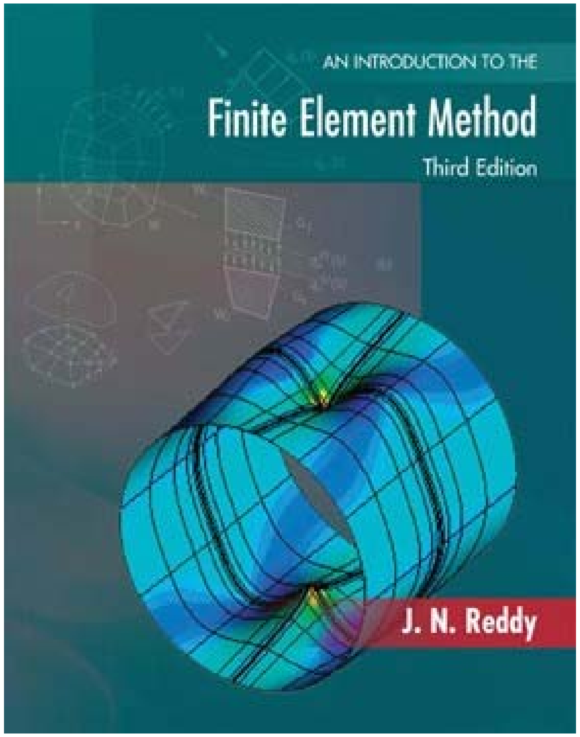
The Finite Element Method

Linear Analysis of FSDT Plates

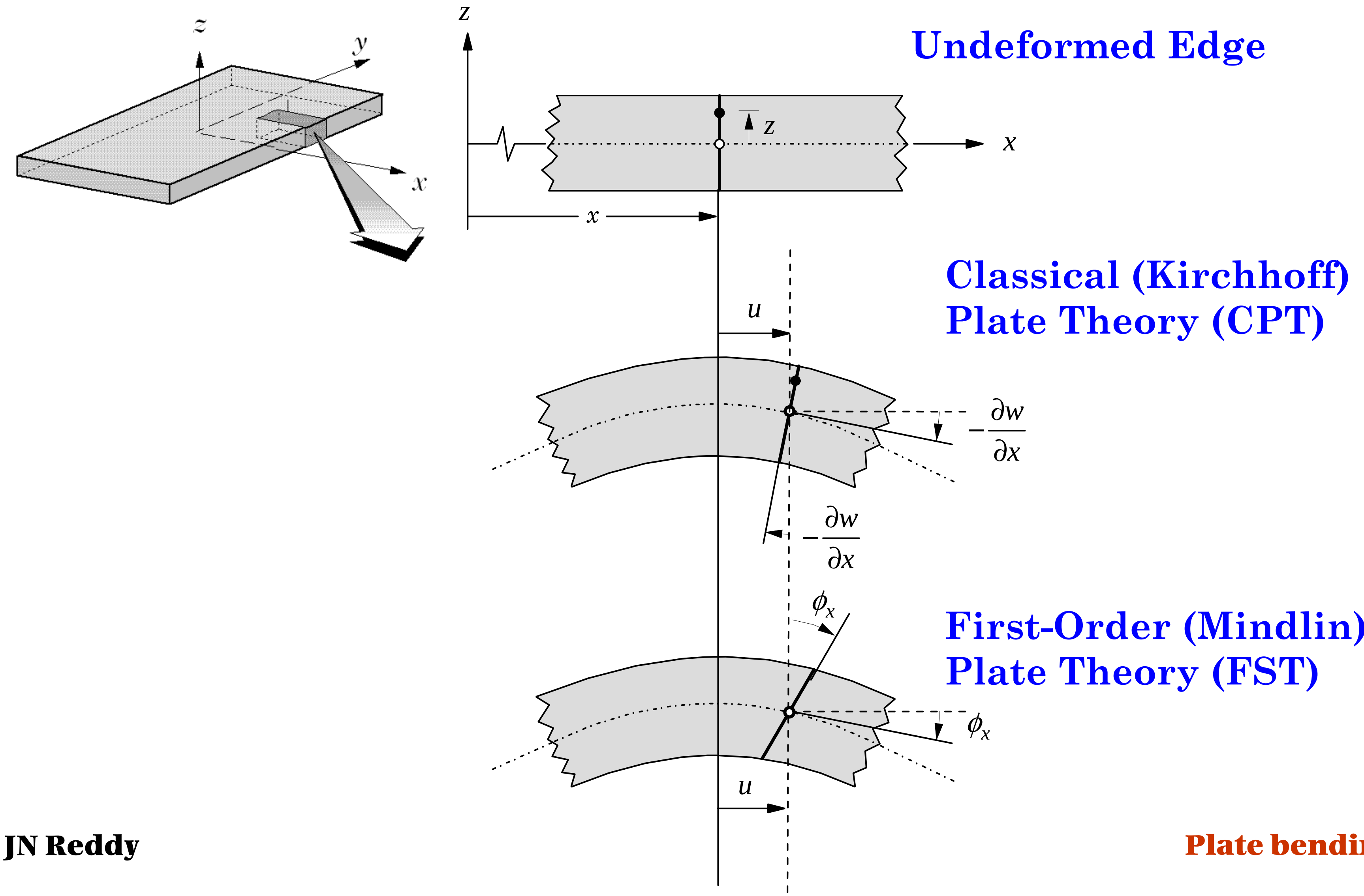
Read: Chapter 12



CONTENTS

- Kinematics of Plate Theories
- Governing Equations of the First-Order Shear Deformation theory (FSDT)
- Finite element models of FSDT
- Shear Locking in FSDT Elements
- Numerical Examples

Kinematics of the Classical and Shear Deformation Plate Theories



THE FIRST-ORDER SHEAR DEFORMATION THEORY

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$

Linear strains

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x} \\ \frac{\partial v}{\partial y} + z\frac{\partial \phi_y}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ \phi_x + \frac{\partial w}{\partial x} \\ \phi_y + \frac{\partial w}{\partial y} \end{Bmatrix}$$

Stress resultants

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz,$$

$$N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz, \quad Q_x = \int_{-h/2}^{h/2} \sigma_{xz}\, dz,$$

$$Q_y = \int_{-h/2}^{h/2} \sigma_{yz}\, dz, \quad M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz,$$

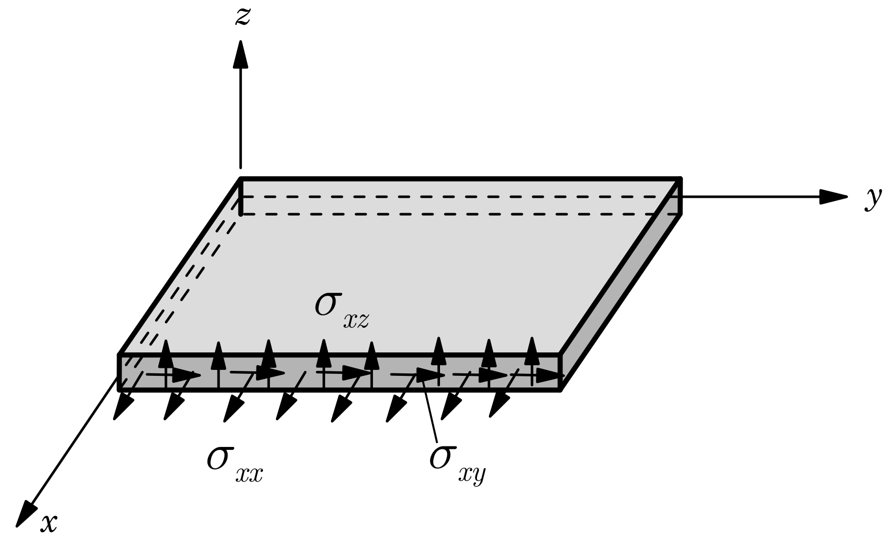
$$M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\, dz, \quad M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz, \; N$$

