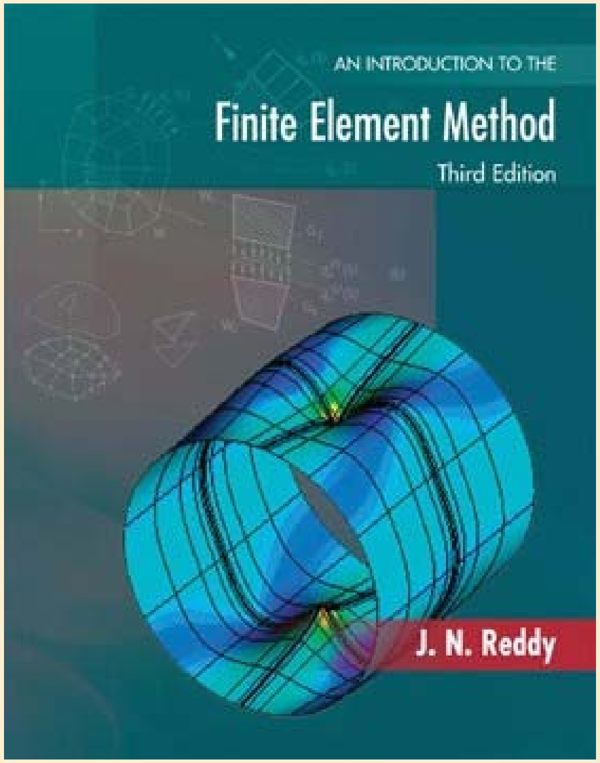
The Finite Element Method

2D Flows of Viscous Incompressible Fluids

Read: Chapter 10



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Governing Equations of Flows of Viscous incompressible Fluids

Equations of motion

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} \;\Rightarrow$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho \frac{D v_x}{Dt}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho \frac{D v_y}{Dt}$$

Material time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \;\Rightarrow\; \frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$$

Conservation of mass

$$\nabla \cdot \mathbf{v} = 0 \;\Rightarrow \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Constitutive relations

$$\boldsymbol{\sigma} = -P\,\mathbf{I} + \boldsymbol{\tau}, \qquad \boldsymbol{\tau} = 2\mu \mathbf{D}$$

Governing Equations of Flows of Viscous incompressible Fluids

Kinematics relations

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right] \Rightarrow D_{xx} = \frac{\partial v_x}{\partial x},\ D_{yy} = \frac{\partial v_y}{\partial y},\ 2D_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

Stress-velocity-pressure relations

$$\sigma_{xx} = 2\mu\frac{\partial v_x}{\partial x} - P,\ \sigma_{yy} = 2\mu\frac{\partial v_y}{\partial y} - P,\ \sigma_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

Boundary conditions

$$(v_x, t_x) \text{ and } (v_y, t_y)$$

$$t_x = \sigma_{xx}n_x + \sigma_{xy}n_y,\quad t_y = \sigma_{xy}n_x + \sigma_{yy}n_y$$

Governing Equations in Terms of Velocities and Pressure

Differential equations

$$\rho \frac{\partial v_x}{\partial t} - \frac{\partial}{\partial x}\Bigg(\underbrace{2\mu \frac{\partial v_x}{\partial x} - P}_{\sigma_{xx}}\Bigg) - \frac{\partial}{\partial y}\Bigg(\underbrace{\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)}_{\sigma_{xy}}\Bigg) - f_x = 0 \quad \textbf{(1)}$$

$$\rho \frac{\partial v_y}{\partial t} - \frac{\partial}{\partial x}\Bigg(\underbrace{\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)}_{\sigma_{xy}}\Bigg) - \frac{\partial}{\partial y}\Bigg(\underbrace{2\mu \frac{\partial v_y}{\partial y} - P}_{\sigma_{yy}}\Bigg) - f_y = 0 \quad \textbf{(2)}$$

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = 0 \quad \textbf{(3)}$$

Boundary conditions

$$(v_x, t_x) \text{ and } (v_y, t_y)$$

Finite Element Models: Some comments

There are two possible finite element models with velocities and possibly pressure as the nodal variables.

The most natural finite element model is that based on Eqs. (1)–(3), and it includes (v_x, v_y, P) as the nodal variables. It is known as the mixed model. It should be noted that P is neither a primary variable nor a secondary variable (although it is a part of (t_x, t_y). Often, pressure P presents numerical difficulties.

An alternative formulation that involves only (v_x, v_y) by treating Eq. (3) as a constraint. This is called the penalty finite element model.

WEAK FORMS OF THE STOKES EQUATIONS

$$0 = \int_{\Omega^e} w_1 \left[\rho \frac{\partial v_x}{\partial t} - \frac{\partial}{\partial x}\left(2\mu \frac{\partial v_x}{\partial x} - P \right) - \frac{\partial}{\partial y}\left(\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right) - f_x \right] dxdy$$

$$= \int_{\Omega^e} \left[\rho w_1 \frac{\partial v_x}{\partial t} - \frac{\partial w_1}{\partial x}\left(2\mu \frac{\partial v_x}{\partial x} - P \right) - \mu \frac{\partial w_1}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) - w_1 f_x \right] dxdy - \oint_{\Gamma^e} w_1 t_x \, ds$$

$$0 = \int_{\Omega^e} w_2 \left[\rho \frac{\partial v_y}{\partial t} - \frac{\partial}{\partial y}\left(2\mu \frac{\partial v_y}{\partial y} - P \right) - \frac{\partial}{\partial x}\left(\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right) - f_y \right] dxdy$$

$$= \int_{\Omega^e} \left[\rho w_2 \frac{\partial v_y}{\partial t} - \frac{\partial w_2}{\partial y}\left(2\mu \frac{\partial v_y}{\partial y} - P \right) - \mu \frac{\partial w_2}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) - w_2 f_y \right] dxdy - \oint_{\Gamma^e} w_2 t_y \, ds$$

$$0 = \int_{\Omega^e} w_3 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dxdy$$

$$w_1 \approx v_x \,;\; w_2 \approx v_y \,;\; w_3 \approx -P$$

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y, \quad t_y = \sigma_{xy} n_x + \sigma_{yy} n_y$$

Mixed Finite Element Model for the steady-static case

$$v_x = \sum_{j=1}^{m} v_{xj}\psi_j\,, \quad v_y = \sum_{j=1}^{m} v_{yj}\psi_j, \quad P = \sum_{j=1}^{n} P_j\phi_j$$

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_x \\ \dot{\mathbf{v}}_y \\ \dot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \\ \mathbf{F}^{3} \end{Bmatrix}$$

$$M_{ij}^{11} = M_{ij}^{22} = \int_{\Omega^e} \rho\,\psi_i\psi_j\,dxdy, \quad M_{ij}^{33} = M_{ij}^{12} = M_{ij}^{21} = M_{ij}^{31} = M_{ij}^{13} = M_{ij}^{23} = M_{ij}^{32} = 0$$

$$K_{ij}^{\alpha\beta} = K_{ji}^{\beta\alpha}, \quad K_{ij}^{11} = \int_{\Omega^e} \mu\left(2\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + \frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy, \qquad K_{ij}^{12} = \int_{\Omega^e} \mu\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x}dxdy$$

