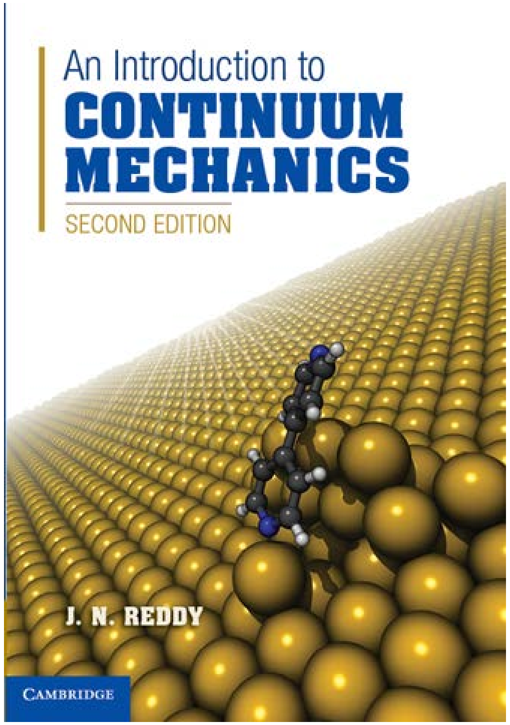
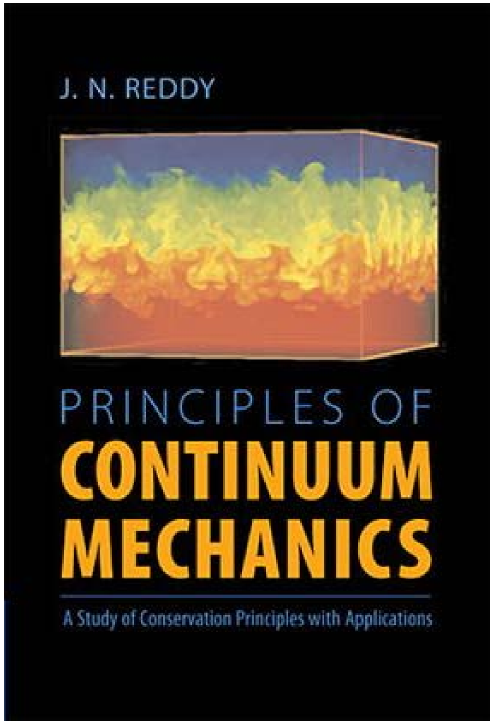
A REVIEW OF VECTORS AND TENSORS



Much of the material included herein is taken from the instructor's two books exhibited here (both published by the Cambridge University Press)



CONTENTS

INTRODUCTION TO
VECTOR AND TENSOR ANALYSIS

In the mathematical description of equations governing a continuous medium, we derive relations between various quantities that characterize the stress and deformation of the continuum by means of the laws of nature (such as Newton's laws, balance of energy, etc). As a means of expressing a natural law, a coordinate system in a chosen frame of reference is often introduced. The mathematical form of the law thus depends on the chosen coordinate system and may appear different in another type of coordinate system. The laws of nature, however, should be independent of the choice of the coordinate system, and we may seek to represent the law in a manner independent of the particular coordinate system. A way of doing this is provided by **vector and tensor analysis**.

VECTOR AND TENSOR ANALYSIS

When vector notation is used, a particular coordinate system need not be introduced. Consequently, the use of vector notation in formulating natural laws leaves them **invariant** to coordinate transformations. A study of physical phenomena by means of vector equations often leads to a deeper understanding of the problem in addition to bringing simplicity and versatility into the analysis.

In basic engineering courses, the term **vector** is used often to imply a **physical vector** that has "magnitude and direction and satisfies the parallelogram law of addition." In mathematics, vectors are more abstract objects than physical vectors. Like physical vectors, **tensors** must satisfy the rules of tensor addition and scalar multiplication.

PHYSICAL VECTORS

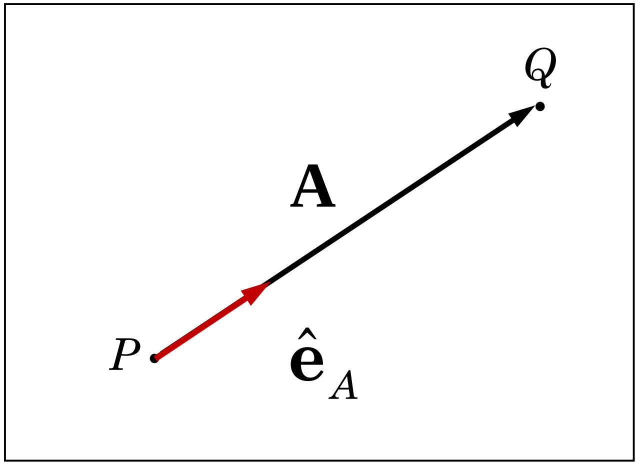
Physical vector: A directed line segment with an arrow head.

Examples: *force, displacement, velocity, weight*

Unit vector along a given vector **A**:

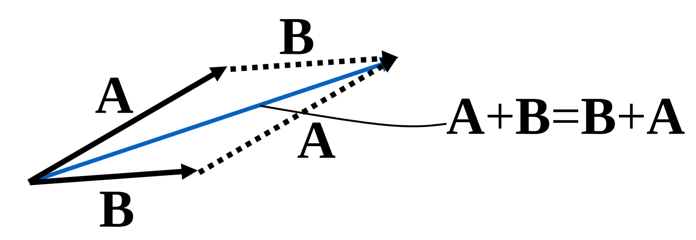
The *unit vector*, $\hat{\mathbf{e}}_A \equiv \dfrac{\mathbf{A}}{A} \;\; (A \neq 0)$

is that vector which has the same direction as **A** but has a magnitude that is unity.



MATHEMATICAL VECTORS

Rules or Axioms



Vector addition:

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B}+\mathbf{A}$ (commutative)

(ii) $(\mathbf{A} + \mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ (associative)

(iii) $\mathbf{A}+\mathbf{0}=\mathbf{A}$ (zero vector)

(iv) $\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$ (negative vector)

Scalar multiplication of a vector:

(i) $\alpha(\beta\mathbf{A})= \alpha\beta(\mathbf{A})$ (associative)

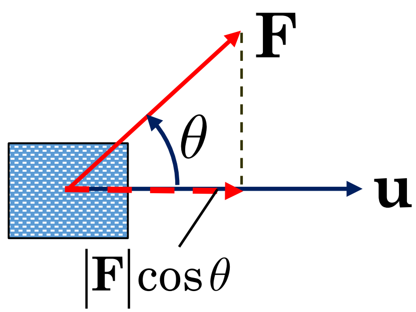
(ii) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A}+ \beta\mathbf{A}$ (distributive w.r.t. scalar addition)

(iii) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$ (distributive w.r.t. vector addition)