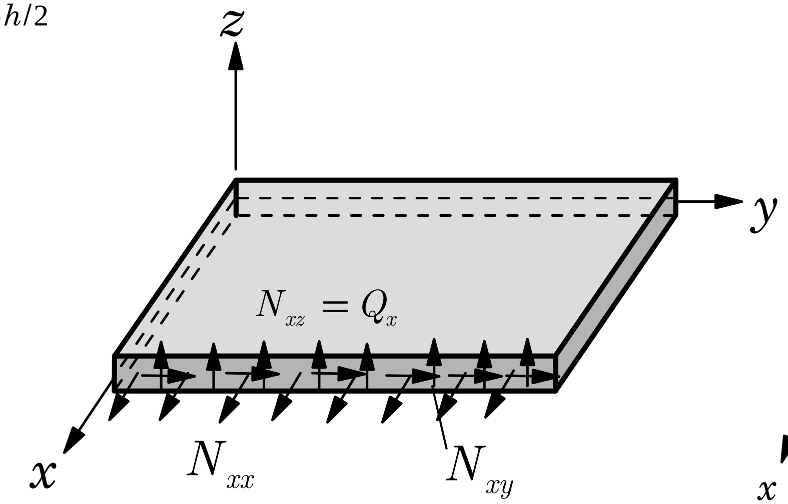
Stresses and Stress Resultants on an edge of a Plate

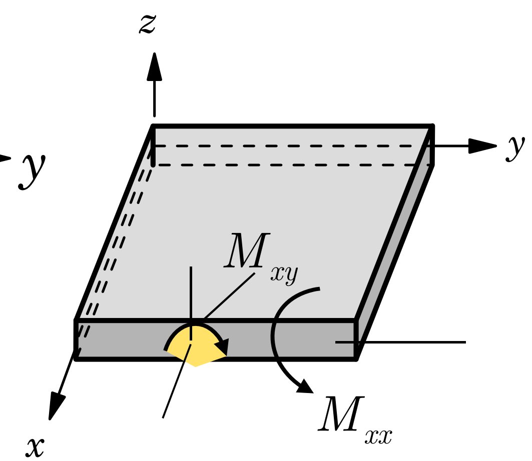
$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz,$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz}\, dz, \quad M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz,$$