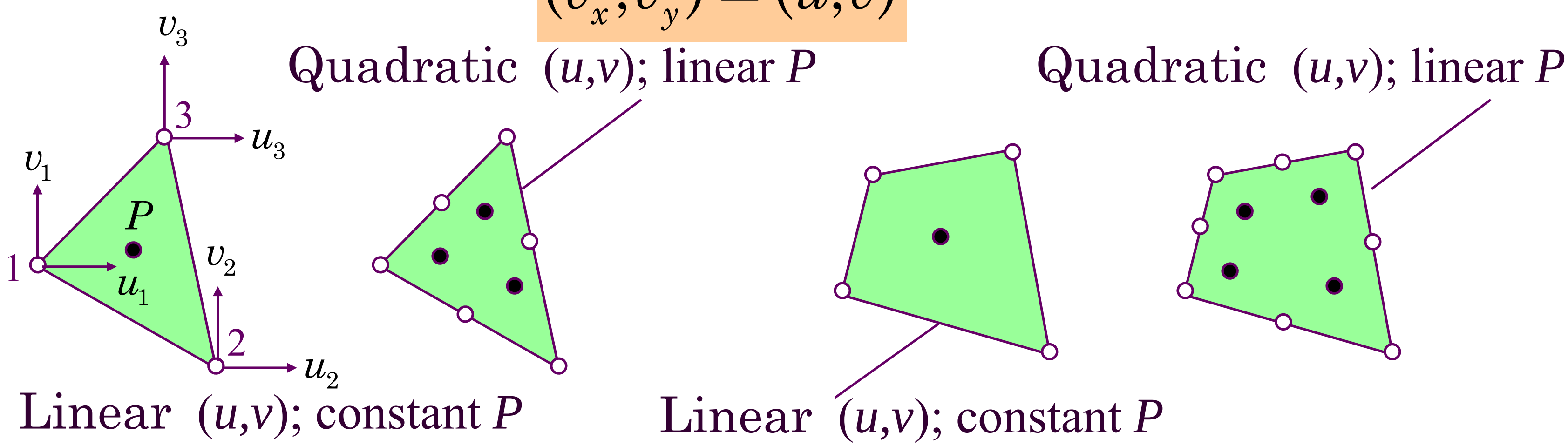
Mixed Finite Element Model for the steady-static case (continued)

$$K_{ij}^{13} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial x} \phi_j \, dxdy, \qquad K_{ij}^{23} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y} \phi_j \, dxdy, \qquad K_{ij}^{33} = 0$$

$$K_{ij}^{22} = \int_{\Omega^e} \mu \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + 2 \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy, \; K_{ij}^{21} = K_{ji}^{12}, \; K_{ij}^{31} = K_{ji}^{13}, \; K_{ij}^{32} = K_{ji}^{23}$$

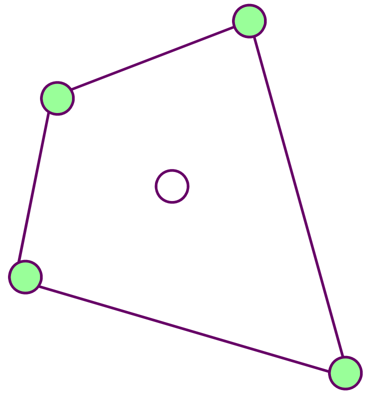
$$F_i^1 = \int_{\Omega^e} f_x \psi_i \, dxdy + \oint_{\Gamma^e} t_x \psi_i \, ds, \qquad F_i^2 = \int_{\Omega^e} f_y \psi_i \, dxdy + \oint_{\Gamma^e} t_y \psi_i \, ds$$

$$(v_x, v_y) = (u, v)$$

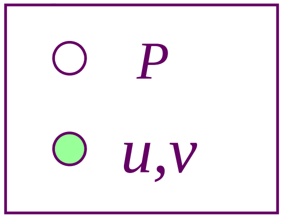


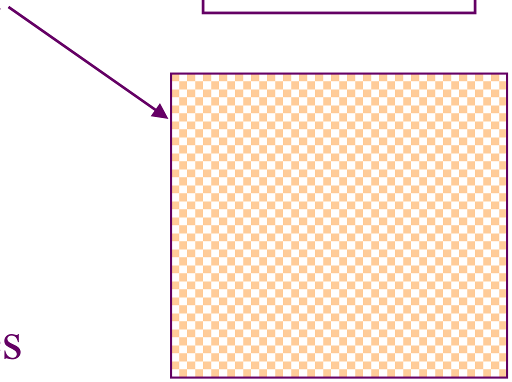
2D Elements Used for Viscous Incompressible Flows

Q4C

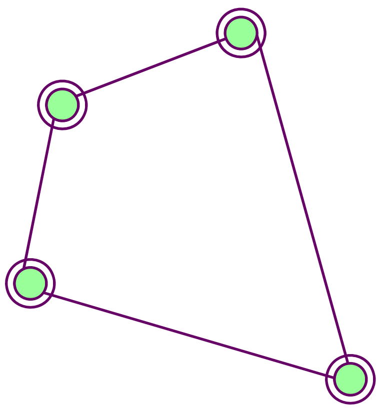


Continuous bilinear displacements/velocities and discontinuous constant pressure **(checker board pattern is observed for pressure)**



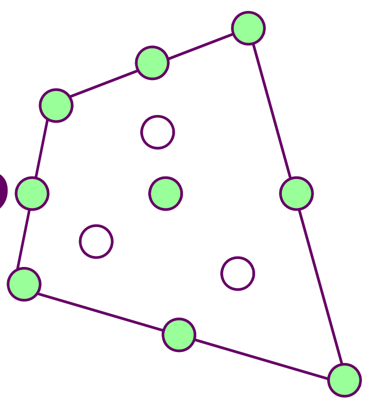


Q4



Continuous bilinear displacements/velocities and continuous bilinear pressure

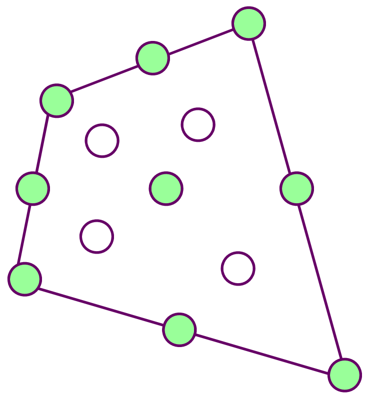
Q9L3-D



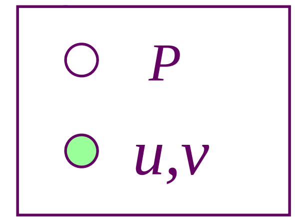
Continuous biquadratic displacements/velocities and discontinuous linear pressure (*works well*)

2D Elements Used for Viscous Incompressible Flows

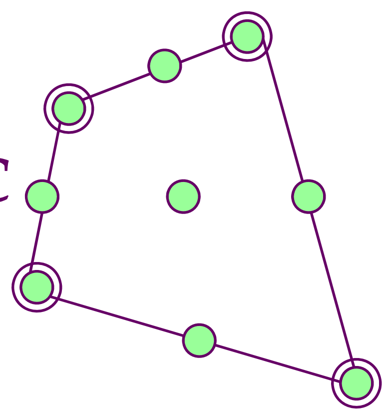
Q9L4-D



Continuous biquadratic displacements/velocities and discontinuous bilinear pressure



