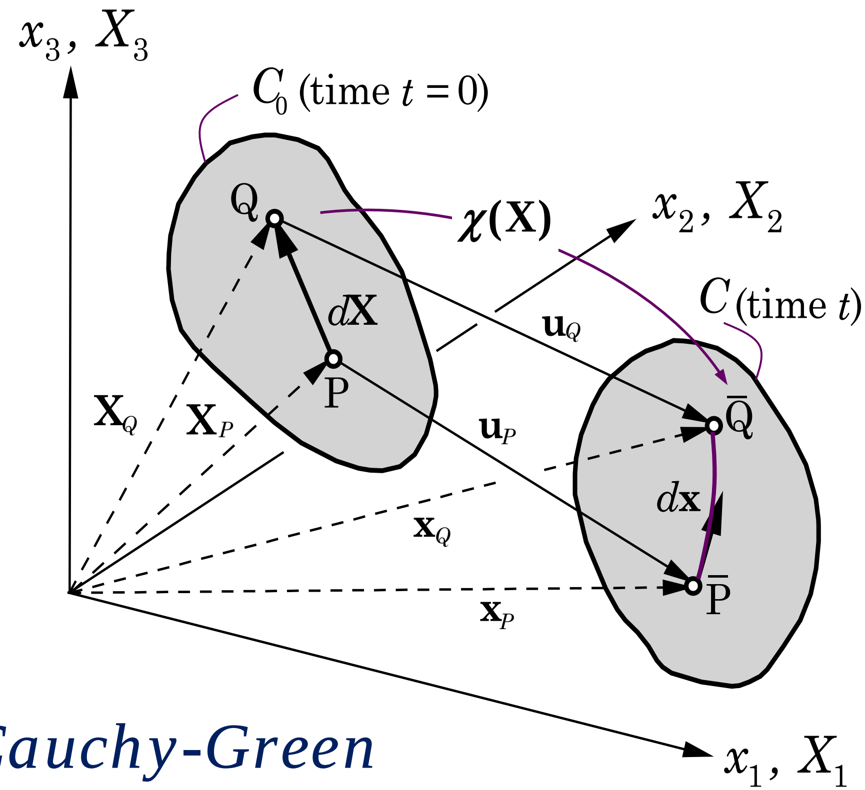


(X_1, X_2)		(x_1, x_2)
$(-1,-1)$	$\rightarrow$	$(2, 3)$
$(1, \; -1)$	$\rightarrow$	$(4, 3)$
$(1, \; 1)$	$\rightarrow$	$(5, 5)$
$(-1, \; 1)$	$\rightarrow$	$(3, \; 5)$

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \left(3.5 + 0.5X_2\right)\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 0\,\hat{\mathbf{e}}_3$$

GREEN-LAGRANGE STRAIN TENSOR

Strain is a measure of geometric changes in a solid body. The elementary mechanics definition of *the change in the length of a line element divided by the original length* cannot be used in multi-dimensions because only the square of the length of a line element (in 2-D or 3-D) can be found. Thus, we have



$$d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} = d\mathbf{X}\cdot\mathbf{F}^{\mathrm{T}}$$

$$(dS)^2 = d\mathbf{X}\cdot d\mathbf{X},$$

$$\begin{aligned}(ds)^2 &= d\mathbf{x}\cdot d\mathbf{x}\\ &= d\mathbf{X}\cdot(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\cdot d\mathbf{X}\\ &\equiv d\mathbf{X}\cdot\mathbf{C}\cdot d\mathbf{X}\end{aligned}$$

where **C** is called the *right Cauchy-Green deformation tensor*.