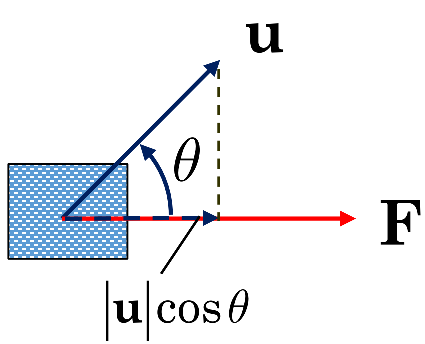
(iv) $1.\,\mathbf{A}=\mathbf{A}\,.\,1$

DOT PRODUCT OF VECTORS

Work done **Magnitude of the force multiplied by the magnitude of the displacement in the direction of the force:**



$$\mathrm{WD} = |\mathbf{F}|\cos\theta \times |\mathbf{u}|$$

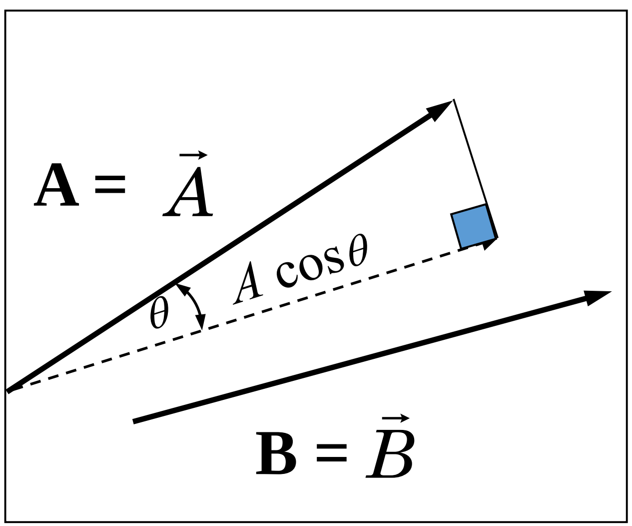


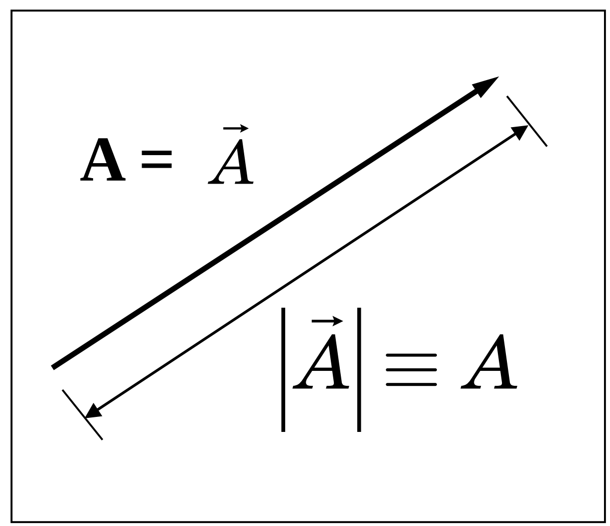
$$\mathrm{WD} = |\mathbf{F}| \times |\mathbf{u}|\cos\theta$$

VECTORS (continued)

Inner product (or scalar product) of two vectors is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta = AB\cos\theta$$

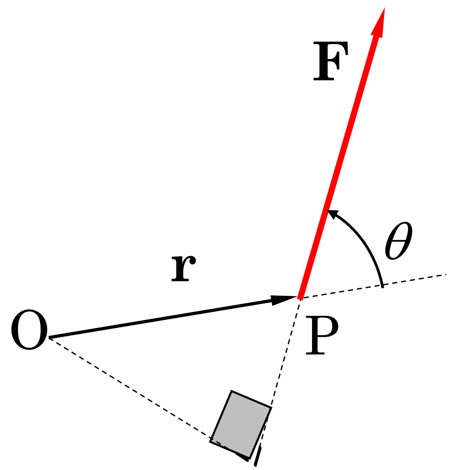


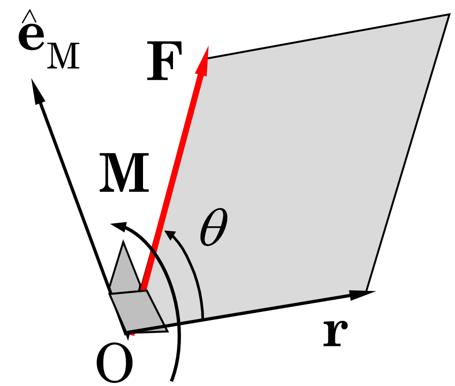


CROSS PRODUCT OF VECTORS

Moment of a force **Magnitude of the force multiplied by the magnitude of the perpendicular distance to the action of the force:**

$$\left|\mathbf{M}\right| = \ell F, \quad \mathbf{M} = \left(r \sin\theta \times F\right)\hat{\mathbf{e}}_M = \mathbf{r}\times\mathbf{F}$$



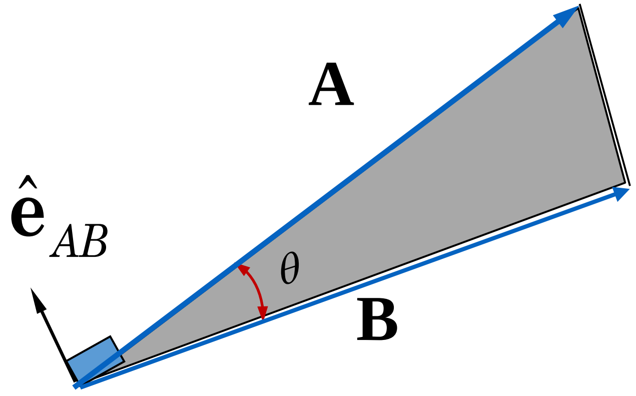


$$\ell = \left|\mathbf{r}\right|\sin\theta = r\sin\theta$$

VECTORS (continued)

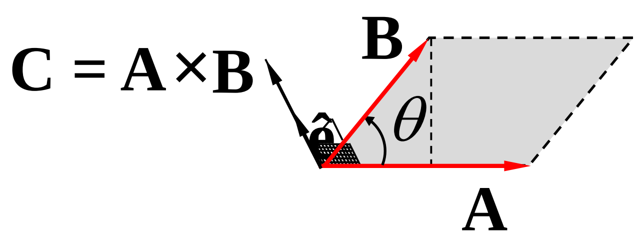
Vector product of two vectors is defined as

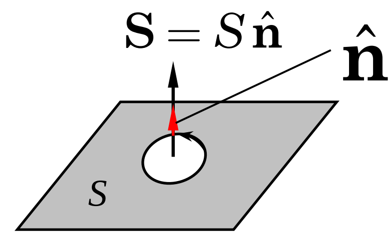
$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \; \hat{\mathbf{e}}_{AB} = AB \sin\theta \; \hat{\mathbf{e}}_{AB}$$

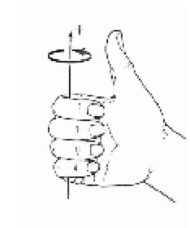


PLANE AREA AS A VECTOR

The magnitude of the vector **C** = **A** × **B** is equal to the area of the parallelogram formed by the vectors **A** and **B**.