Q9L4-C



Continuous biquadratic displacements/velocities continuous linear pressure (*works well*)

Penalty Function Method-*algebraic*

Problem: Find the minimum of the function $F(x, y)$ subject to the constraint $G(x, y) = 0$

$$dF \equiv \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

Lagrange multiplier method

$$F_L(x, y, \lambda) \equiv F(x, y) + \lambda G(x, y)$$

$$\begin{aligned} dF_L &\equiv \frac{\partial F_L}{\partial x}dx + \frac{\partial F_L}{\partial y}dy + \frac{\partial F_L}{\partial \lambda}d\lambda = 0 \\ &= \left(\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y}\right)dy + G(x, y)d\lambda \end{aligned}$$

$$\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y} = 0, \quad G(x, y) = 0$$

Penalty Function Method-*algebraic* (continued)

Penalty function method

$$F_P(x,y) = F(x,y) + \frac{\gamma}{2}[G(x,y) - 0]^2$$

$$\begin{aligned} dF_P &\equiv \frac{\partial F_P}{\partial x}dx + \frac{\partial F_P}{\partial y}dy = 0 \\ &= \left(\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y}\right)dy \end{aligned}$$

$$\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y} = 0$$

$$dF_L \equiv \frac{\partial F_L}{\partial x}dx + \frac{\partial F_L}{\partial y}dy + \frac{\partial F_L}{\partial \lambda}d\lambda = 0$$

$$= \left(\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y}\right)dy + G(x,y)d\lambda$$

$$\boxed{\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x} = 0,} \quad \boxed{\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y} = 0,} \quad G(x,y) = 0$$

$$dF_P \equiv \frac{\partial F_P}{\partial x}dx + \frac{\partial F_P}{\partial y}dy = 0$$

$$= \left(\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y}\right)dy$$

$$\boxed{\frac{\partial F}{\partial x} + \gamma G(x,y)\frac{\partial G}{\partial x} = 0,} \quad \boxed{\frac{\partial F}{\partial y} + \gamma G(x,y)\frac{\partial G}{\partial y} = 0}$$

Approximation of the Lagrange multiplied can be computed in the penalty method from

$$\boxed{\lambda_\gamma = \gamma G(x_\gamma, y_\gamma)}$$

Penalty Function Method-*algebraic*

(An Example)

$$F(x,y) = 2x^2 + y^2 - 8x + y + 1, \quad G(x,y) \equiv 2x - y = 0$$

Lagrange Multiplier Method

$$4x - 8 + 2\lambda = 0, \quad 2y + 1 - \lambda = 0, \quad 2x - y = 0$$

$$x = 0.5, \qquad y = 1.0, \qquad \lambda = 3.0$$

Penalty Function Method

$$4x - 8 + 2\gamma(2x - y) = 0, \qquad 2y + 1 - \gamma(2x - y) = 0$$

$$x_\gamma = \frac{8 + 3\gamma}{4 + 6\gamma}, \qquad y_\gamma = \frac{3\gamma - 1}{2 + 3\gamma}$$

Clearly, as $\gamma \to \infty$, we have

$$\lim_{\gamma\to\infty} x_\gamma = 0.5 = x, \qquad \lim_{\gamma\to\infty} y_\gamma = 1.0 = y$$

Penalty Function Method-*algebraic*

(Example - continued)

Table: A comparison of the penalty solution with the exact for various values of the penalty parameter γ.

γ	1.0	10.0	25.0	50.0	100.0	1000.0
x_γ	1.1	0.5938	0.5390	0.5197	0.5099	0.5010
y_γ	0.4	0.9063	0.9610	0.9803	0.9901	0.9990
$G(x_\gamma, y_\gamma)$	1.8	0.2813	0.1169	0.0592	0.0298	0.0030
λ_γ	1.8	2.8125	2.9221	2.9605	2.9801	2.9980

$$\lambda_\gamma = \gamma\, G(x_\gamma, y_\gamma)$$

Penalty Function Method – *continuum (Beam Example)*

Original Problem: Minimize the total potential energy

$$\Pi^e(w_0) = \int_{x_a}^{x_b} \left[\frac{D_{xx}}{2} \left(\frac{d^2 w_0}{dx^2} \right)^2 - q w_0 \right] dx - w_0(x_a)\bar{Q}_1^e - w_0(x_b)\bar{Q}_3^e$$

$$- \left(- \frac{dw_0}{dx} \right)_{x_a} \bar{Q}_2^e - \left(- \frac{dw_0}{dx} \right)_{x_b} \bar{Q}_4^e$$

New Problem: Minimize the total potential energy

$$\Pi^e(w_0, \varphi) = \int_{x_a}^{x_b} \left[\frac{D_{xx}}{2} \left(\frac{d\varphi}{dx} \right)^2 - q w_0 \right] dx - w_0(x_a)\bar{Q}_1^e - w_0(x_b)\bar{Q}_3^e$$

$$+ \varphi(x_a)\bar{Q}_2^e + \varphi(x_b)\bar{Q}_4^e$$

subject to the constraint

$$G(w_0, \varphi) \equiv \frac{dw_0}{dx} - \varphi(x) = 0$$

Penalty Function Method:
Beam Example - continued

$$\mathrm{I}_P^e(w_0,\varphi) = \Pi^e(w_0,\varphi) + \frac{1}{2}\int_{x_a}^{x_b} \gamma(x)\left[G(w_0,\varphi)\right]^2 dx$$

$$= \int_{x_a}^{x_b}\left[\frac{D_{xx}}{2}\left(\frac{d\varphi}{dx}\right)^2 - qw_0\right] dx - w_0(x_a)\bar{Q}_1^e - w_0(x_b)\bar{Q}_3^e$$

$$+ \varphi(x_a)\bar{Q}_2^e + \varphi(x_b)\bar{Q}_4^e + \frac{1}{2}\int_{x_a}^{x_b}\gamma\left[\frac{dw_0}{dx} - \varphi\right]^2 dx$$

$$0 = \int_{x_a}^{x_b}\left[D_{xx}\frac{d\varphi}{dx}\frac{d\delta\varphi}{dx} + \gamma\left(\frac{dw_0}{dx} - \varphi\right)\left(\frac{d\delta w_0}{dx} - \delta\varphi\right) - q\delta w_0\right] dx$$

$$- \delta w_0(x_a)\bar{Q}_1^e - \delta w_0(x_b)\bar{Q}_3^e + \delta\varphi(x_a)\bar{Q}_2^e + \delta\varphi(x_b)\bar{Q}_4^e$$

Penalty Function Method:
Beam Example - continued

$$w_0 = \sum_{j=1}^{m} w_j \psi_j^{(1)}, \quad \varphi = \sum_{j=1}^{n} s_j \psi_j^{(2)}$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{s\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} \gamma \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{12} = -\int_{x_a}^{x_b} \gamma \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(D_{xx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + \gamma \psi_i^{(2)} \psi_j^{(2)} \right) dx$$

$$F_i^1 = \int_{x_a}^{x_b} q \psi_i^{(1)}\, dx + w_0(x_a) \bar{Q}_1^e + w_0(x_b) \bar{Q}_3^e$$

$$F_i^2 = -\varphi(x_a) \bar{Q}_2^e - \varphi(x_b) \bar{Q}_4^e$$

PENALTY FORMULATION for flows of viscous incompressible fluids
(for the Steady-State Case)