GREEN-LAGRANGE STRAIN TENSOR

Define the Green-Lagrange strain tensor **E** as

$$(ds)^2 - (dS)^2 \equiv 2d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X}$$

where **E** is

$$\mathbf{E} = \tfrac{1}{2}\left(\mathbf{F}^{\mathbf{T}}\cdot\mathbf{F} - \mathbf{I}\right) = \tfrac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) = \tfrac{1}{2}\left[(\mathbf{I} + \nabla_0\mathbf{u})\cdot(\mathbf{I} + \nabla_0\mathbf{u})^{\mathbf{T}} - \mathbf{I}\right]$$

$$= \tfrac{1}{2}\left[\nabla_0\mathbf{u} + (\nabla_0\mathbf{u})^{\mathbf{T}} + (\nabla_0\mathbf{u})\cdot(\nabla_0\mathbf{u})^{\mathbf{T}}\right]$$

The rectangular Cartesian component form of **E** is

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right)$$

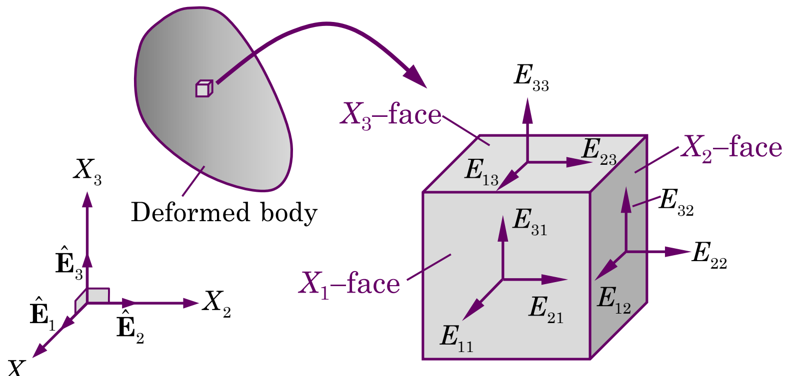
GREEN-LAGRANGE STRAIN TENSOR

The rectangular Cartesian components in explicit form are given by

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right],$$

$$E_{22} = \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right]$$

$$E_{33} = \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right]$$



E_{IJ} – Strain in the I^{th} direction on the J^{th} face

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right)$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3}\right)$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right)$$

TRANSFORMATION RELATIONS AND INVARIANTS

The strain transformation equations are the same as those for any second-order tensor:

$$\bar{E}_{ij} = \ell_{im}\ell_{jn}E_{mn} \Rightarrow [\bar{E}] = [L][E][L]^{\mathrm{T}}$$

The components of a tensor are not invariant, that is, the components depend on the coordinate system chosen. However, certain combinations of the components remain invariant under coordinate transformations. For example the trace (i.e., sum of the diagonal elements) of a second-order tensor is an invariant quantity. There are many invariants. The principal invariants of strain tensor are:

$$J_1 = E_{ii}, \quad J_2 = \tfrac{1}{2}(E_{ii}E_{jj} - E_{ij}E_{ij}), \; J_3 = |\mathbf{E}|$$

AN EXAMPLE

Problem statement:

For the deformation mapping given in an earlier example,

$$x_1 = 3.5 + X_1 + 0.5X_2, \quad x_2 = 4 + X_2, \; x_3 = X_3.$$

$$X_1 = -1.5 + x_1 - 0.5x_2, \; X_2 = -4 + x_2, X_3 = x_3.$$

determine the Cartesian components of the right Cauchy-Green deformation tensor **C** and the Green-Lagrange strain tensor **E**.

Solution: We have

$$[C] = [F]^{\mathrm{T}}[F] = \begin{bmatrix} 1.0 & 0 & 0 \\ 0.5 & 1.0 & 0 \\ 0.0 & 0.0 & 1 \end{bmatrix} \begin{bmatrix} 1.0 & 0.5 & 0 \\ 0.0 & 1.0 & 0 \\ 0.0 & 0.0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.50 & 0 \\ 0.5 & 1.25 & 0 \\ 0.0 & 0.00 & 1 \end{bmatrix}$$

$$[E] = \frac{1}{2}\left([C] - [I]\right) = \frac{1}{2} \begin{bmatrix} 0.0 & 0.50 & 0 \\ 0.5 & 0.25 & 0 \\ 0.0 & 0.00 & 0 \end{bmatrix}$$

INFITESIMAL STRAIN TENSOR

If **E** is of the order $O(\epsilon)$ in $\nabla_0 \mathbf{u}$, then we mean

$$\frac{\partial u_I}{\partial X_J} = O(\epsilon) \text{ as } \epsilon \to 0$$

If terms of the order $O(\epsilon^2)$ can be omitted, then

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J} \right)$$

can be approximated as

$$E_{IJ} \approx \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} \right) = O(\epsilon) \text{ as } \epsilon^2 \to 0$$