$$M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz$$

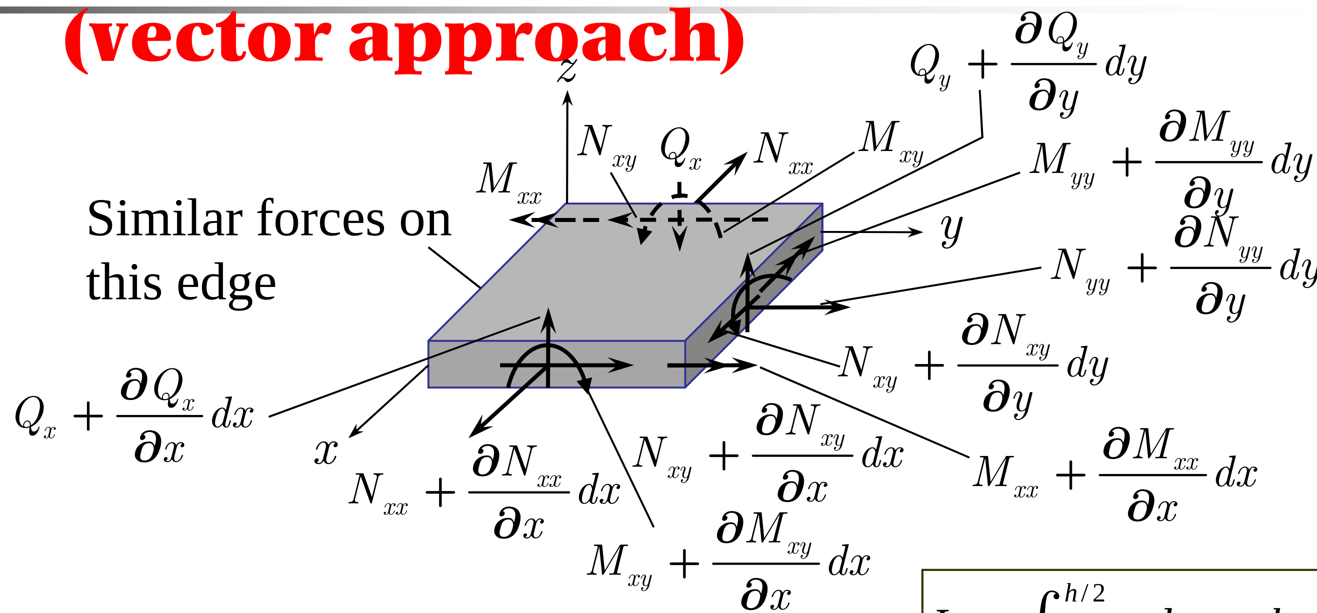






EQUATIONS OF EQUILIBRIUM
(vector approach)

Element of dimensions dx, dy, and h



$$I_0 = \int_{-h/2}^{h/2} \rho \, dz = \rho h,$$

$$I_2 = \int_{-h/2}^{h/2} \rho \, z^2 dz = \frac{\rho h^3}{12}$$

Equations of motion (CPT)

$$\sum F_x = 0: \; \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\sum F_y = 0: \; \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\sum F_z = 0: \; \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\sum M_y = 0: \; \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = -I_2 \frac{\partial^3 w}{\partial t^2 \partial x}$$

$$\sum M_x = 0: \; \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = -I_2 \frac{\partial^3 w}{\partial t^2 \partial y}$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY

Equations of motion

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2}$$

Weak forms

$v_1 = \delta u,\ \ v_2 = \delta v,\ \ v_3 = \delta w,\ \ v_4 = \delta\phi_x,\ \ v_5 = \delta\phi_y$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} + I_0 \delta u \frac{\partial^2 u}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta u N_n \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_{xy} + \frac{\partial \delta v}{\partial y} N_{yy} + I_0 \delta v \frac{\partial^2 v}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta v N_{ns} \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta w}{\partial x} Q_x + \frac{\partial \delta w}{\partial y} Q_y + I_0 \delta w \frac{\partial^2 w}{\partial t^2} - \delta w q \right) dxdy - \oint_{\Gamma^e} \delta w Q_n \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_x}{\partial x} M_{xy} + \delta\phi_x Q_x + I_2 \delta\phi_x \frac{\partial^2 \phi_x}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta\phi_x M_n \, ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_y}{\partial x} M_{yy} + \delta\phi_y Q_y + I_2 \delta\phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} \delta\phi_y M_{ns} \, ds$$

$$Q_n = Q_x n_x + Q_y n_y, \quad M_{nn} = M_{xx} n_x + M_{xy} n_y, \quad M_{ns} = M_{xy} n_x + M_{yy} n_y$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY

There are five equations in five unknowns $(u, v, w, \phi_x, \phi_y)$. However, for linear theories of plates, the inplane displacements (u, v) are uncoupled with the bending variables (w, ϕ_x, ϕ_y). Thus, the first two equations govern the inplane stretching (plane elasticity) and the last three equations govern bending of the plate. Here we consider only bending of plates (having already discussed plane elasticity).

$$Q_x = K_s \int_{-h/2}^{h/2} \sigma_{xz} dz = K_s A_{55} \left(\phi_x + \frac{\partial w}{\partial x} \right)$$

$$Q_y = K_s \int_{-h/2}^{h/2} \sigma_{yz} dz = K_s A_{44} \left(\phi_y + \frac{\partial w}{\partial y} \right)$$

$$M_{xx} = D_{11} \frac{\partial \phi_x}{\partial x} + D_{12} \frac{\partial \phi_y}{\partial y}, \quad M_{yy} = D_{12} \frac{\partial \phi_x}{\partial x} + D_{22} \frac{\partial \phi_y}{\partial y}$$

$$M_{xy} = D_{66} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)$$

LINEAR FINITE ELEMENT MODELS OF THE FSDT (continued)

Finite element approximation of bending deformation

$$w(x,y,t) = \sum_{j=1}^{m} w_j(t)\psi_j^{(1)}(x,y)$$

$$\phi_x(x,y,t) = \sum_{j=1}^{n} S_{xj}(t)\psi_j^{(2)}(x,y), \quad \phi_y(x,y,t) = \sum_{j=1}^{n} S_{yj}(t)\psi_j^{(2)}(x,y)$$