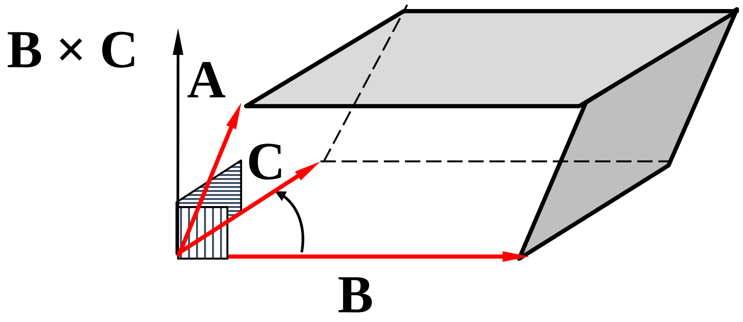




In fact, the vector **C** may be considered to represent both the *magnitude* and *the direction* of the product of **A** and **B**. Thus, a plane area in space may be looked upon as possessing a direction in addition to a magnitude, the directional character arising out of the need to specify an orientation of the plane area in space. Representation of an area as a vector has many uses in mechanics, as will be seen in the sequel.

SCALAR TRIPLE PRODUCT

The product **A** . (**B** × **C**) is a scalar and it is termed *the scalar triple product*. It can be seen from the figure that the product **A** . (**B** × **C**) , except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A**, **B**, and **C**.



A . (**B** × **C**)

EXERCISES ON VECTORS

1. If two vectors are such that $\mathbf{A}\cdot\mathbf{B}=0$ what can we conclude?

2. If two vectors are such that $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ what can we conclude?

3. Prove that $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$

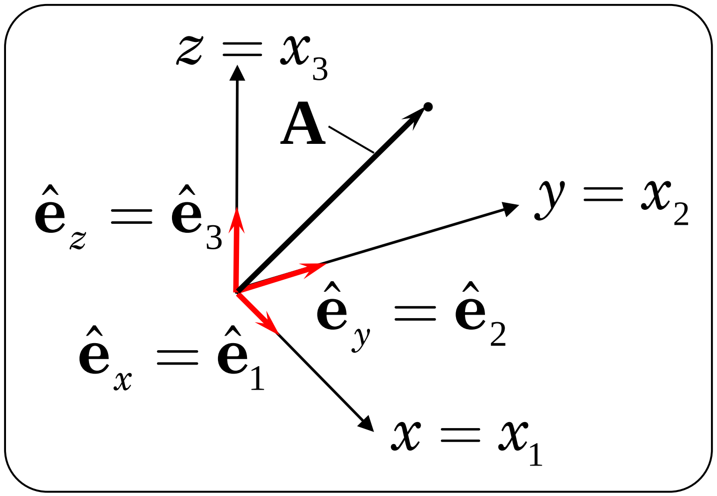
4. If three vectors are such that $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=0$ what can we conclude?

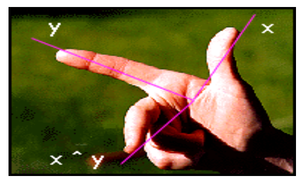
5. The velocity vector in a flow field is $\mathbf{v}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ (m/ s). Determine (a) the velocity vector $\mathbf{v}_n$ normal to the plane $\mathbf{n}=3\hat{\mathbf{i}}-4\hat{\mathbf{k}}$ passing through the point, (b) the angle between $\mathbf{v}$ and $\mathbf{v}_n$, (c) tangential velocity vector on the plane, and (d) The mass flow rate across the plane through an area $A=0.15\ \mathrm{m}^2$ if the fluid density is $\rho=10^3\ \mathrm{kg/\ m}^3$ and the flow is uniform.

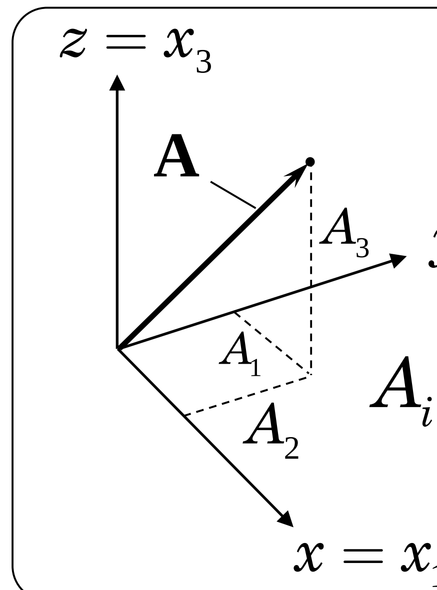
COMPONENTS OF VECTORS

Components of a vector

$$\mathbf{A} = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z$$
$$= A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{n}} = n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y + n_z\hat{\mathbf{e}}_z$$
$$= n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2 + n_3\hat{\mathbf{e}}_3$$







$$A_i = \mathbf{A}\cdot\hat{\mathbf{e}}_i$$

$$\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 = 0,\quad \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3 = 0,$$
$$\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 = 1,\quad \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3 = 0,\quad \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3 = 1,$$
$$\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 = 0,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3,$$
$$\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2$$

AN EXERCISE

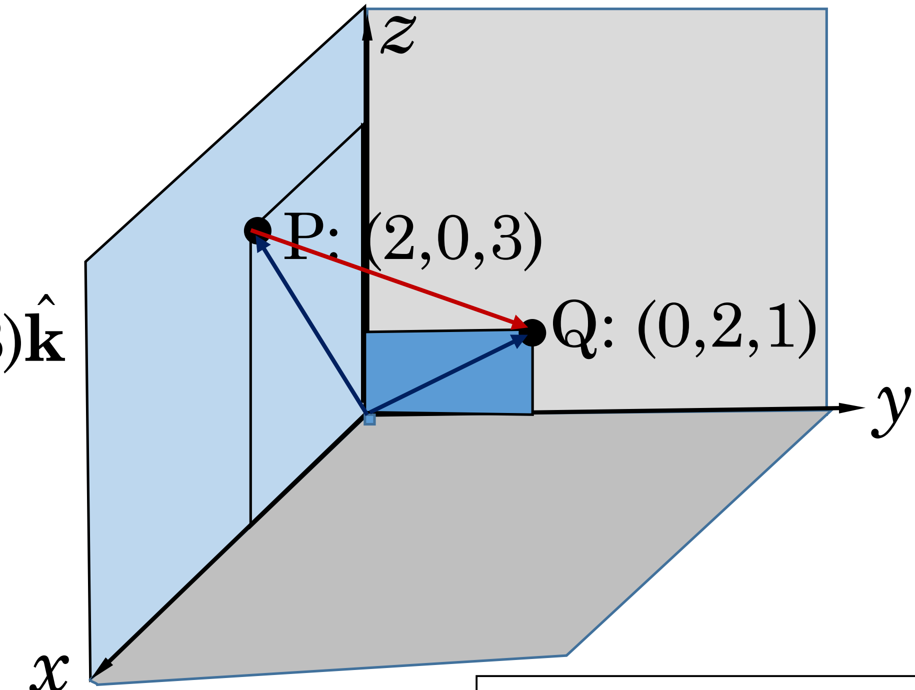
<u>**Problem**</u>**:** Find the vector connecting point P: (2,0,3) to point Q: (0,2,1).