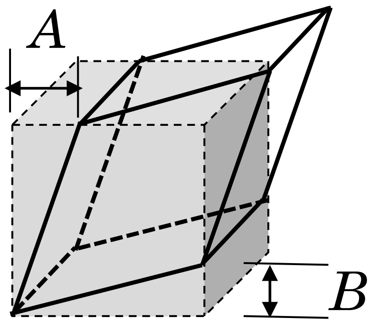
$$\mathbf{E} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^{\mathrm{T}} \right],$$ **the infinitesimal strain tensor**

EXERCISES ON KINEMATICS

1. If the deformation mapping of a body is given by

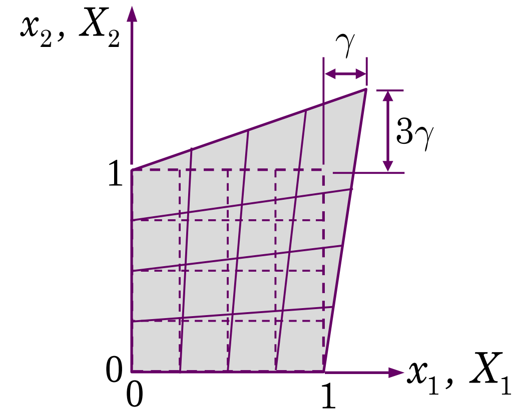
$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + AX_2)\hat{\mathbf{e}}_1 + (X_2 + BX_1)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3$$

where A and B are constants, determine (a) the displacement components in the material description, (b) the displacement components in the spatial description, and (c) the components of the Green-Lagrange strain tensor.



EXERCISES ON KINEMATICS

2. Consider a unit square block of material of thickness h (into the plane of the paper), as shown in figure below. If the block is subjected to a loading that deforms the square block into the shape shown (with no change in the thickness),



(a) determine the deformation mapping, assuming that it is a complete polynomial in X_1 and X_2 up to the term X_1X_2, (b) compute the components of the right Cauchy--Green deformation tensor **C** and Green-Lagrange strain tensor **E** at the point **X**=(1,1,0), and (c) compute the principal strains and directions at **X**=(1,1,0) for $\gamma = 1$.

EXERCISES ON KINEMATICS

3. Given the following displacement vector in a material description using a cylindrical coordinate system

$$\mathbf{u} = Ar\,\hat{\mathbf{e}}_r + Brz\,\hat{\mathbf{e}}_\theta + C\sin\theta\,\hat{\mathbf{e}}_z$$

where A, B, and C are constants, determine the infinitesimal strains. Here (r,θ,z) denote the material coordinates.

4. The two-dimensional displacement field in a body is given by

$$u_1(\mathbf{X}) = X_1\left[X_1^2X_2 + c_1\left(2c_2^3 + 3c_2^2X_2 - X_2^3\right)\right],$$

$$u_2(\mathbf{X}) = -X_2\left(2c_2^3 + \frac{3}{2}c_2^2X_2 - \frac{1}{4}X_2^3 + \frac{3}{2}c_1X_1^2X_2\right)$$

where c_1 and c_2 are constants. Find the linear and nonlinear Green-Lagrange strains.

DECOMPOSITION OF DISPLACEMENT GRADIENT TENSOR

The displacement gradient tensor, $\nabla_0 \mathbf{u}$, as the sum of symmetric $\mathbf{e}$ and skew-symmetric $\boldsymbol{\Omega}$ tensors:

$$\left(\nabla_0 \mathbf{u}\right)^{\mathrm{T}} = \tfrac{1}{2}\left[(\nabla_0 \mathbf{u})^{\mathrm{T}} + \nabla_0 \mathbf{u}\right] + \tfrac{1}{2}\left[(\nabla_0 \mathbf{u})^{\mathrm{T}} - \nabla_0 \mathbf{u}\right] \equiv \mathbf{e} + \boldsymbol{\Omega}$$

$$\mathbf{e} = \tfrac{1}{2}\left[(\nabla_0 \mathbf{u})^{\mathrm{T}} + \nabla_0 \mathbf{u}\right], \quad \boldsymbol{\Omega} = \tfrac{1}{2}\left[(\nabla_0 \mathbf{u})^{\mathrm{T}} - \nabla_0 \mathbf{u}\right]$$

