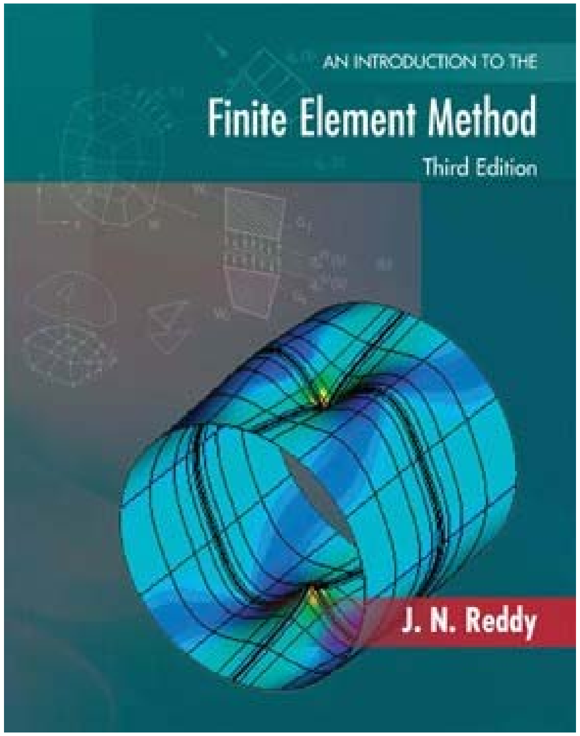
The Finite Element Method

3D Problems Heat Transfer and Elasticity

Read: Chapter 14



CONTENTS

- Finite element models of 3-D Heat Transfer
- Finite element model of 3-D Elasticity
- Typical 3-D Finite Elements

JN Reddy

3-D HEAT TRANSFER

Governing Equation

$$-\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)-\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)=g \quad \text{in} \quad \Omega$$