Solution: The points P and Q are shown in the figure using their coordinates. We have

$$\vec{P} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{k}}, \quad \vec{Q} = 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\vec{P}\vec{Q} = (0-2)\hat{\mathbf{i}} + (2-0)\hat{\mathbf{j}} + (1-3)\hat{\mathbf{k}}$$

$$= -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$



SUMMATION CONVENTION

Omit the summation sign and understand that a <u>repeated index</u> is to be <u>summed</u> over its range:

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

Dummy index

$$= \sum_{i=1}^{3} A_i\hat{\mathbf{e}}_i = A_i\hat{\mathbf{e}}_i \text{ (summation convention)}$$

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_i\right)\hat{\mathbf{e}}_i = \left(\mathbf{A}\cdot\hat{\mathbf{e}}_j\right)\hat{\mathbf{e}}_j$$

Dummy indices

<u>Scalar product</u>

$$\mathbf{A}\cdot\mathbf{B} = \left(A_i\hat{\mathbf{e}}_i\right)\cdot\left(B_j\hat{\mathbf{e}}_j\right)$$

$$= A_iB_j\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right)$$

$$= A_iB_j\delta_{ij} = A_iB_i$$

$$\delta_{ij} \equiv \left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right) = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

SUMMATION CONVENTION (continued)

Vector product

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\ \hat{\mathbf{e}}_{AB}$$
$$= \left(A_i\hat{\mathbf{e}}_i\right)\times\left(B_j\hat{\mathbf{e}}_j\right) = A_iB_j\left(\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\right)$$
$$= A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j = \varepsilon_{ijk}\hat{\mathbf{e}}_k$$

$$\varepsilon_{ijk} \equiv \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k = \begin{cases} 0, \text{ if any two indices are the same} \\ 1, \textit{if } i\neq j\neq k, \text{ and they permute in a natural order} \\ -1, \textit{if } i\neq j\neq k, \text{and they permute } \textit{opposite} \text{ to a natural order}\end{cases}$$

SUMMATION CONVENTION (continued)

Contraction of indices:

The Kronecker delta modifies (or contracts) the subscripts in the coefficients of an expression in which it appears:

$$A_i\delta_{ij} = A_j, \quad A_iB_j\delta_{ij} = A_iB_i = A_jB_j, \quad \delta_{ij}\delta_{ik} = \delta_{jk}$$

Correct expressions:

$$F_i = A_iB_jC_j, \; G_k = H_k(2-3A_iB_i) + P_jQ_jF_k$$

Free indices

Incorrect expressions:

$$A_i = B_jC_k, \; A_i = B_j \text{ and } F_k = A_iB_jC_k$$

SUMMATION CONVENTION (continued)

One must be careful when substituting a quantity with an index into an expression with indices or solving for one quantity with index in terms of the others with indices in an equation. For example, consider the equations

$$p_i = a_i b_j c_j \text{ and } c_k = d_i e_i q_k$$

It is correct to write

$$a_i = \frac{p_i}{b_j c_j}$$

but it is incorrect to write

$$b_j c_j = \frac{p_i}{a_i}$$

which has a totally different meaning

$$b_j c_j = \frac{p_i}{a_i} = \frac{p_1}{a_1} + \frac{p_2}{a_2} + \frac{p_3}{a_3}$$

SUMMATION CONVENTION (continued)

The permutation symbol and the Kronecker delta prove to be very useful in establishing vector identities. Since a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to prove it in one coordinate system. The following identity is useful:

$$\boxed{\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$$

The determinant of the matrix of a second-order tensor can be expressed as

$$|\mathbf{S}| = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} = (s_{1i}\hat{\mathbf{e}}_i)\cdot(s_{2j}\hat{\mathbf{e}}_j)\times(s_{3k}\hat{\mathbf{e}}_k) = \varepsilon_{ijk}\, s_{1i}\, s_{2j}\, s_{3k}$$

EXERCISES ON INDEX NOTATION

Exercise-1: Check which one of the following expressions are valid:

(a) $a_m b_s = c_m(d_r - f_r)$; (b) $a_m b_s = c_m(d_s - f_s)$

(c) $a_i = b_j c_i (d_i - f_i)$; (d) $x_m x_m = r^2$

(e) $a_i = 3$; (f) $\delta_{ij}\,\delta_{jk}\,\delta_{ki} = ?$

Exercise-2: Prove $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \varepsilon_{ijk} A_i B_j C_k = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Exercise-3: Simplify the expression $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$

Exercise-4: Simplify the expression $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

Exercise-5: Rewrite the expression $\varepsilon_{mni} A_i B_j C_m D_n \hat{\mathbf{e}}_j$ in vector form

SECOND-ORDER TENSORS

A second-order tensor is one that has two basis vectors standing next to each other, and they satisfy the same rules as those of a vector (hence, mathematically, tensors are also called vectors). A second-order tensor and its *transpose* can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{S} = S_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ji}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i; \qquad \mathbf{S}^{\mathrm{T}} = S_{ji}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = S_{ij}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i$$

A second-order tensor is <u>symmetric</u> only if

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \Leftrightarrow S_{ij} = S_{ji}$$

Second-order <u>identity tensor</u> has the form

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

SECOND-ORDER TENSORS

We note that $\mathbf{S}\cdot\mathbf{T} \neq \mathbf{T}\cdot\mathbf{S}$ (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors) because

$$\mathbf{S}\cdot\mathbf{T} = \left(S_{ij}\ \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(T_{kl}\ \hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{jl}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

$$\mathbf{T}\cdot\mathbf{S} = \left(T_{ij}\ \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(S_{kl}\ \hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = T_{ij}S_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{jl}T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$$

We also note that (where $\mathbf{S}$ and $\mathbf{T}$ are second-order tensors and $\mathbf{A}$ is a vector)

$$\mathbf{S}\times\mathbf{T} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(T_{kl}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right) = S_{ij}T_{kl}\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right)\hat{\mathbf{e}}_l = S_{ij}T_{kl}\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p\hat{\mathbf{e}}_l$$

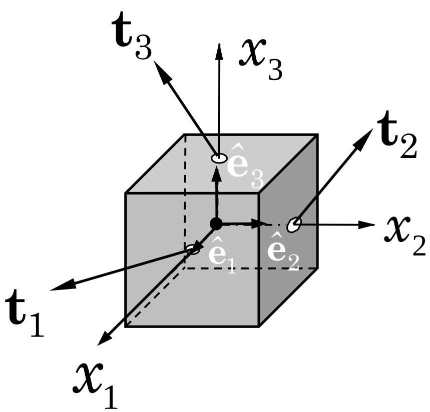
$$\mathbf{S}\cdot\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\cdot\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right) = S_{ij}A_j\hat{\mathbf{e}}_i$$

$$\mathbf{S}\times\mathbf{A} = \left(S_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\right)\times\left(A_k\,\hat{\mathbf{e}}_k\right) = S_{ij}A_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\right) = S_{ij}A_k\varepsilon_{jkp}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_p$$

CAUCHY STRESS TENSOR

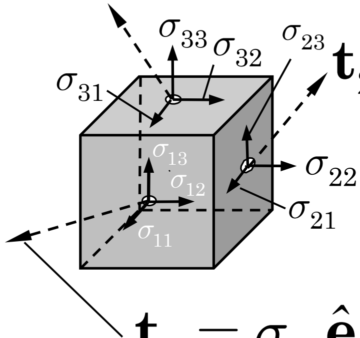
Stress tensor is a good example of a second-order tensor. The two basis vectors represent the direction and the plane on which they act. The Cauchy stress tensor is defined by the Cauchy formula (to be established in the sequel):