For example, we have

$$\mathbf{u} = (X_1 - X_3)^2 \hat{\mathbf{e}}_1 + (X_2 + X_3)^2 \hat{\mathbf{e}}_2 - X_1 X_2 \hat{\mathbf{e}}_3$$

$$\frac{\partial u_1}{\partial X_1} = 2(X_1 - X_3), \; \frac{\partial u_1}{\partial X_2} = 0, \; \frac{\partial u_1}{\partial X_3} = -2(X_1 - X_3),$$

$$\frac{\partial u_2}{\partial X_1} = 0, \; \frac{\partial u_2}{\partial X_2} = 2(X_2 + X_3), \; \frac{\partial u_2}{\partial X_3} = 2(X_2 + X_3),$$

$$\frac{\partial u_3}{\partial X_1} = -X_2, \; \frac{\partial u_3}{\partial X_2} = -X_1, \; \frac{\partial u_3}{\partial X_3} = 0.$$

DECOMPOSITION OF DISPLACEMENT GRADIENT TENSOR

For symmetric and skew-symmetric parts are

$$
\left(\nabla_0 \mathbf{u}\right)^{\mathrm{T}} = \begin{bmatrix} 2(X_1 - X_3) & 0 & -X_2 \\ 0 & 2(X_2 + X_3) & -X_1 \\ -2(X_1 - X_3) & 2(X_2 + X_3) & 0 \end{bmatrix}
$$

$$
= \frac{1}{2}\begin{bmatrix} 4(X_1 - X_3) & 0 & -2(X_1 - X_3) - X_2 \\ 0 & 4(X_2 + X_3) & 2(X_2 + X_3) - X_1 \\ -2(X_1 - X_3) - X_2 & 2(X_2 + X_3) - X_1 & 0 \end{bmatrix}
$$

$$
+ \frac{1}{2}\begin{bmatrix} 0 & 0 & 2(X_1 - X_3) - X_2 \\ 0 & 0 & -2(X_2 + X_3) - X_1 \\ -2(X_1 - X_3) + X_2 & 2(X_2 + X_3) + X_1 & 0 \end{bmatrix}
$$

COMPATIBILITY CONDITIONS FOR SOLID CONTINUA

The task of computing strains from a given displacement field is a straightforward exercise. However, sometimes we face the problem of finding the displacements from a given strain field. This is not as straightforward because there are six independent partial differential equations (i.e., strain-displacement relations) for only three unknown displacements, which would in general over-determine the solution. We will find some conditions, known as *Saint-Venant's compatibility equations*, that will ensure the computation of a unique displacement field from a given strain field. The derivation is presented for infinitesimal strains.

COMPATIBILITY CONDITIONS FOR 2-D SOLID CONTINUA

We begin with infinitesimal strains in two dimensions. We have the following three strain-displacement relations:

$$\frac{\partial u_1}{\partial X_1} = \varepsilon_{11}, \ \frac{\partial u_2}{\partial X_2} = \varepsilon_{22}, \ \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 2\varepsilon_{12}$$

If the given data $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ are compatible, any two of the three equations should yield the same displacement components. For example, consider the following infinitesimal strain field:

$$\varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon_{12} = X_1 X_2$$

In terms of the displacement components u_1 and u_2, we have

COMPATIBILITY CONDITIONS FOR 2-D SOLID CONTINUA

$$\frac{\partial u_1}{\partial X_1} = 0, \quad \frac{\partial u_2}{\partial X_2} = 0, \quad \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 2X_1X_2$$

Integration of the first two equations gives

$$u_1 = f(X_2), \ \ u_2 = g(X_1)$$

On substitution into the shear strain, we obtain

$$\frac{df}{dX_2} + \frac{dg}{dX_1} = 2X_1X_2$$ which cannot be satisfied.

If ε_{12} is specified as, $\varepsilon_{12} = c_1X_1 + c_2X_2$ it would be possible to determine *f* and *g*, and then u_1 and u_2. Thus, not all arbitrarily specified strain fields are compatible.