Consider the weak forms

$$0 = \int_{\Omega^e} \left[\frac{\partial w_1}{\partial x} \left(2\mu \frac{\partial v_x}{\partial x} - P \right) - \mu \frac{\partial w_1}{\partial y} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) - w_1 f_x \right] dxdy$$
$$- \oint_{\Gamma^e} w_1 \; t_x \, ds$$

$$0 = \int_{\Omega^e} \left[\mu \frac{\partial w_2}{\partial x} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \frac{\partial w_2}{\partial y} \left(2\mu \frac{\partial v_y}{\partial y} - P \right) - w_2 f_y \right] dxdy$$
$$- \oint_{\Gamma^e} w_2 \; t_y \; ds$$

$$0 = - \int_{\Omega^e} w_3 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dxdy, \qquad \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$$

Penalty Finite Element Formulation (continued)

Now suppose that the velocity field satisfies the constraint

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$$

Then adding the three weak statements, we obtain

$$0 = \int_{\Omega^e} \left[2\mu \frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x} + 2\mu \frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y} + \mu\left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x}\right)\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \right.$$
$$\left. - P\underbrace{\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\right)}_{\mathbf{0}} - w_3\underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)}_{\mathbf{0}} - w_1 f_x - w_2 f_y \right] dxdy$$
$$- \oint_{\Gamma^e} \left(w_1 t_x + w_2 t_y\right) ds$$

Penalty Finite Element Formulation
- continued

Thus, the weak form of the problem, subjected to the constraint is

$$0=\int_{\Omega^e}\left[2\mu\frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x}+2\mu\frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y}+\mu\left(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x}\right)\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right.$$
$$\left.-w_1 f_x-w_2 f_y\right]dxdy-\oint_{\Gamma^e}\left(w_1 t_x+w_2 t_y\right)ds$$

Then, the modified weak form with the constraint is

$$0=\int_{\Omega^e}\left[2\mu\frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x}+2\mu\frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y}+\mu\left(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x}\right)\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right.$$
$$\left.-w_1 f_x-w_2 f_y\right]dxdy-\oint_{\Gamma^e}\left(w_1 t_x+w_2 t_y\right)ds$$
$$+\int_{\Omega^e}\gamma\left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}\right)\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)dxdy$$

Penalty expression

Penalty Finite Element Formulation
- continued

The weak form of the problem can be separated into the following two statements

$$0 = \int_{\Omega^e} \left[2\mu \frac{\partial w_1}{\partial x}\frac{\partial v_x}{\partial x} + \mu \frac{\partial w_1}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \gamma \frac{\partial w_1}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) - w_1 f_x \right] dxdy - \oint_{\Gamma^e} w_1 t_x \, ds$$

$$0 = \int_{\Omega^e} \left[2\mu \frac{\partial w_2}{\partial y}\frac{\partial v_y}{\partial y} + \mu \frac{\partial w_2}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \gamma \frac{\partial w_2}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) - w_2 f_y \right] dxdy - \oint_{\Gamma^e} w_2 t_y \, ds$$

These statements form the basis of the penalty finite element model.

Penalty Finite Element Formulation
(continued)

Alternatively, the pressure (negative of the Lagrange multiplier) in the governing equations can be replaced by

$$P_\gamma = -\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

we obtain

$$-\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right) - \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \frac{\partial}{\partial x}\left[\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] - f_x = 0$$

$$-\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right) - \frac{\partial}{\partial y}\left[\gamma\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] - f_y = 0$$

The weak forms of these equations are precisely the same as those on the previous slide.

Penalty Finite Element Model

$$v_x = \sum_{j=1}^{m} v_{xj}\psi_j(x,y), \quad v_y = \sum_{j=1}^{m} v_{yj}\psi_j(x,y)$$

Substitution into the weak forms (adding inertia terms) yields the equations

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_x \\ \dot{\mathbf{v}}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix}$$

$$M_{ij}^{11} = M_{ij}^{22} = \int_{\Omega^e} \rho\, \psi_i \psi_j \, dxdy$$

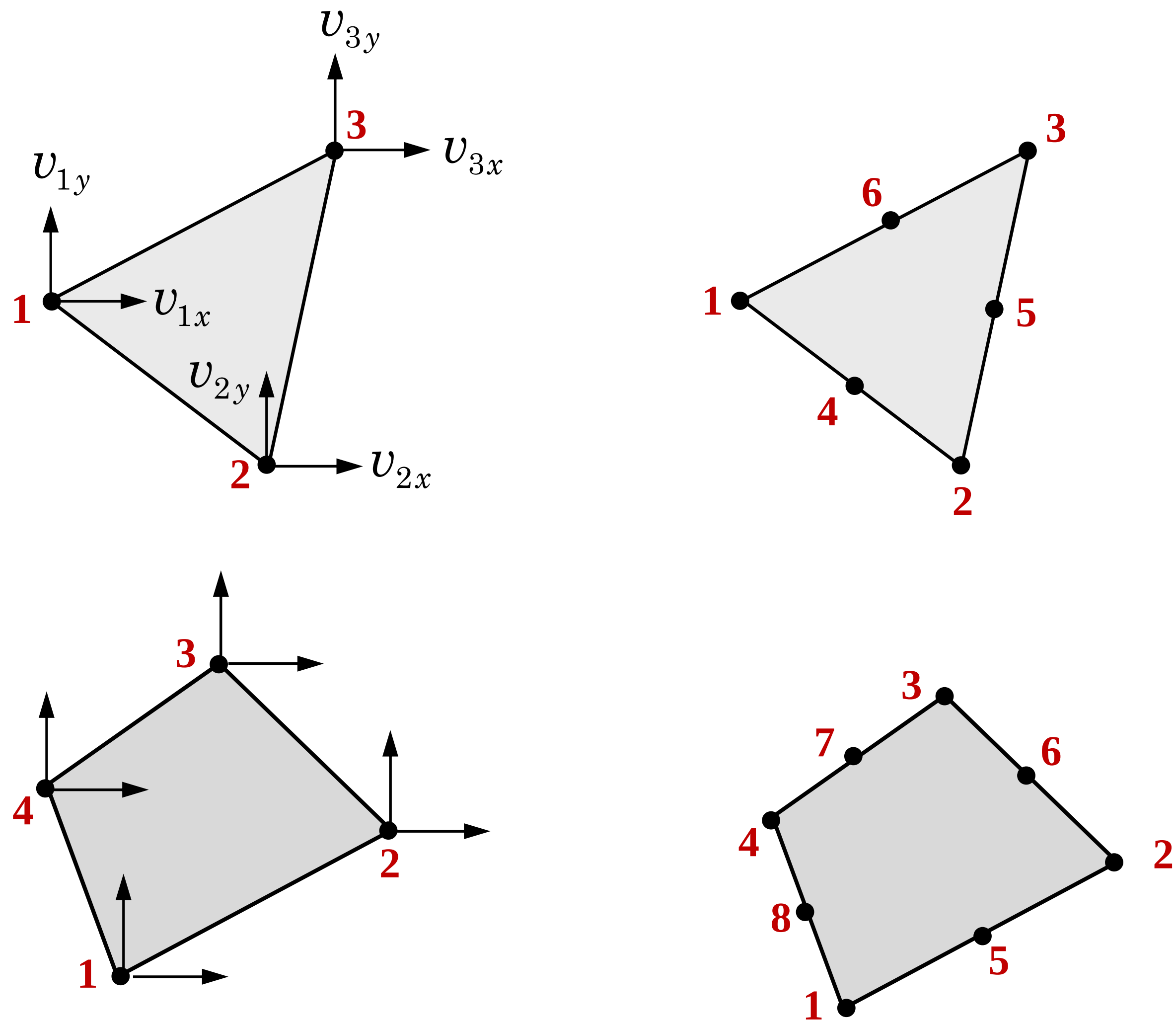
$$K_{ij}^{11} = \int_{\Omega^e} \mu \left(2\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial x} dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial y} dxdy, \quad K_{ij}^{21} = K_{ji}^{12}$$