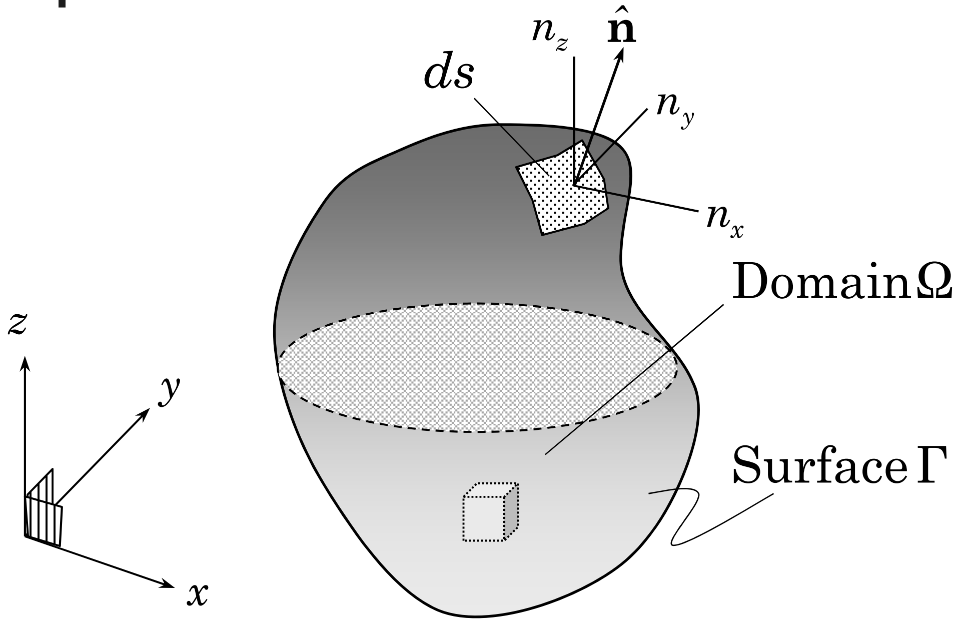
Boundary Conditions

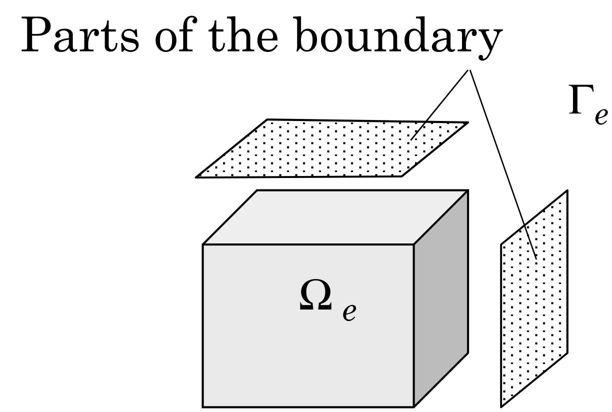
$$T=\hat{T} \quad \text{on} \quad \Gamma_1,$$

$$k_x\frac{\partial T}{\partial x}n_x+k_y\frac{\partial T}{\partial y}n_y+k_z\frac{\partial T}{\partial z}n_z+\beta(T-T_\infty)=\hat{q} \quad \text{on} \quad \Gamma_2$$

where k_x, k_y and k_z are conductivities of an orthotropic solid in the three coordinate directions, g is the internal heat generation per unit volume in a three-dimensional domain Ω, and $\hat{T}$ and $\hat{q}$ are specified functions of position on the portions Γ_1 and Γ_2, respectively, of the surface Γ of the domain (see Fig.1); β is the convection coefficient and T_∞ is the ambient temperature.

3-D HEAT TRANSFER (continued)





A six-face 3-D finite element

Weak Form

$$0 = \int_{\Omega_e} w \left[-\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) - g \right] d\mathbf{x}$$

$$= \int_{\Omega_e} \left[k_x \frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + k_y \frac{\partial w}{\partial y}\frac{\partial T}{\partial y} + k_z \frac{\partial w}{\partial z}\frac{\partial T}{\partial z} - wg \right] d\mathbf{x}$$

$$+ \oint_{\Gamma^e} \beta w T \, ds - \oint_{\Gamma^e} w \left(q_n + \beta T_\infty\right) ds \qquad (3)$$

3-D HEAT TRANSFER (continued)

Finite element approximation

$$T = \sum_{j=1}^{n} T_j \psi_j^e(x, y, z)$$

Finite element model

$$\mathbf{K}^e \mathbf{T}^e = \mathbf{f}^e + \mathbf{Q}^e$$

where

$$K_{ij}^e = \int_{\Omega_e} \left(k_x \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + k_y \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} + k_z \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial z} \right) d\mathbf{x} + \oint_{\Gamma^e} \beta \psi_i^e \psi_j \, ds$$

$$f_i^e = \int_{\Omega_e} f \psi_i^e \, d\mathbf{x}, \quad Q_i^e = \oint_{\Gamma^e} (q_n + \beta T_\infty) \psi_i^e \, ds$$

3-D ELASTICITY

Equations of Motion

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

Strain-Displacement Relations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

3-D ELASTICITY (continued)

Constitutive Relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

The material axes are assumed coincide with the global axes and the material is orthotropic with respect to the global axes.

Boundary Conditions

$$\left.\begin{aligned} t_x \equiv \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z = \hat{t}_x \\ t_y \equiv \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z = \hat{t}_y \\ t_z \equiv \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z = \hat{t}_z \end{aligned}\right\} \quad \text{on } \Gamma_\sigma \quad \underline{\text{or}} \quad \mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u$$

3-D ELASTICITY (continued)

MATRIX FORM OF THE GOVERNING EQUATIONS

Notation

$$\mathbf{D}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial z & 0 & \partial/\partial y \\ 0 & \partial/\partial y & 0 & 0 & \partial/\partial z & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial x & \partial/\partial y & 0 \end{bmatrix}$$

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad \varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