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \text{ or } t_i = n_j \sigma_{ji} = \sigma_{ji} n_j$$



$$\mathbf{t}_3 = \sigma_{31}\hat{\mathbf{e}}_1 + \sigma_{32}\hat{\mathbf{e}}_2 + \sigma_{33}\hat{\mathbf{e}}_3$$

$$\mathbf{t}_2 = \sigma_{21}\hat{\mathbf{e}}_1 + \sigma_{22}\hat{\mathbf{e}}_2 + \sigma_{23}\hat{\mathbf{e}}_3$$



$$\mathbf{t}_1 = \sigma_{11}\hat{\mathbf{e}}_1 + \sigma_{12}\hat{\mathbf{e}}_2 + \sigma_{13}\hat{\mathbf{e}}_3$$

$$\mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_j \;, \quad \sigma = \hat{\mathbf{e}}_i\mathbf{t}_i = \hat{\mathbf{e}}_i\sigma_{ij}\hat{\mathbf{e}}_j = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

HIGHER-ORDER TENSORS

A n^{th}-order tensor is one that has n basis vectors standing next to each other, and they satisfy the same rules as those of a vector. A n^{th}-order tensor **T** can be expressed in terms of rectangular Cartesian base vectors as

$$\mathbf{T} = T_{\underbrace{ijk\ldots p}_{n \text{ subs}}} \underbrace{\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \cdots \hat{\mathbf{e}}_p}_{n \text{ base vectors}};$$

The permutation tensor is a <u>third-order</u> tensor

$$\boldsymbol{\varepsilon} = \varepsilon_{ijk} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k$$

The elasticity tensor is a <u>fourth-order</u> tensor

$$\mathbf{C} = C_{ijkl} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \hat{\mathbf{e}}_l$$

TRANSFORMATION OF TENSOR COMPONENTS

A second-order Cartesian tensor **S** may be represented in barred $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and unbarred (x_1, x_2, x_3) Cartesian coordinate systems as

$$\mathbf{S} = s_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = \bar{s}_{mn}\hat{\bar{\mathbf{e}}}_m\hat{\bar{\mathbf{e}}}_n$$

The unit base vectors in the unbarred and barred systems are related by

$$\hat{\mathbf{e}}_j = \ell_{ij}\hat{\bar{\mathbf{e}}}_i \text{ and } \hat{\bar{\mathbf{e}}}_i = \ell_{ij}\hat{\mathbf{e}}_j, \quad \ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j$$

Thus the components of a second-order tensor transform according to

$$\boxed{\bar{s}_{ij} = \ell_{im}\ell_{jn}s_{mn} \;\Rightarrow\; [\bar{S}] = [L][S][L]^{\mathrm{T}}}$$

The components of a *4*th-order tensor transform as

$$\boxed{\bar{c}_{ijkl} = \ell_{im}\ell_{jn}\ell_{kp}\ell_{lq}\, c_{mnpq}}$$

INVARIANTS OF A SECOND-ORDER TENSOR

When a combination of components of a tensor remains the same in all coordinate systems, they are said to be invariant. There are many invariants of a tensor. For example,

$$(1) \qquad S_{ii} = S_{11} + S_{22} + S_{33}$$

is invariant, because

$$\bar{S}_{ii} = S_{mn} l_{im} l_{in} = S_{mn} \delta_{mn} = S_{mm} = S_{ii}$$

The determinant of a second –order tensor remains the same in all coordinate systems. The determinant of a second-order tensor can be expressed as (exercises to the reader)

$$\text{(a)} \ |\mathbf{S}| = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} = \tfrac{1}{6} s_{ir} s_{js} s_{kt} \varepsilon_{ijk} \varepsilon_{rst}, \quad \text{(b)} \ \varepsilon_{rst} |\mathbf{S}| = \varepsilon_{ijk} s_{ir} s_{js} s_{kt}$$

INVARIANTS OF A SECOND-ORDER TENSOR

$$\left|\overline{\mathbf{S}}\right| = \tfrac{1}{6}\overline{s}_{ir}\overline{s}_{js}\overline{s}_{kt}\varepsilon_{ijk}\varepsilon_{rst} = \tfrac{1}{6}\left(s_{mn}l_{im}l_{rn}\right)\left(s_{pq}l_{jp}l_{sq}\right)\left(s_{ab}l_{ka}l_{tb}\right)\varepsilon_{ijk}\varepsilon_{rst}$$

$$= \tfrac{1}{6}s_{mn}s_{pq}s_{ab}l_{rn}l_{sq}l_{tb}\left(l_{im}l_{jp}l_{ka}\varepsilon_{ijk}\right)\varepsilon_{rst}$$

$$= \tfrac{1}{6}s_{mn}s_{pq}s_{ab}\varepsilon_{mpa}\left(l_{rn}l_{sq}l_{tb}\varepsilon_{rst}\right)$$

$$= \tfrac{1}{6}s_{mn}s_{pq}s_{ab}\varepsilon_{mpa}\varepsilon_{nqb} \Rightarrow$$ (2) $\left|\mathbf{S}\right|$ is invariant

$$\overline{s}_{ij}\overline{s}_{ij} = \left(s_{mn}l_{im}l_{jn}\right)\left(s_{pq}l_{ip}l_{jq}\right) = s_{mn}s_{pq}l_{im}l_{ip}\,l_{jn}l_{jq}$$

$$= s_{mn}s_{pq}\delta_{mp}\delta_{nq} = s_{mn}s_{mn}$$

(3) Thus $s_{ij}s_{ij}$ is invariant

The well-known three invariants are:

(1) $I_1 = s_{ii}$, (2) $I_2 = \tfrac{1}{2}\left(s_{ii}s_{jj} - s_{ij}s_{ij}\right)$, (3) $I_3 = \left|\mathbf{S}\right|$

EIGEN VALUES OF A SECOND-ORDER TENSOR

A second-order tensor can be viewed as an operator that changes a vector into another vector (by means of the dot product). Then it is possible that there may exist certain vectors that have only their lengths, not their orientations, changed when operated upon by the tensor: If such vectors exist, then we have

$$\mathbf{S}\cdot\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}}$$

Such vectors are called characteristic vectors, or eigenvectors, associated with $\mathbf{S}$. The parameter is called the eigenvalue, and it characterizes the change in length. Thus, we have

$$\left(\mathbf{S} - \lambda\mathbf{I}\right)\cdot\hat{\mathbf{n}} = \mathbf{0} \qquad (1)$$

EIGEN VALUES OF A SECOND-ORDER TENSOR

The vanishing of the determinant, $\left|\mathbf{S} - \lambda\mathbf{I}\right| = 0$, yields a cubic equation for λ, called the *characteristic equation:*

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where $I_1, I_2, \text{and}\, I_3$ are the invariants of $\mathbf{S}$ as defined

$$I_1 = S_{ii}, \quad I_2 = \tfrac{1}{2}(S_{ii}S_{jj} - S_{ij}S_{ij}), \quad I_3 = |\mathbf{S}|,$$

$$I_2 = \tfrac{1}{2}\left(I_1^2 - S_{11}^2 - S_{22}^2 - S_{33}^2 - S_{12}^2 - S_{13}^2 - S_{23}^2 - S_{21}^2 - S_{31}^2 - S_{32}^2\right)$$

The eigenvector $\hat{\mathbf{n}}^{(i)}$ associated with any particular eigenvalue λ_i is calculated using Eq. (1), which gives only two independent relations among the three components $n_1^{(i)}, n_2^{(i)}, \text{and}\, n_3^{(i)}$. The third equation is provided by

$$(n_1^{(i)})^2 + (n_2^{(i)})^2 + (n_3^{(i)})^2 = 1$$

AN EXAMPLE OF AN EIGENVALUE PROBLEM

Problem: Given the matrix of a second-order tensor

$$[S]=\begin{bmatrix}57 & 0 & 24\\ 0 & 50 & 0\\ 24 & 0 & 43\end{bmatrix}$$

Find the eigenvalues and eigenvectors associated with the minimum eigenvalue.