COMPATIBILITY CONDITIONS FOR 2-D SOLID CONTINUA

The compatibility of a given strain field can be established as follows. Differentiate the first equation with respect to X_2 twice, the second equation with respect to X_1 twice, and the third equation with respect to X_1 and X_2 each, and obtain

$$\frac{\partial^3 u_1}{\partial X_1 \partial X_2^2} = \frac{\partial^2 \varepsilon_{11}}{\partial X_2^2}, \quad \frac{\partial^3 u_2}{\partial X_2 \partial X_1^2} = \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2}, \quad \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} + \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}$$

Using the first two equations in the third equation, we arrive at the following relation among the three strains: Equation (3.7.4) is called the strain compatibility condition for 2-D:

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}$$

AN EXAMPLE

Problem Statement:

Given the following two-dimensional, infinitesimal strain field:

$$\varepsilon_{11} = c_1 X_1 \left(X_1^2 + X_2^2 \right), \ \ \varepsilon_{22} = \frac{1}{3} c_2 X_1^3, \ \ \varepsilon_{12} = c_3 X_1^2 X_2$$

where c_1, c_2, and c_3 are constants, determine if the strain field is compatible.

Solution:

Using the 2-D compatibility condition, we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} - 2 \frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} = 2c_1 X_1 + 2c_2 X_1 - 4c_3 X_1$$

Thus the strain field is <u>not</u> compatible, unless

$$c_1 + c_2 - 2c_3 = 0$$

COMPATIBILITY CONDITIONS FOR 3-D SOLID CONTINUA

The six strain compatibility conditions for 3-D solid continua for infinitesimal deformation are as follows:

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}, \quad \frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3}, \quad \frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_1 \partial X_3} + \frac{\partial^2 \varepsilon_{13}}{\partial X_2^2} = \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_2 \partial X_3}, \quad \frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_3^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_3}$$

In index notation they can be expressed as (out of 81 only 6 are distinctly different)

$$\frac{\partial^2 \varepsilon_{mn}}{\partial X_i \partial X_j} + \frac{\partial^2 \varepsilon_{ij}}{\partial X_m \partial X_n} = \frac{\partial^2 \varepsilon_{im}}{\partial X_j \partial X_n} + \frac{\partial^2 \varepsilon_{jn}}{\partial X_i \partial X_m}$$

AN EXAMPLE

Problem Statement:

Given the following three-dimensional, infinitesimal strain field:

$$\varepsilon_{11} = X_1^2,\ \ \varepsilon_{22} = X_2^2,\ \ \varepsilon_{12} = X_1 X_3,$$

$$\varepsilon_{22} = X_3,\ \ \varepsilon_{23} = X_3^2,\ \ \varepsilon_{33} = c,\ \text{a constant}$$

determine if the strain field is compatible.

Solution: Using the first compatibility condition, we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} = 0 + 0 - 0$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_3} = 0 + 0 - 2$$

Thus the strain field is <u>not</u> compatible.

AN INTRODUCTION TO FLUID MECHANICS

Fluid mechanics is concerned with the motion of gases and liquids and their interaction with the surroundings. *At the outset one must note that fluid mechanics is seldom concerned with fluids alone but rather with effects of fluids on the surroundings with which they come in contact*. A fluid state of matter is characterized by the relative mobility of its molecules. The intermolecular forces are weaker in liquids and extremely small in gases. The stress in a fluid is proportional to the gradient of velocity, and the proportionality parameter is known as the *viscosity*. It is a measure of the intermolecular forces exerted as layers of fluid attempt to slide past one another.

DESCRIPTION OF FLUID CONTINUA

The flight of birds in the air and the motion of fish in the water can be understood by the principles of fluid mechanics. Such understanding helps us design airplanes and ships. The formation of tornadoes, hurricanes, and thunderstorms can also be explained with the help of the equations of fluid mechanics, which are expressed using the **spatial description**. That is, in fluid mechanics we are interested in the spatial location which the fluid is instantly occupies, and we do not keep track of where the fluid particles come from and where they go. Also, there is no concept of displacement in fluid mechanics. Thus, much of the preceding discussion is not relevant for study of fluid flows.