Semidiscrete finite element model

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{w}} \\ \ddot{\mathbf{S}}_x \\ \ddot{\mathbf{S}}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{w} \\ \mathbf{S}_x \\ \mathbf{S}_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \end{Bmatrix}$$

$$\mathbf{M}\ddot{\boldsymbol{\Delta}} + \mathbf{K}\boldsymbol{\Delta} = \mathbf{F}$$

$$M_{ij}^{11} = I_0 M_{ij}, \quad M_{ij}^{22} = M_{ij}^{33} = I_2 M_{ij}, \quad M_{ij} = \int_{\Omega_e} \psi_i \psi_j \; dx\, dy$$

$$K_{ij}^{11} = \int_{\Omega_e} \left(A_{55} \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + A_{44} \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dx \; dy$$

$$K_{ij}^{12} = \int_{\Omega_e} A_{55} \frac{\partial \psi_i}{\partial x} \psi_j \; dx\, dy$$

$$K_{ij}^{13} = \int_{\Omega_e} A_{44} \frac{\partial \psi_i}{\partial y} \psi_j \; dx\, dy$$

Use reduced integration to avoid **shear locking**

FINITE ELEMENT MODELS OF FSDT (continued)

Stiffness coefficients (continued)

$$K_{ij}^{22} = \int_{\Omega_e} \left(D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{55} \psi_i \psi_j \right) dx\, dy$$

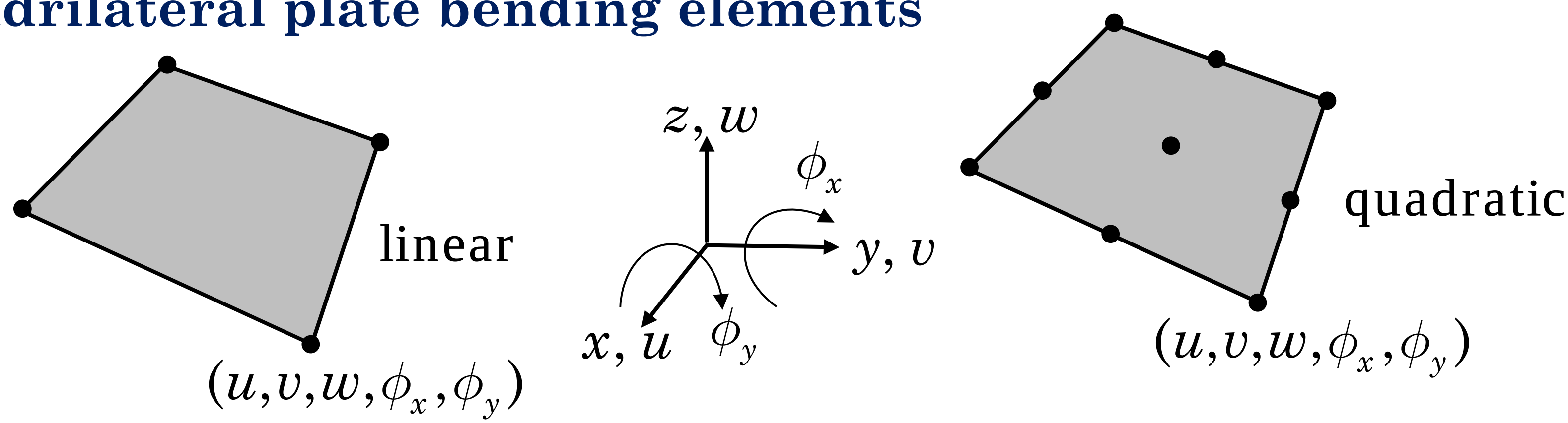
$$K_{ij}^{23} = \int_{\Omega_e} \left(D_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dx\, dy$$

$$K_{ij}^{33} = \int_{\Omega_e} \left(D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{44} \psi_i \psi_j \right) dx\, dy$$

$$F_i^1 = \int_{\Omega_e} q \psi_i \, dx\, dy + \oint_{\Gamma_e} Q_n \psi_i \, ds$$

$$F_i^2 = \oint_{\Gamma_e} \hat{M}_{nn} \psi_i \, ds, \quad F_i^3 = \oint_{\Gamma_e} \hat{M}_{ns} \psi_i \, ds$$