Solution: we set

$$\begin{vmatrix}57-\lambda & 0 & 24\\ 0 & 50-\lambda & 0\\ 24 & 0 & 43-\lambda\end{vmatrix}=(57-\lambda)(50-\lambda)(43-\lambda)-(50-\lambda)24\times 24$$

and obtain the eigenvalues $\lambda_1=25,\ \lambda_2=50,\ \lambda_3=75.$ The eigenvector is given by

$$\mathbf{n}^{(1)}=\pm\left(\frac{3}{5}\hat{\mathbf{e}}_1-\frac{4}{5}\hat{\mathbf{e}}_3\right)$$

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

"Del" operator: $\nabla \equiv \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i} = \hat{\mathbf{e}}_1 \dfrac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \dfrac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \dfrac{\partial}{\partial x_3}$

"Laplace" operator:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

"Gradient" operation:

$$\nabla F \equiv \hat{\mathbf{e}}_i \frac{\partial F}{\partial x_i}, \text{ where } F \text{ is a scalar function}$$

Grad F defines both the direction and magnitude of the maximum rate of increase of F at any point.

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

$$\nabla F = \hat{\mathbf{n}} \frac{\partial F}{\partial n}, \text{ where } \hat{\mathbf{n}} \text{ is a unit vector normal to the surface } F = \text{constant}$$

$$\text{We also have } \hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} \text{ and } \frac{\partial F}{\partial n} = \hat{\mathbf{n}} \cdot \nabla F$$

"Divergence" operation:

$$\nabla \cdot \mathbf{G} \equiv \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\hat{\mathbf{e}}_j G_j \right) = \frac{\partial G_i}{\partial x_i}, \text{ where } \mathbf{G} \text{ is a } \textit{vector} \text{ function}$$

The divergence of a vector function represents the volume density of the outward flux of the vector field.

THE DEL OPERATOR AND ITS PROPERTIES IN RECTANGULAR CARTESIAN SYSTEM

"Curl" operation:

$$\nabla \times \mathbf{G} \equiv \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \times \left(\hat{\mathbf{e}}_j G_j \right) = \varepsilon_{ijk} \frac{\partial G_j}{\partial x_i} \hat{\mathbf{e}}_k,$$

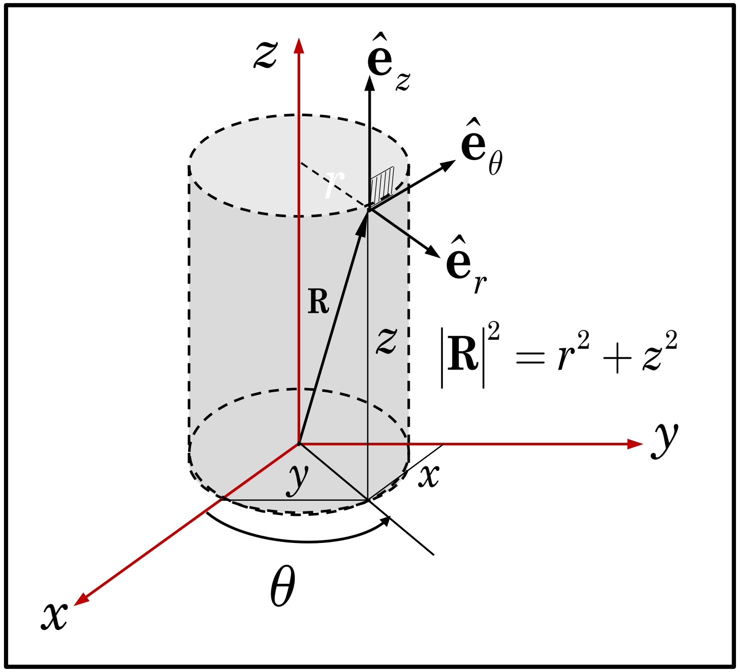
where $\mathbf{G}$ is a *vector* function.

The curl of a vector function represents its rotation. If the vector field is the velocity of a fluid, curl of the velocity represents the rotation of the fluid at the point.

CYLINDRICAL COORDINATE SYSTEM

$$\begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$



$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$

"Del" operator in cylindrical coordinates

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

CYLINDRICAL COORDINATE SYSTEM

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\left[\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r\frac{\partial A_z}{\partial z}\right]$$

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\right]$$

Here **A** is a vector:

$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$$

Verify these relations to yourself

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{e}}_z$$

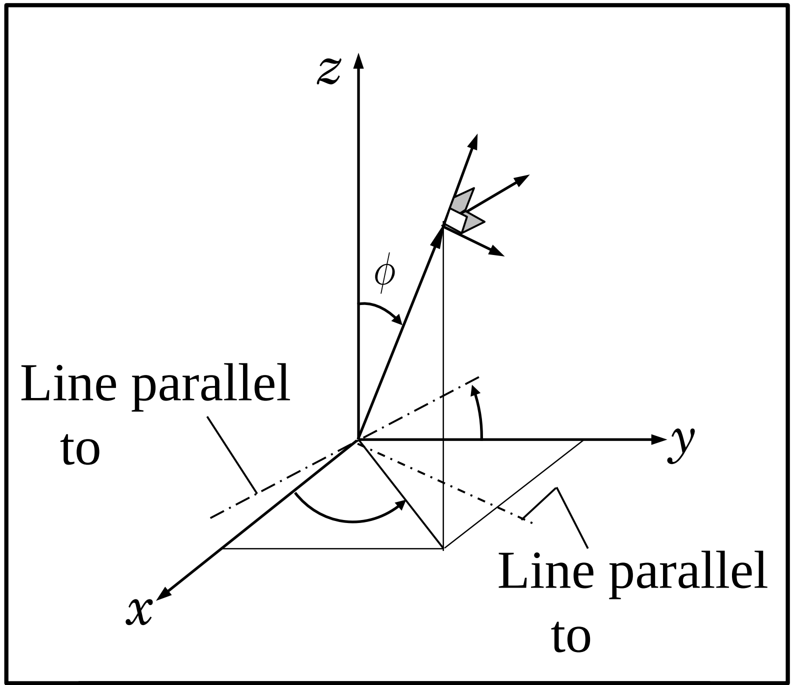
$$\nabla \mathbf{A} = \frac{\partial A_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{\partial A_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - A_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \frac{\partial A_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \frac{\partial A_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r$$

$$+ \frac{1}{r}\left(A_r + \frac{\partial A_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial A_z}{\partial \theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial A_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial A_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z$$