Governing equations

$$\mathbf{D}^{\mathrm{T}}\sigma + \mathbf{f} = \rho\ddot{\mathbf{u}} \qquad \varepsilon = \mathbf{D}\mathbf{u}, \qquad \sigma = \mathbf{C}\varepsilon$$

3-D ELASTICITY (continued)

Principle of virtual displacements (in matrix form)

$$0 = \int_{\Omega_e} \left[(\mathbf{D}\delta\mathbf{u})^{\mathrm{T}} \mathbf{C} \, (\mathbf{Du}) + \rho\,\delta\mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}} \right] d\mathbf{x} - \int_{\Omega_e} (\delta\mathbf{u})^{\mathrm{T}} \mathbf{f} \; d\mathbf{x} - \oint_{\Gamma_e} (\delta\mathbf{u})^{\mathrm{T}} \mathbf{t} \; ds$$

Finite element approximation (in matrix form)

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \mathbf{\Psi}\mathbf{\Delta}, \quad \mathbf{w} = \delta\mathbf{u} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \mathbf{\Psi}\delta\mathbf{\Delta}$$

$$\mathbf{\Psi} = \begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & 0 & 0 & \ldots & \psi_n & 0 & 0 \\ 0 & \psi_1 & 0 & 0 & \psi_2 & 0 & \ldots & \psi_n & 0 & \\ 0 & 0 & \psi_1 & 0 & 0 & \psi_2 & 0 & \ldots & 0 & \psi_n \end{bmatrix}$$

$$\mathbf{\Delta} = \{ u_x^1 \quad u_y^1 \quad u_z^1 \quad u_x^2 \quad u_y^2 \quad u_z^2 \quad \ldots \quad u_x^n \quad u_y^n \quad u_z^n \}^{\mathrm{T}}$$

3-D ELASTICITY (continued)

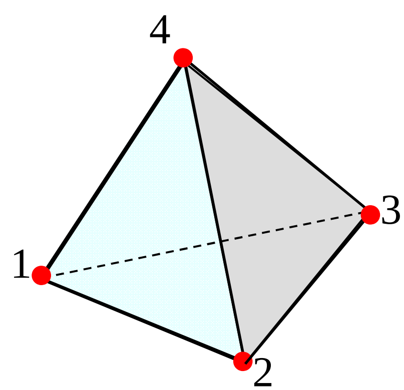
Finite Element Model

$$\mathbf{M}^e\ddot{\boldsymbol{\Delta}}^e + \mathbf{K}^e\boldsymbol{\Delta}^e = \mathbf{F}^e + \mathbf{Q}^e$$

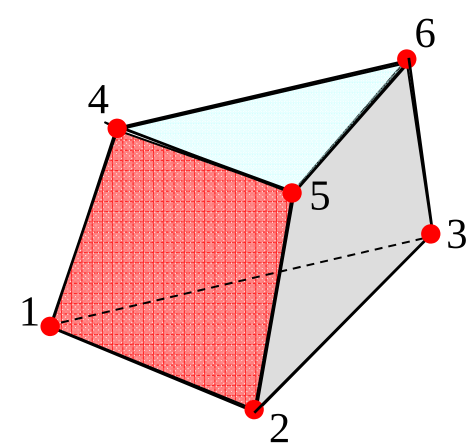
where

$$\mathbf{K}^e = \int_{\Omega_e} \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\, d\mathbf{x}, \quad \mathbf{M}^e = \int_{\Omega_e} \rho \quad \boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}^e\, d\mathbf{x}$$

$$\mathbf{F}^e = \int_{\Omega_e} \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{f}\, d\mathbf{x}, \qquad \mathbf{Q}^e = \oint_{\Gamma_e} \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{t}\, ds$$