$$K_{ij}^{22} = \int_{\Omega^e} \mu \left(\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + 2\frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dxdy + \int_{\Omega^e} \gamma \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} dxdy$$

$$F_i^1 = \int_{\Omega^e} f_x \psi_i \, dxdy + \oint_{\Gamma^e} t_x \psi_i \, ds, \; F_i^2 = \int_{\Omega^e} f_y \psi_i \, dxdy + \oint_{\Gamma^e} t_y \psi_i \, ds$$

Elements Used for Penalty FE Model



Computational Aspects of the Penalty FEM

General form of the Penalty FEM:

$$\left(\mu[K^1] + \rho[K^2] + \gamma[K^3]\right)\{\Delta\} = \{F\}$$

Element 'locking':

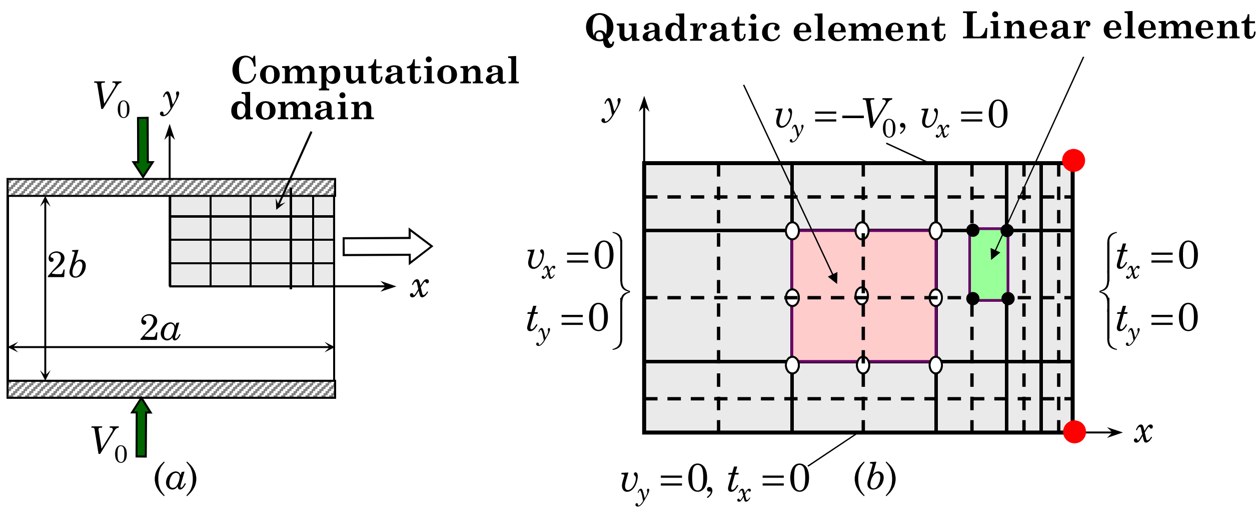
$$\lim \gamma \to 0 \;\; \left(\mu[K^1] + \rho[K^2] + \gamma[K^3]\right)\{\Delta\} = \{F\} \;\rightarrow\; \gamma[K^3]\{\Delta\} = \{F\}$$

Choice of the penalty parameter:

$$\gamma = 10^4\mu \text{ to } \;\gamma = 10^{12}\mu$$

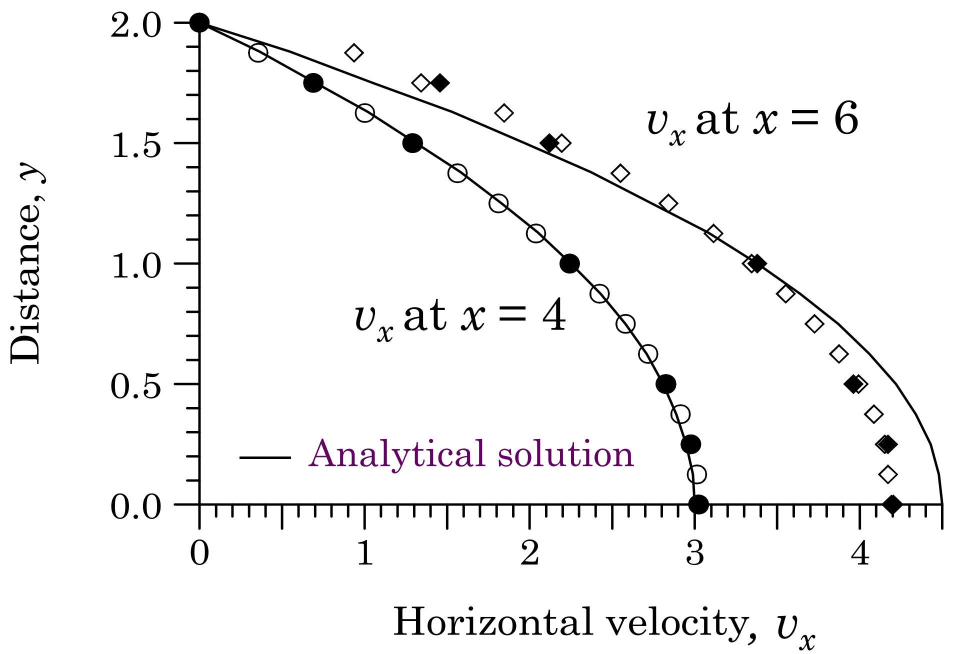
Numerical Examples

Viscous fluid squeezed between parallel plates

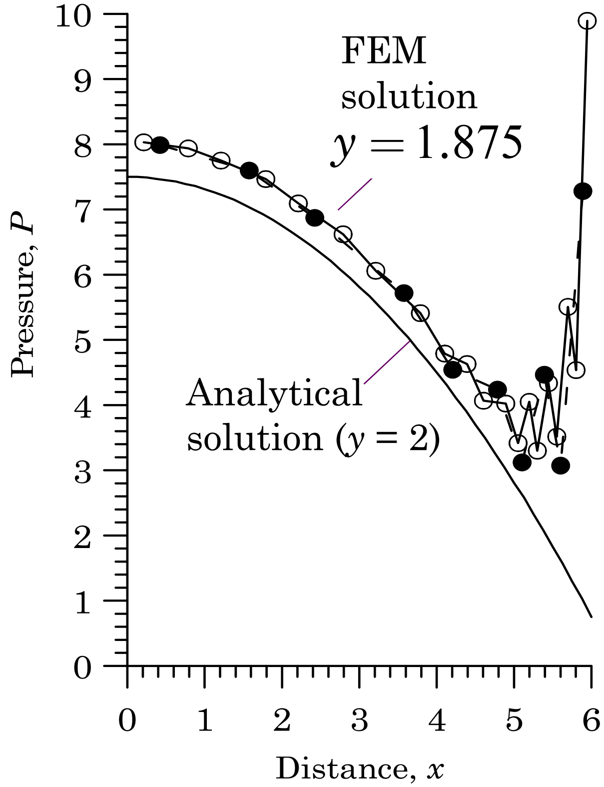


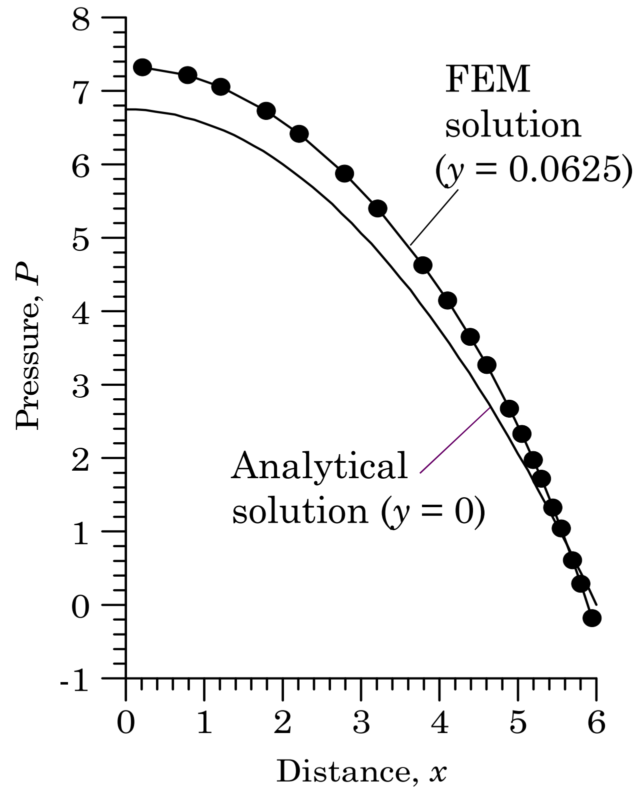
- **<u>Note</u>**: There are singularities in the application of the boundary conditions

Viscous fluid squeezed between parallel plates: Velocity field

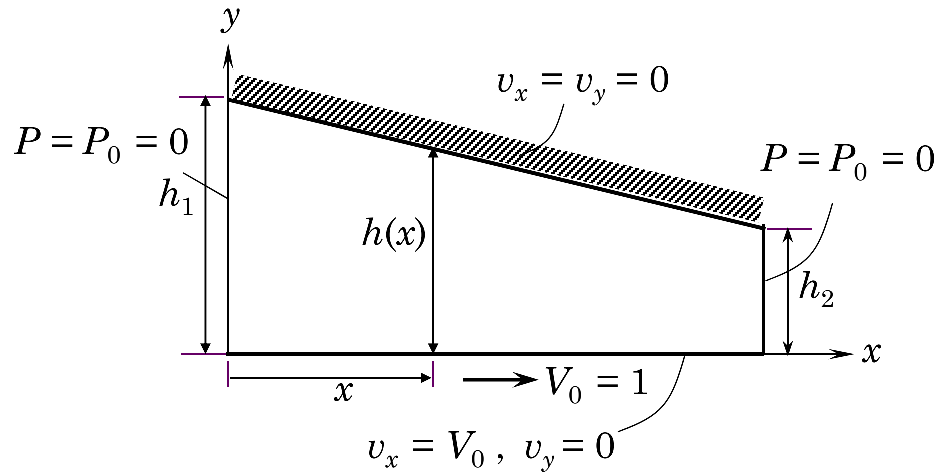


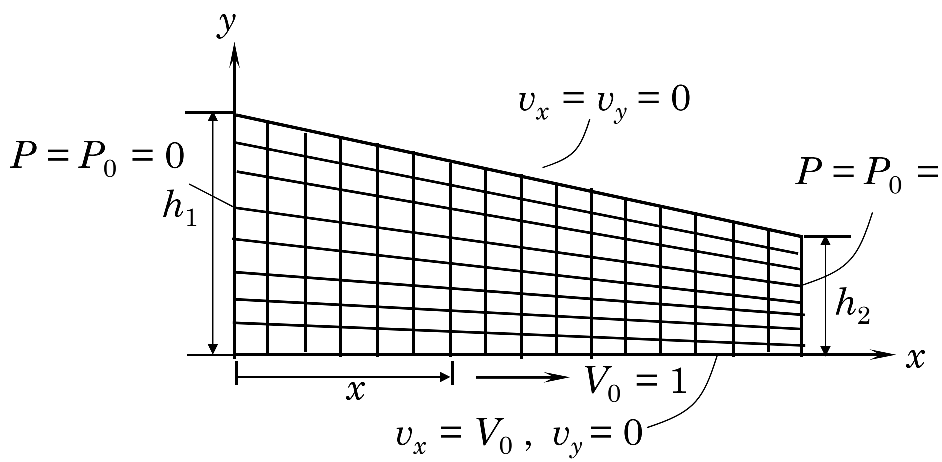
Viscous fluid squeezed between parallel plates: Pressure field