SPHERICAL COORDINATE SYSTEM

$$\begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix}$$



$$\mathbf{A} = A_R\hat{\mathbf{e}}_R + A_\phi\hat{\mathbf{e}}_\phi + A_\theta\hat{\mathbf{e}}_\theta$$

$$\frac{\partial \hat{\mathbf{e}}_R}{\partial \phi} = \hat{\mathbf{e}}_\phi, \quad \frac{\partial \hat{\mathbf{e}}_R}{\partial \theta} = \sin\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\hat{\mathbf{e}}_R, \quad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \theta} = \cos\phi\,\hat{\mathbf{e}}_\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\sin\phi\,\hat{\mathbf{e}}_R - \cos\phi\,\hat{\mathbf{e}}_\phi$$

"Del" operator

$$\nabla = \hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta}$$

SPHERICAL COORDINATE SYSTEM

$$\nabla^2 = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right]$$

$$\nabla\cdot\mathbf{A} = \frac{2A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{A} = \frac{1}{R\sin\phi}\left[\frac{\partial(\sin\phi A_\theta)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_R + \left[\frac{1}{R\sin\phi}\frac{\partial A_R}{\partial\theta} - \frac{1}{R}\frac{\partial(RA_\theta)}{\partial R}\right]\hat{\mathbf{e}}_\phi + \frac{1}{R}\left[\frac{\partial(RA_\phi)}{\partial R} - \frac{\partial A_R}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$$

$$\nabla\mathbf{A} = \frac{\partial A_R}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_R + \frac{\partial A_\phi}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\phi + \frac{1}{R}\left(\frac{\partial A_R}{\partial\phi} - A_\phi\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_R + \frac{\partial A_\theta}{\partial R}\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_R}{\partial\theta} - A_\theta\sin\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_R$$

$$+\frac{1}{R}\left(A_R + \frac{\partial A_\phi}{\partial\phi}\right)\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial A_\theta}{\partial\phi}\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial A_\phi}{\partial\theta} - A_\theta\cos\phi\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\phi$$

$$+\frac{1}{R\sin\phi}\left(A_R\sin\phi + A_\phi\cos\phi + \frac{\partial A_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$$

EXERCISES ON VECTOR CALCULUS

Establish the following identities (using rectangular Cartesian components and index notation):

1. $\nabla(r) = \dfrac{\mathbf{r}}{r}$
2. $\nabla(r^n) = nr^{n-2}\mathbf{r}$
3. $\nabla \times (\nabla F) = \mathbf{0}$
4. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
5. $\nabla \cdot (\nabla F \times \nabla G) = 0$
6. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
7. $\nabla \cdot \left(\mathbf{A} \times \mathbf{B}\right) = \left(\nabla \times \mathbf{A}\right) \cdot \mathbf{B} - \left(\nabla \times \mathbf{B}\right) \cdot \mathbf{A}$
8. $\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A} \cdot \nabla \mathbf{A}.$

Exercise: Check appropriate box

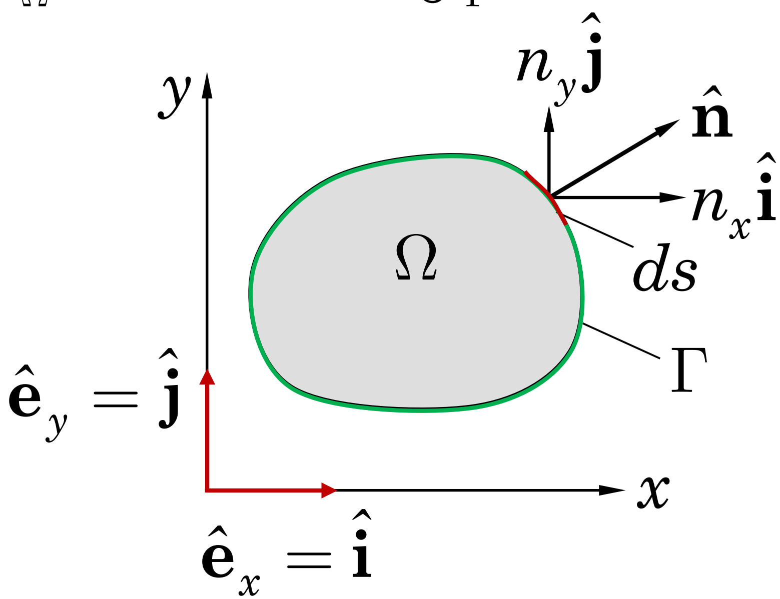
Quantity	Vector	Scalar	Nonsense
$\nabla\times(\nabla f)$	☐	☐	☐
$\nabla\cdot(\nabla\times\mathbf{F})$	☐	☐	☐
$\nabla\cdot(\nabla\times f)$	☐	☐	☐
$\nabla\times(\nabla\times\mathbf{F})$	☐	☐	☐
$\nabla\cdot(\nabla\cdot\mathbf{F})$	☐	☐	☐
$\nabla\times(\nabla\cdot\mathbf{F})$	☐	☐	☐

INTEGRAL THEOREMS
involving the del operator

$$\int_\Omega \nabla\phi \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\phi \, ds \quad \text{(Gradient theorem)}$$

$$\int_\Omega \nabla\cdot\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{A} \, ds \quad \text{(Divergence theorem)}$$

$$\int_\Omega \nabla\times\mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\times\mathbf{A} \, ds \quad \text{(Curl theorem)}$$

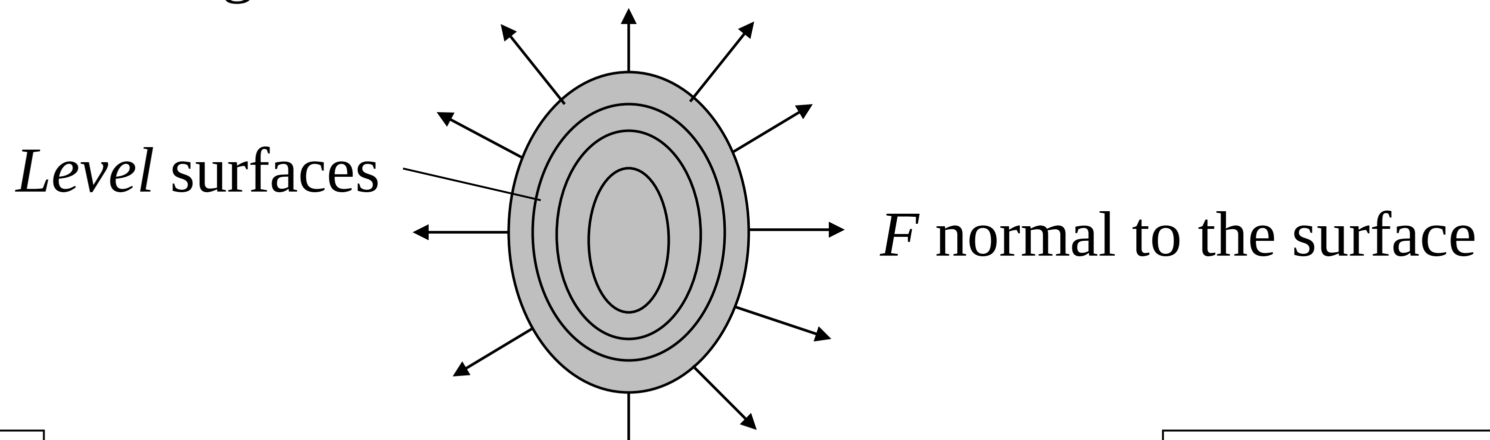


$$\begin{aligned}\hat{\mathbf{n}} &= n_x\,\hat{\mathbf{i}} + n_y\,\hat{\mathbf{j}} \\ &= n_x\,\hat{\mathbf{e}}_x + n_y\,\hat{\mathbf{e}}_y \\ &= n_1\,\hat{\mathbf{e}}_1 + n_2\,\hat{\mathbf{e}}_2\end{aligned}$$

THE GRADIENT THEOREM

$$\oint_{\Gamma} \hat{\mathbf{n}}\, F \; d\Gamma = \int_{\Omega} \nabla F \, d\Omega$$

The **gradient** of a function F represents the rate of change of F with respect to the coordinate directions. the partial derivative with respect to *x*, for example, gives the rate of change of *F* in the *x* direction.