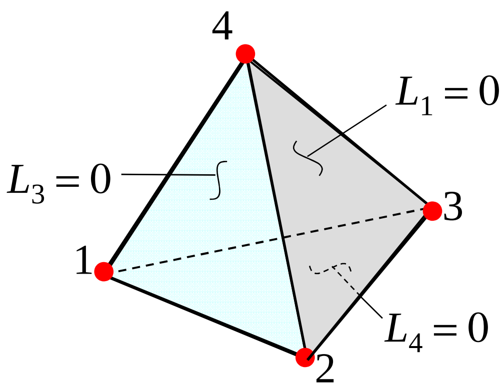


At each node (u,v,w)



TYPICAL 3-D FINITE ELEMENTS

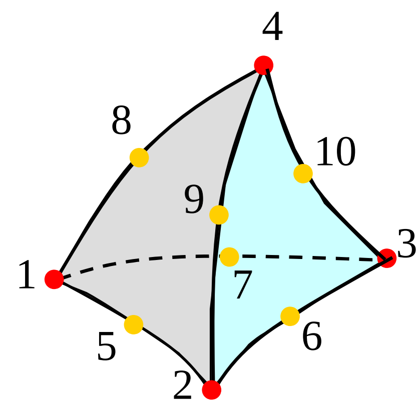
Linear tetrahedral element



$$u = a_0 + a_1 x + a_2 y + a_3 z$$

$$\{\boldsymbol{\Psi}^e\} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix}$$

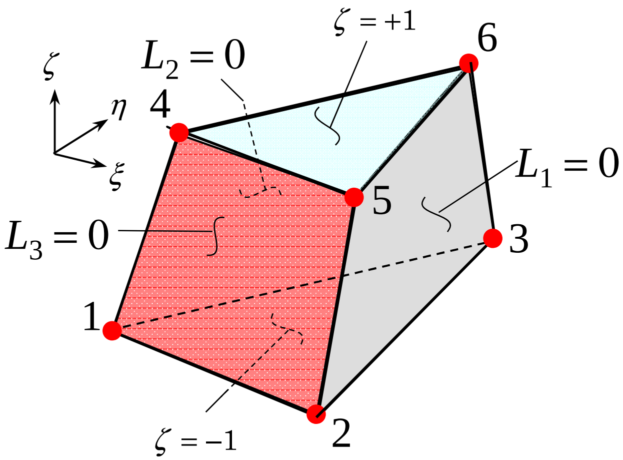
Quadratic tetrahedral element



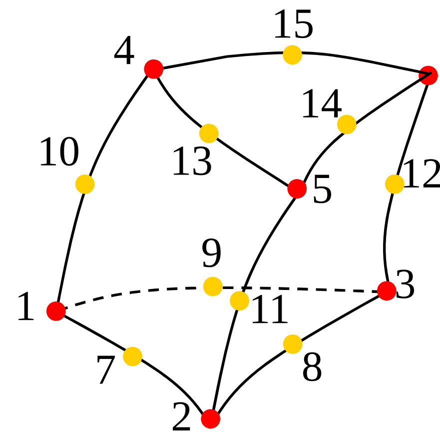
$$\{\boldsymbol{\Psi}^e\} = \begin{Bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ L_4(2L_4 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \\ 4L_1L_4 \\ 4L_2L_4 \\ 4L_3L_4 \end{Bmatrix}$$

TYPICAL 3-D FINITE ELEMENTS

Linear prism element



Quadratic prism element

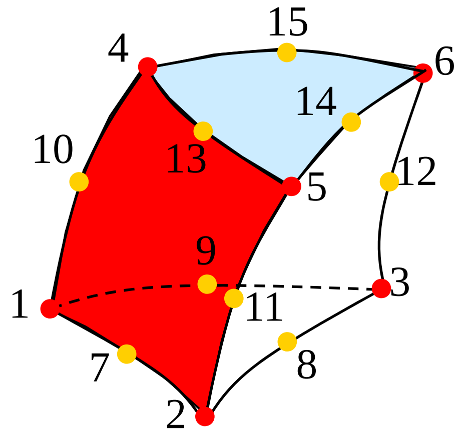


$$u = a_0 + a_1 x + a_2 y + a_3 z + a_4 xz + a_5 yz$$

$$\{\boldsymbol{\Psi}^e\} = \frac{1}{2}\left\{\begin{array}{c} L_1(1-\zeta) \\ L_2(1-\zeta) \\ L_3(1-\zeta) \\ L_1(1+\zeta) \\ L_2(1+\zeta) \\ L_3(1+\zeta) \end{array}\right\}$$

TYPICAL 3-D FINITE ELEMENTS (cont...)