TYPES OF FLUIDS AND FLOWS

Fluids are classified based on characteristics of the fluid properties or the basic nature of the flow. An ***inviscid fluid*** is one where the viscosity is assumed to be zero. An ***incompressible fluid*** is one with constant density, and an ***incompressible flow*** is one in which density variations compared to a reference density are negligible. An inviscid and incompressible fluid is termed an *ideal* or a *perfect* fluid. A ***real fluid*** is one with finite viscosity, and it may or may not be incompressible. An **ideal fluid** is one for which $\rho = \text{constant}$ and $\mu = 0$. When the stress is linearly related to the strain rate, the fluid is said to be *Newtonian*. A ***non-Newtonian fluid*** is one which does not obey a linear stress-strain rate relation. A non-Newtonian constitutive relation can be of algebraic power-law, differential, or integral type.

VARIOUS TYPES OF FLOWS

Uniform flow occurs when the convective term $\mathbf{v} \cdot \nabla \mathbf{v}$ is negligible. The velocity field does not change with space in the direction of uniform flow. A uniform flow can be unsteady. In a nonuniform flow, velocity field is a function of position in all directions.

In a **laminar flow** fluid particles move smoothly; at high velocities, random motion, called **turbulent motion**, can occur.

Potential flows are irrotational flows of ideal fluids. Potential flows allow use of the velocity potential or stress function to write the governing equations in terms of a single variable.

VARIOUS TYPES OF FLOWS (continued)

When the curl of the velocity field in a flow is zero, then the flow is called **irrotational (**i.e., the fluid does not rotate but only translates**):** $\nabla \times \mathbf{v} = \mathbf{0}$

When the divergence of the velocity field in a flow is zero, then the fluid is called **incompressible (**i.e., the volume does not change but only its shape changes**):** $\nabla \cdot \mathbf{v} = 0$

The flow is said to be **steady** if the velocity field is not a function of time: $(\partial \mathbf{v} / \partial t) = \mathbf{0}$; when $(\partial \mathbf{v} / \partial t) \neq \mathbf{0}$, the flow is **unsteady**.

There is no concept of displacement in fluid mechanics. A **pathline** is the line traced out by a given fluid particle as it moves one point to another (the pathline is a Lagrangian concept).

DECOMPOSITION OF VELOCITY GRADIENT

In fluid mechanics, the velocity vector $\mathbf{v}(\mathbf{x},t)$ is the variable of interest. Similar to the displacement gradient tensor, we can write the *velocity gradient tensor* $\mathbf{L} \equiv (\nabla\mathbf{v})^{\mathrm{T}}$ as the sum of symmetric **D** and skew-symmetric **W** tensors:

$$\mathbf{L} \equiv (\nabla\mathbf{v})^{\mathrm{T}} = \tfrac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} + \nabla\mathbf{v}\right] + \tfrac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} - \nabla\mathbf{v}\right] \equiv \mathbf{D} + \mathbf{W}$$

where **D** is called the *rate of deformation tensor* and **W** is called the *vorticity tensor* or *spin tensor*:

$$\mathbf{D} = \tfrac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} + \nabla\mathbf{v}\right], \quad \mathbf{W} = \tfrac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} - \nabla\mathbf{v}\right]$$

The Green-Lagrange strain tensor **E** and the rate of deformation **D** are related (if and when needed):

$$\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F} \quad \text{or} \quad \mathbf{D} = \mathbf{F}^{-\mathrm{T}} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1}$$

EXERCISES ON KINEMATICS OF FLUID CONTINUA

1. Show that the components of rate of deformation tensor **D** in the cylindrical coordinate system are:

$$D_{rr} = \frac{\partial v_r}{\partial r}, \; D_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right), \; D_{rz} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$D_{\theta\theta} = \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, \; D_{z\theta} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right), \; D_{zz} = \frac{\partial v_z}{\partial z}$$

2. Show that the components of the vorticity tensor **W** in the cylindrical coordinate system are:

$$W_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r}\right) = -W_{\theta r}, \quad W_{rz} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = -W_{zr}$$

$$W_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) = -W_{\theta z}, \quad W_{rr} = W_{\theta\theta} = W_{zz} = 0$$