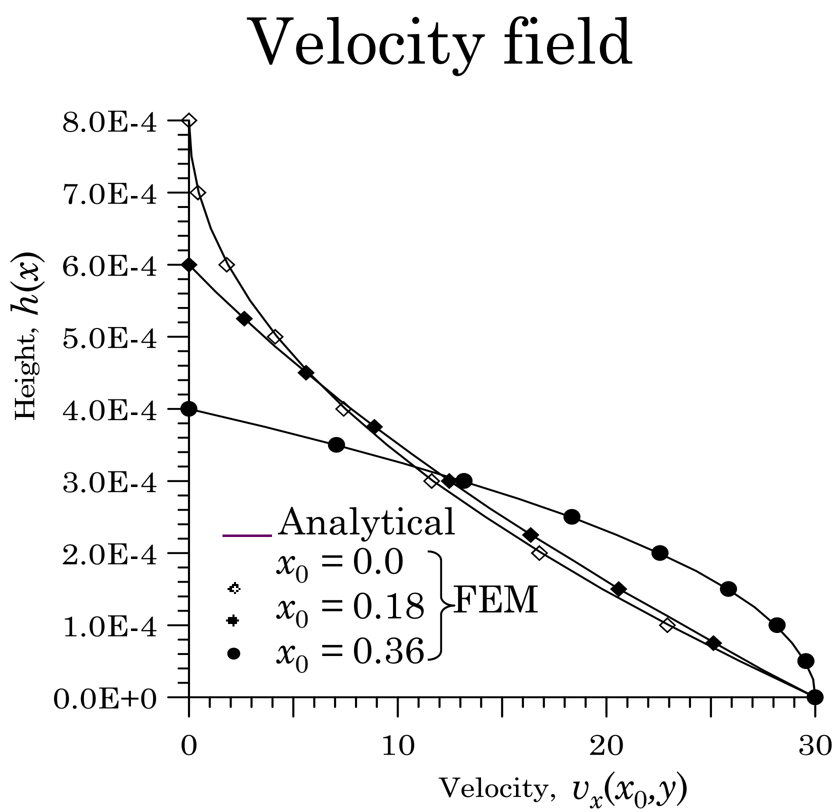


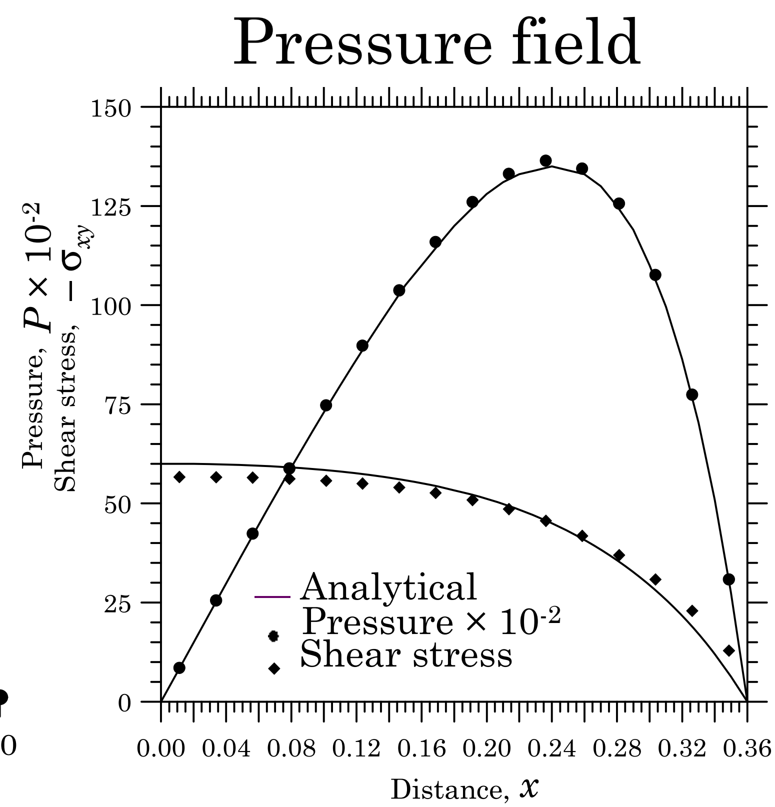
2. Flow of a viscous lubricant in a slider bearing