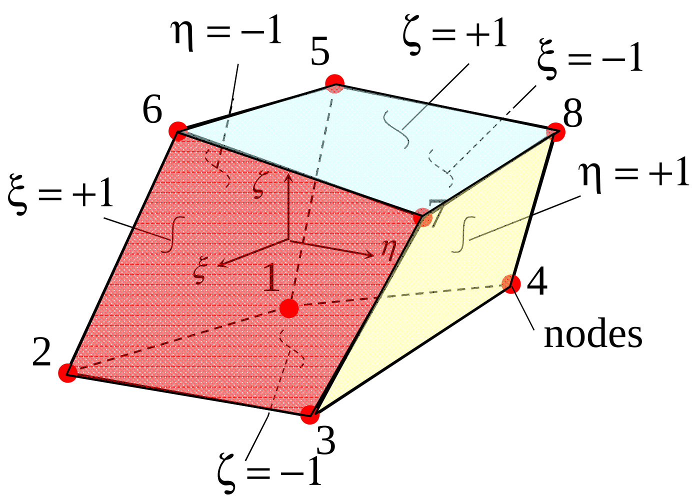
Quadratic Prism Element



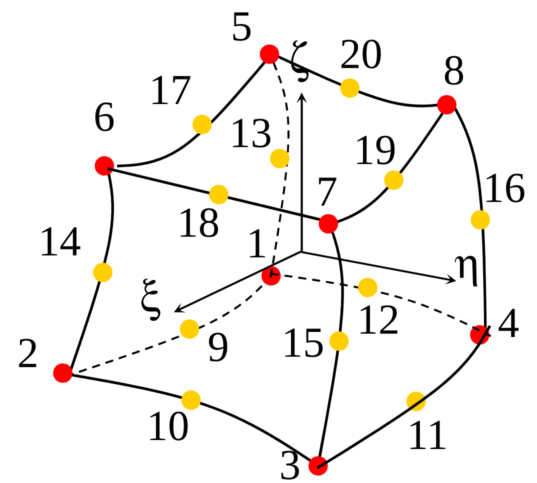
$$\{\boldsymbol{\Psi}^e\} = \frac{1}{2}\left\{\begin{array}{c} L_1[(2L_1-1)(1-\zeta)-(1-\zeta^2)] \\ L_2[(2L_2-1)(1-\zeta)-(1-\zeta^2)] \\ L_3[(2L_3-1)(1-\zeta)-(1-\zeta^2)] \\ L_1[(2L_1-1)(1+\zeta)-(1-\zeta^2)] \\ L_2[(2L_2-1)(1+\zeta)-(1-\zeta^2)] \\ L_3[(2L_3-1)(1+\zeta)-(1-\zeta^2)] \\ 4L_1L_2(1-\zeta) \\ 4L_2L_3(1-\zeta) \\ 4L_3L_1(1-\zeta) \\ 2L_1(1-\zeta^2) \\ 2L_2(1-\zeta^2) \\ 2L_3(1-\zeta^2) \\ 4L_1L_2(1+\zeta) \\ 4L_2L_3(1+\zeta) \\ 4L_3L_1(1+\zeta) \end{array}\right\}$$

TYPICAL 3-D FINITE ELEMENTS (cont…)