GRADIENT OF A SCALAR FUNCTION

$$\nabla F = \text{a first-order tensor, that is, a } \textit{vector}$$

$$\nabla = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla F = \frac{\partial F}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial F}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial F}{\partial z}\hat{\mathbf{e}}_x$$

in rectangular Cartesian system

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

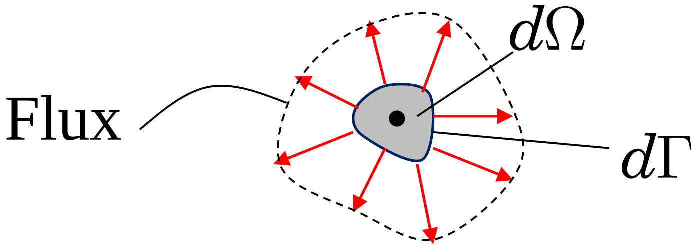
$$\nabla F = \hat{\mathbf{e}}_r \frac{\partial F}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial F}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial F}{\partial z}$$

in cylindrical coordinate system

THE DIVERGENCE THEOREM

$$\oint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega$$

The **divergence** represents the volume density of the outward flux of a **vector** field **F** from an infinitesimal volume $d\Omega$ around a given point. It is a local measure of its "outgoingness."



DIVERGENCE OF FIRST-ORDER TENSORS

$$\nabla \cdot \mathbf{F} = \text{a zeroth tensor, that is, a scalar}$$

$$\nabla = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

in rectangular Cartesian system

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\left[\frac{\partial (rF_r)}{\partial r} + \frac{\partial F_\theta}{\partial \theta} + r\frac{\partial F_z}{\partial z}\right]$$

in cylindrical coordinate system

DIVERGENCE OF SECOND-ORDER TENSORS

$$\nabla \cdot \mathbf{S} = \text{a first-order tensor, that is, a vector}$$

$$\nabla \cdot \mathbf{S} = \left(\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial F_{zx}}{\partial z} \right) \hat{\mathbf{e}}_x + \left(\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial F_{zy}}{\partial z} \right) \hat{\mathbf{e}}_y$$

$$+ \left(\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) \hat{\mathbf{e}}_z$$

in rectangular Cartesian system

$$\nabla \cdot \mathbf{S} = \frac{1}{r} \left[\frac{\partial (r S_{rr})}{\partial r} + \frac{\partial S_{\theta r}}{\partial \theta} + r \frac{\partial S_{zr}}{\partial z} - S_{\theta\theta} \right] \hat{\mathbf{e}}_r$$

$$+ \frac{1}{r} \left[\frac{\partial (r S_{r\theta})}{\partial r} + \frac{\partial S_{\theta\theta}}{\partial \theta} + r \frac{\partial S_{z\theta}}{\partial z} + S_{\theta r} \right] \hat{\mathbf{e}}_\theta$$

$$+ \frac{1}{r} \left[\frac{\partial (r S_{rz})}{\partial r} + \frac{\partial S_{\theta z}}{\partial \theta} + r \frac{\partial S_{zr}}{\partial z} \right] \hat{\mathbf{e}}_z$$

in cylindrical coordinate system

CURL (STOKES's) THEOREM

$$\oint_{\Gamma} \hat{\mathbf{n}} \times \mathbf{F} \, d\Gamma = \int_{\Omega} \nabla \times \mathbf{F} \, d\Omega$$

The **curl** of a vector **F** describes the infinitesimal rotation of **F**. A physical interpretation is as follows. Suppose the vector field describes the velocity field **F** = **v** of a fluid flow, say, in a large tank of liquid, and a small spherical ball is located within the fluid (the center of the ball being fixed at a certain point but free to rotate about an axis perpendicular to the plane of the flow). If the ball has a rough surface, the fluid flowing past the ball will make it rotate. The rotation axis (oriented according to the right hand rule) points in the direction of the curl of the field at the center of the ball, and the angular speed of the rotation is half the magnitude of the curl at this point.

Curl of a Vector Function in (x,y,z) system

$$\nabla \times \mathbf{A} = \text{a first-order tensor, that is, a vector}$$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{\partial A_x}{\partial x}\overbrace{\left(\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_x\right)}^{0} + \frac{\partial A_y}{\partial x}\overbrace{\left(\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y\right)}^{\hat{\mathbf{e}}_z} + \frac{\partial A_z}{\partial x}\overbrace{\left(\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_z\right)}^{-\hat{\mathbf{e}}_y} \\
&+ \frac{\partial A_x}{\partial y}\overbrace{\left(\hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_x\right)}^{-\hat{\mathbf{e}}_z} + \frac{\partial A_y}{\partial y}\overbrace{\left(\hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_y\right)}^{0} + \frac{\partial A_z}{\partial y}\overbrace{\left(\hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z\right)}^{\hat{\mathbf{e}}_x} \\
&+ \frac{\partial A_x}{\partial z}\overbrace{\left(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x\right)}^{\hat{\mathbf{e}}_y} + \frac{\partial A_y}{\partial z}\overbrace{\left(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_y\right)}^{-\hat{\mathbf{e}}_x} + \frac{\partial A_z}{\partial z}\overbrace{\left(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_z\right)}^{0} \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{e}}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{e}}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{e}}_z
\end{aligned}$$

EXERCISES ON INTEGRAL IDENTITIES

Establish the following identities using the integral theorems:

1. $\text{volume} = \frac{1}{6}\oint_\Gamma \nabla(r^2)\cdot\hat{\mathbf{n}}\,d\Gamma = \frac{1}{3}\oint_\Gamma \mathbf{r}\cdot\hat{\mathbf{n}}\,d\Gamma$

2. $\int_\Omega \nabla^2\phi\,d\Omega = \oint_\Gamma \frac{\partial\phi}{\partial n}\,d\Gamma$

3. $\int_\Omega \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)d\Omega = \oint_\Gamma \phi\frac{\partial\psi}{\partial n}\,d\Gamma$

4. $\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d\Omega = \oint_\Gamma \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)d\Gamma$

5. $\int_\Omega \left(\phi\nabla^4\psi - \nabla^2\phi\nabla^2\psi\right)d\Omega = \oint_\Gamma \left[\phi\frac{\partial}{\partial n}(\nabla^2\psi) - \nabla^2\psi\frac{\partial\phi}{\partial n}\right]d\Gamma$