Quadrilateral plate bending elements



FINITE ELEMENT MODELS OF FSDT (continued)

$$\mathbf{M}\ddot{\mathbf{\Delta}} + \mathbf{K}\mathbf{\Delta} = \mathbf{F}$$

Fully Discretized Finite Element Model

$$\hat{\mathbf{K}}_{s+1}\mathbf{\Delta}_{s+1} = \hat{\mathbf{F}}_{s,s+1}$$

$$\hat{\mathbf{K}}_{s+1} = \mathbf{K}_{s+1} + a_3\mathbf{M}_{s+1}$$
$$\hat{\mathbf{F}}_{s,s+1} = \mathbf{F}_{s+1} + \mathbf{M}_{s+1}(a_3\mathbf{\Delta}_s + a_4\dot{\mathbf{\Delta}}_s + a_5\ddot{\mathbf{\Delta}}_s)$$
$$a_3 = \frac{2}{\gamma(\Delta t)^2}, \quad a_4 = \frac{2}{\gamma\Delta t}, \quad a_5 = \frac{1}{\gamma} - 1$$

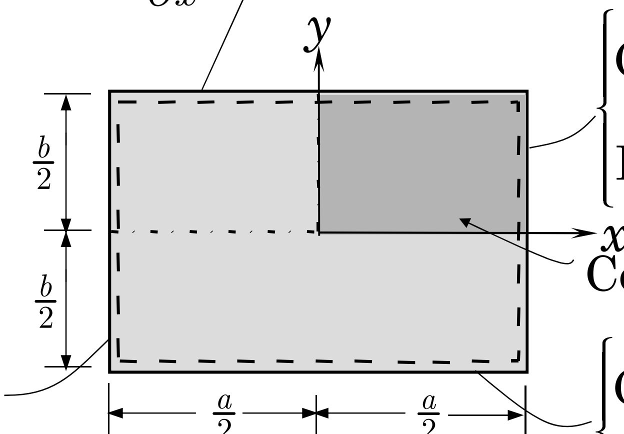
Computation of Velocities and Accelerations

$$\ddot{\mathbf{\Delta}}_{s+1} = a_3(\mathbf{\Delta}_{s+1} - \mathbf{\Delta}_s) - a_4\dot{\mathbf{\Delta}} - a_5\ddot{\mathbf{\Delta}}_s$$
$$\dot{\mathbf{\Delta}}_{s+1} = \dot{\mathbf{\Delta}}_s + a_2\ddot{\mathbf{\Delta}}_s + a_1\ddot{\mathbf{\Delta}}_{s+1}$$

where $a_1 = \alpha\Delta t$ and $a_2 = (1-\alpha)\Delta t$.

TYPICAL SIMPLY SUPPORT CONDITIONS
for Pure Bending case

CPT: $w = \frac{\partial w}{\partial x} = 0$; FSDT: $w = \phi_x = 0$



$$\begin{cases} \text{CPT:} \quad w = \frac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{cases}$$

$$\begin{cases} \text{CPT:} \quad w = \frac{\partial w}{\partial x} = 0 \\ \text{FSDT:} \quad w = \phi_x = 0 \end{cases}$$

$$\left. \begin{matrix} \text{CPT:} \quad w = \frac{\partial w}{\partial y} = 0 \\ \text{FSDT:} \quad w = \phi_y = 0 \end{matrix} \right\}$$

Symmetry conditions: $\begin{cases} \text{CPT:} \quad \frac{\partial w}{\partial x} = 0 \text{ at } x = 0; \; \frac{\partial w}{\partial y} = 0 \text{ at } y = 0 \\ \text{FSDT:} \quad \phi_x = 0 \text{ at } x = 0; \quad \phi_y = 0 \text{ at } y = 0 \end{cases}$