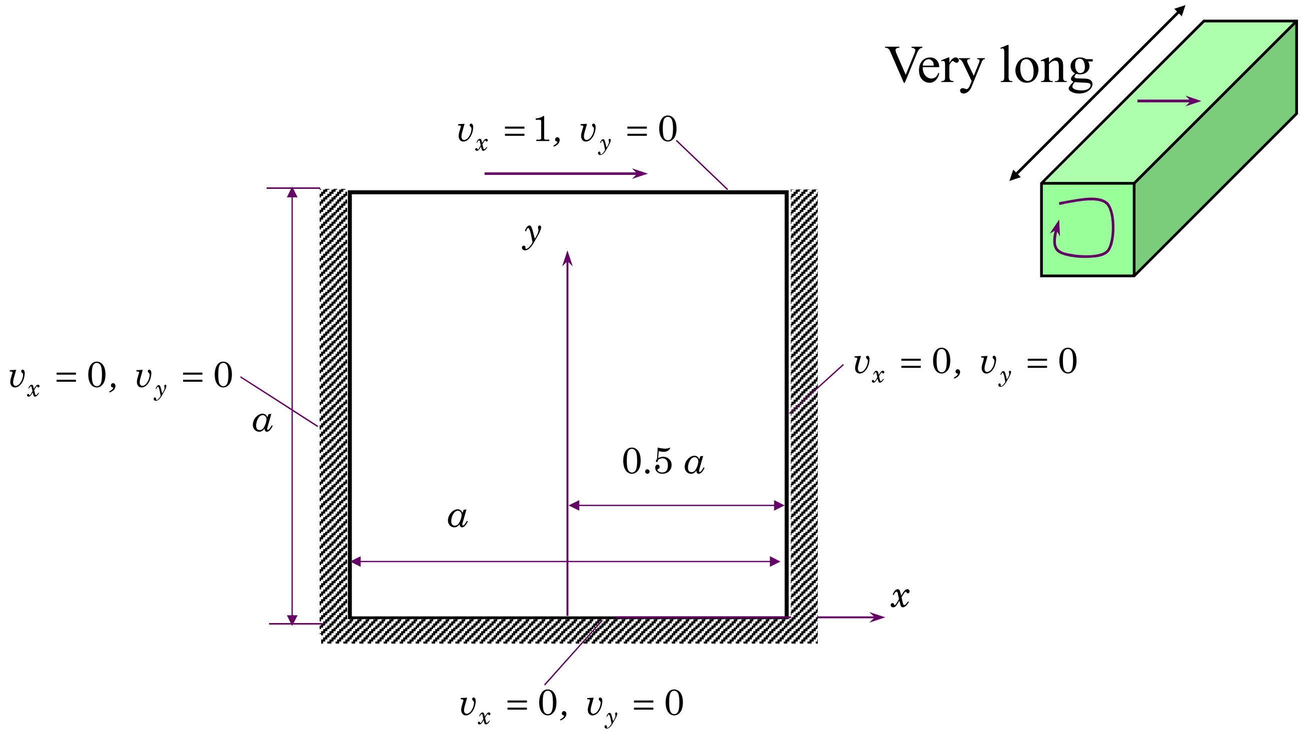


Flow of a viscous lubricant in a slider bearing

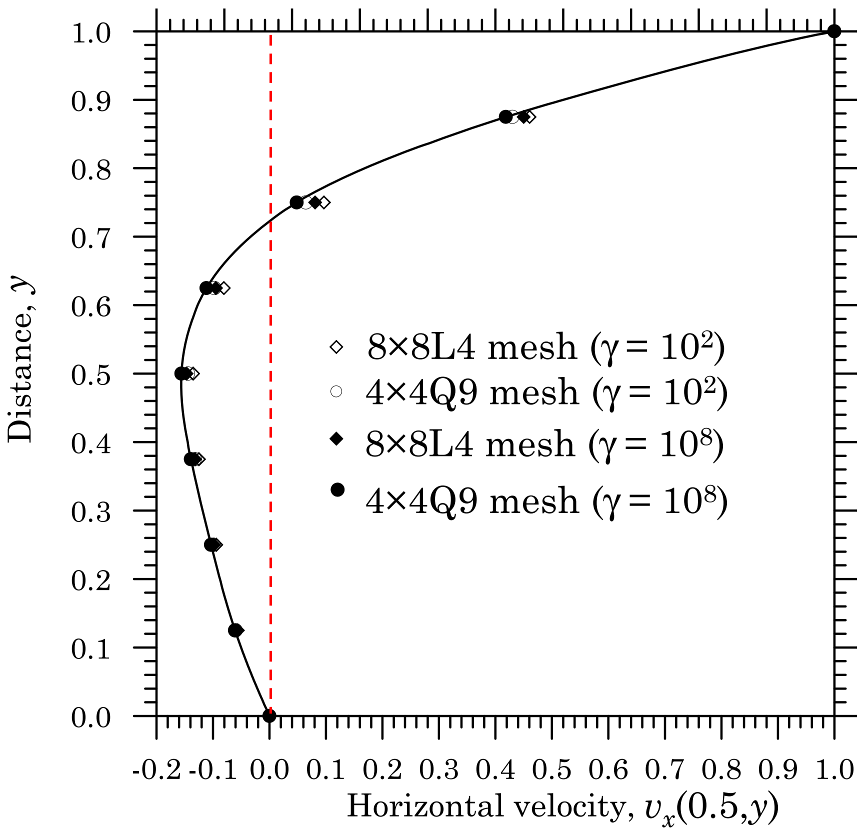


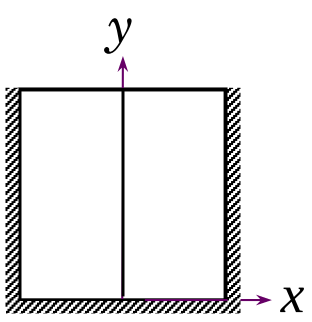


3.Wall-driven cavity flow

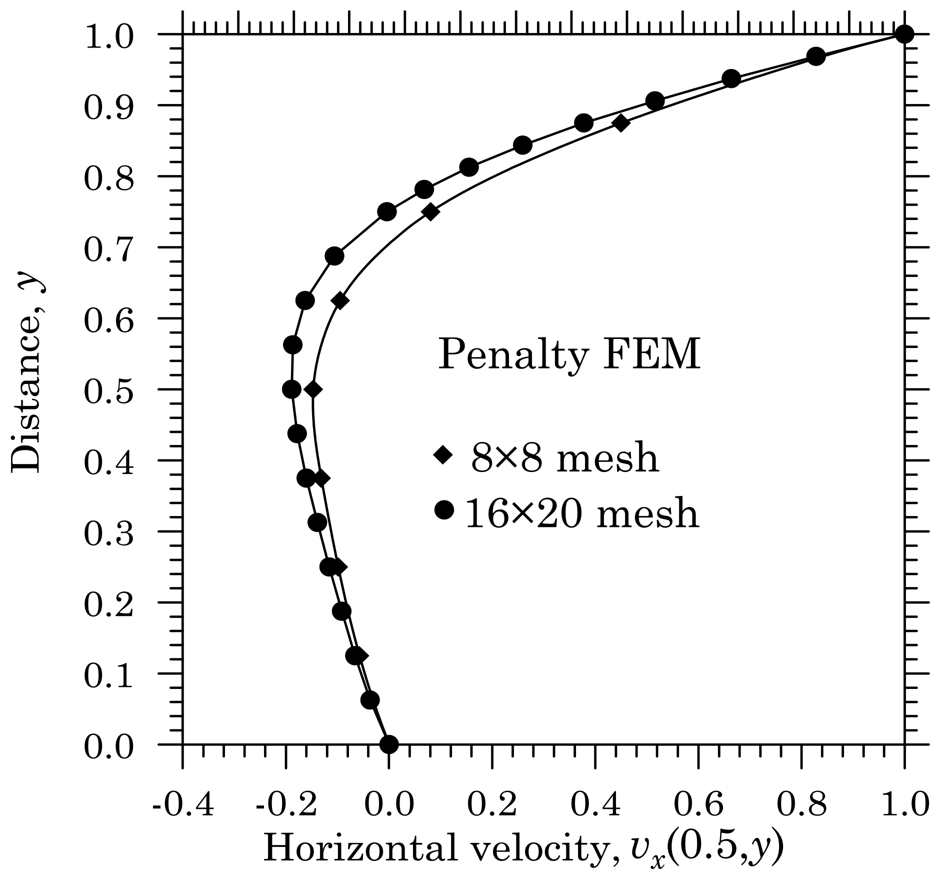


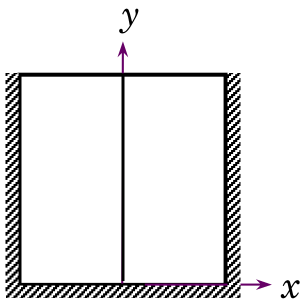
Wall-driven cavity flow – velocity profiles





Wall-driven cavity flow – velocity profiles





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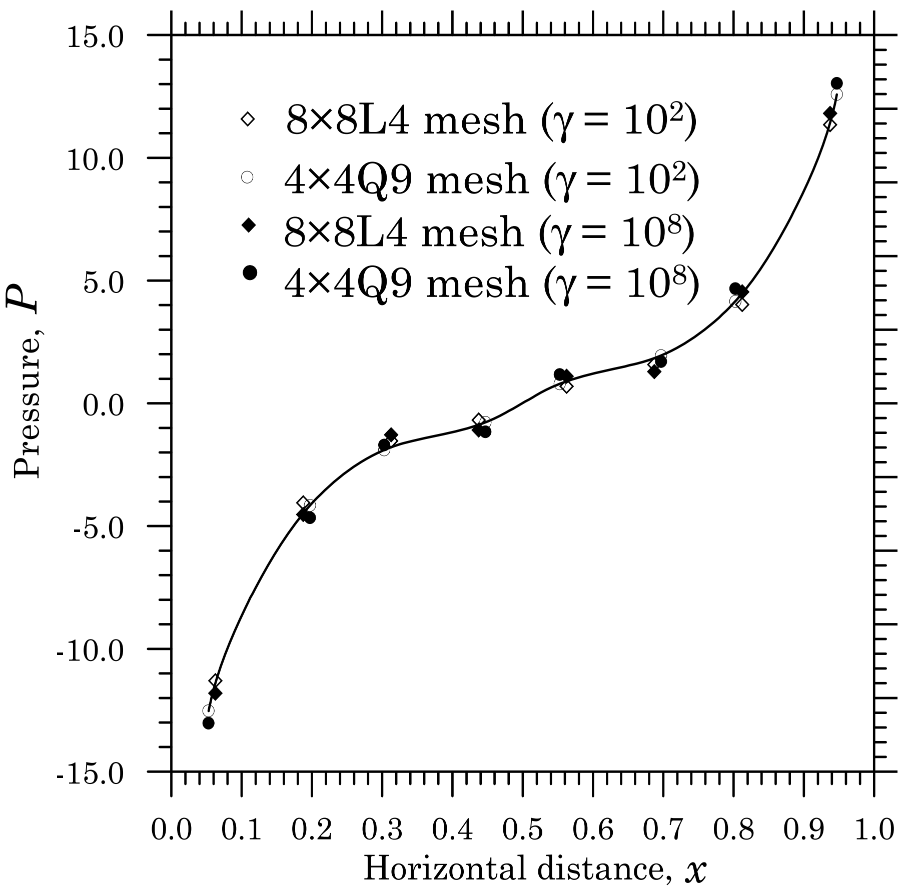
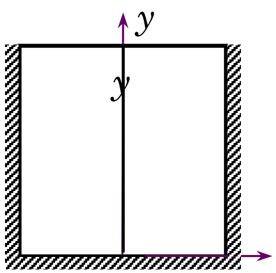
Wall-driven cavity flow – Pressure profile





SUMMARY

The following topics were covered:

- **Governing equations of flows of incompressible fluids**
- **Mixed (velocity-pressure) finite element model**
- **Penalty function method - *algebraic problem***
- **Penalty finite element model of viscous incompressible fluids**
- **Numerical results**