Linear brick element



Quadratic brick element

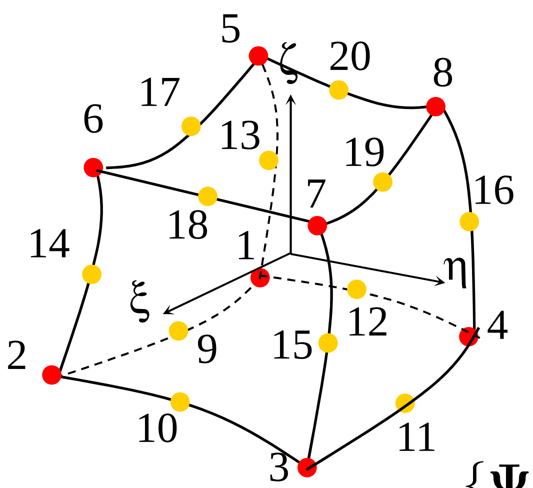


$$u = a_0 + a_1 x + a_2 y + a_3 z + a_4 yz + a_5 xz + a_6 xy + a_7 xyz$$

$$\{\mathbf{\Psi}^e\} = \frac{1}{8}\left\{\begin{array}{c} (1-\xi)(1-\eta)(1-\zeta) \\ (1+\xi)(1-\eta)(1-\zeta) \\ (1+\xi)(1+\eta)(1-\zeta) \\ (1-\xi)(1+\eta)(1-\zeta) \\ (1-\xi)(1-\eta)(1+\zeta) \\ (1+\xi)(1-\eta)(1+\zeta) \\ (1+\xi)(1+\eta)(1+\zeta) \\ (1-\xi)(1+\eta)(1+\zeta) \end{array}\right\}$$

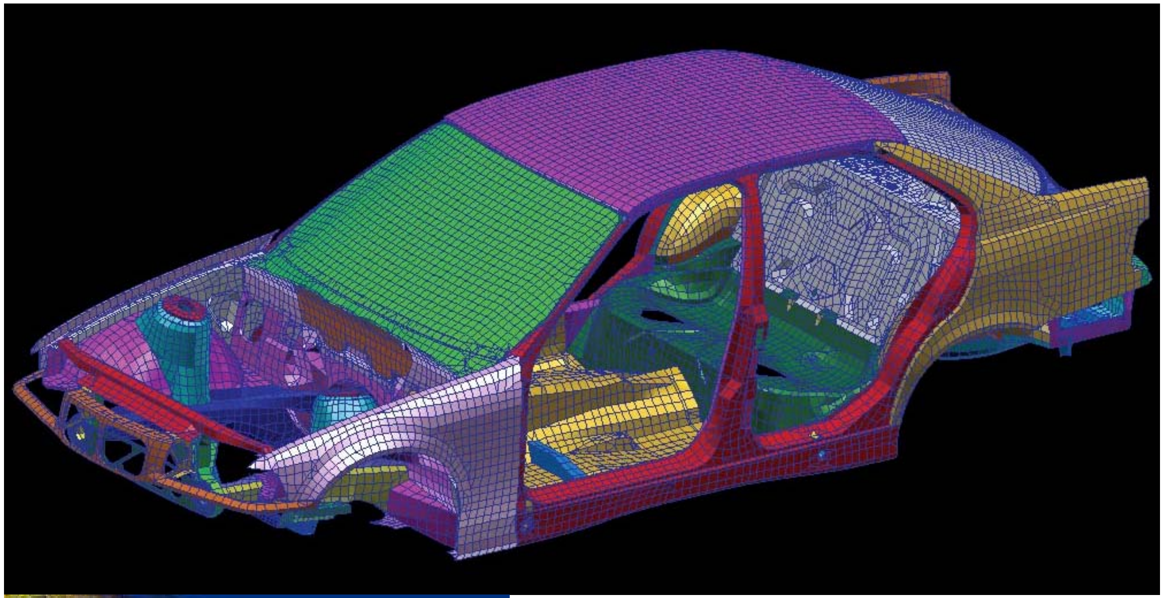
TYPICAL 3-D FINITE ELEMENTS (cont…)

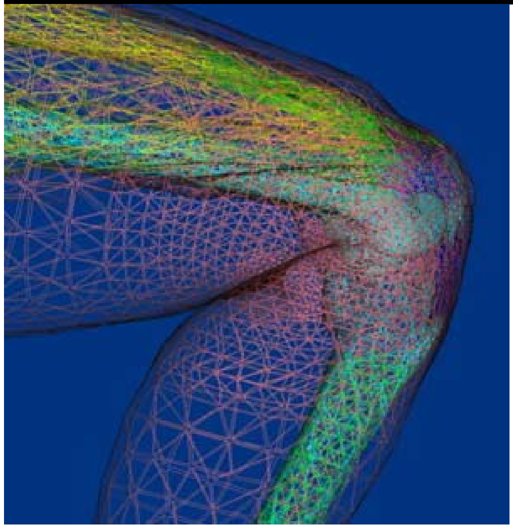
Quadratic Brick Element

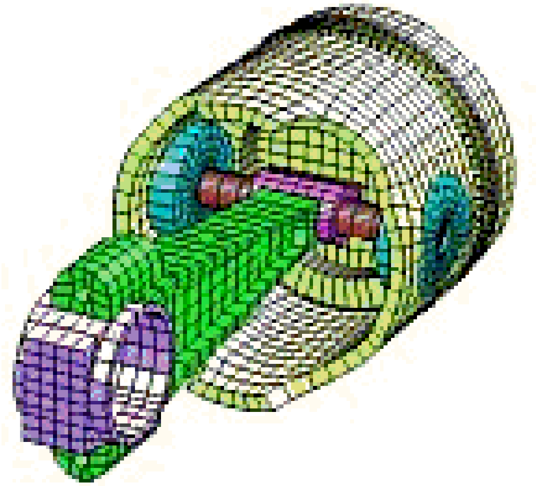


$$\{\Psi^e\} = \frac{1}{8}\left\{\begin{array}{c}
(1-\xi)(1-\eta)(1-\zeta)(-\xi-\eta-\zeta-2)\\
(1+\xi)(1-\eta)(1-\zeta)(\xi-\eta-\zeta-2)\\
(1+\xi)(1+\eta)(1-\zeta)(\xi+\eta-\zeta-2)\\
(1-\xi)(1+\eta)(1-\zeta)(-\xi+\eta-\zeta-2)\\
(1-\xi)(1-\eta)(1+\zeta)(-\xi-\eta+\zeta-2)\\
(1+\xi)(1-\eta)(1+\zeta)(\xi-\eta+\zeta-2)\\
(1+\xi)(1+\eta)(1+\zeta)(\xi+\eta+\zeta-2)\\
(1-\xi)(1+\eta)(1+\zeta)(-\xi+\eta+\zeta-2)\\
2(1-\xi^2)(1-\eta)(1-\zeta)\\
2(1+\xi)(1-\eta^2)(1-\zeta)\\
2(1-\xi^2)(1+\eta)(1-\zeta)\\
2(1-\xi)(1-\eta^2)(1-\zeta)\\
2(1-\xi)(1-\eta)(1-\zeta^2)\\
2(1+\xi)(1-\eta)(1-\zeta^2)\\
2(1+\xi)(1+\eta)(1-\zeta^2)\\
2(1-\xi)(1+\eta)(1-\zeta^2)\\
2(1-\xi^2)(1-\eta)(1+\zeta)\\
2(1+\xi)(1-\eta^2)(1+\zeta)\\
2(1-\xi^2)(1+\eta)(1+\zeta)\\
2(1-\xi)(1-\eta^2)(1+\zeta)
\end{array}\right\}$$

TYPICAL 3-D or SHELL FINITE ELEMENT MESHES







TYPICAL 3-D FINITE ELEMENT MESHES