The effect of reduced integration, thickness, and mesh refinement

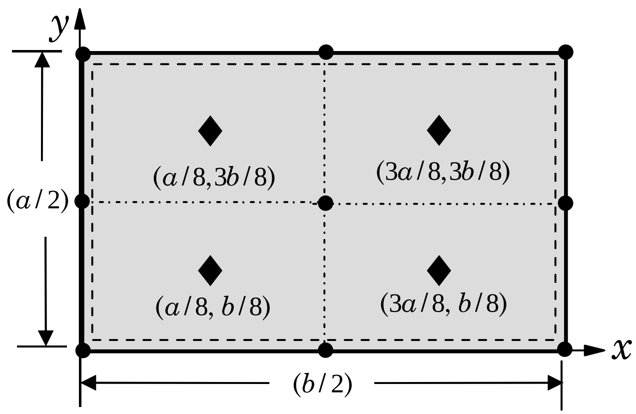
on the center deflections and stresses of a simply supported, isotropic ($\nu = 0.25$) square plate under a uniform transverse load of intensity q_0.

F – full integration
M – Mixed integration

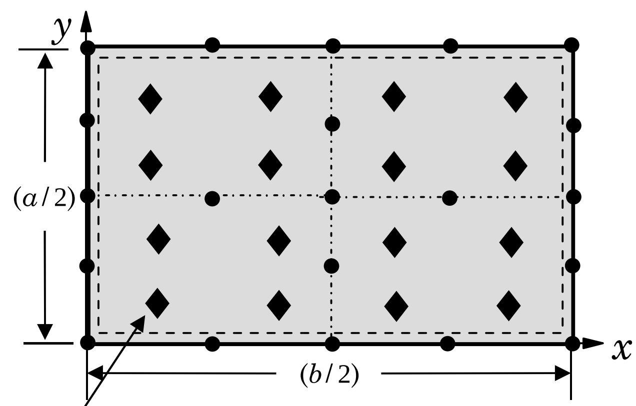
		1 × 1 linear		2 × 2 linear		4 × 4 linear		2 × 2 quadratic		Exact‡	
a/h	Integ.	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{w}$	$\bar{\sigma}_x$
10	F	0.964	0.018	2.474	0.119	3.883	0.216	4.770	0.290	4.791	0.276
	M	3.950	0.095	4.712	0.235	4.773	0.266	4.799	0.272		
20	F	0.270	0.005	0.957	0.048	2.363	0.138	4.570	0.268	4.625	0.276
	M	3.669	0.095	4.524	0.235	4.603	0.266	4.633	0.272		
40	F	0.070	0.001	0.279	0.014	0.944	0.056	4.505	0.270	4.584	0.276
	M	3.599	0.095	4.375	0.235	4.560	0.266	4.592	0.271		
50	F	0.005	0.000	0.182	0.009	0.652	0.039	4.496	0.267	4.579	0.276
	M	3.590	0.095	4.472	0.235	4.555	0.266	4.587	0.271		
100	F	0.011	0.000	0.047	0.002	0.182	0.011	4.482	0.266	4.572	0.276
	M	3.579	0.095	4.465	0.235	4.548	0.266	4.580	0.272		
CPT(N)		5.643	0.260	4.857	0.274	4.643	0.276	—	—	4.570	0.276
CPT(C)		4.638	0.262	4.574	0.272	4.570	0.275	—	—	4.570	0.276

†$\bar{w} = wEh^3 \times 10^2/q_0a^4$, $\bar{\sigma}_x = \sigma_x(A, A, \pm h)h^2/q_0a^2$, $A = \frac{1}{4}a$ (1 × 1 linear), $\frac{1}{8}a$ (2 × 2 linear), $\frac{1}{16}a$ (4 × 4 linear), $0.05283a$ (2 × 2 quadratic).

Gauss Point Locations for Stress Computation



2×2 Mesh of
4-node (linear) elements



2×2 Mesh of
9-node (quadratic) elements

$$\left(\frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}}, \frac{b}{8}\frac{(\sqrt{3}-1)}{\sqrt{3}}\right) = (0.05283a, 0.05283b)$$

Remarks on FSDT Elements

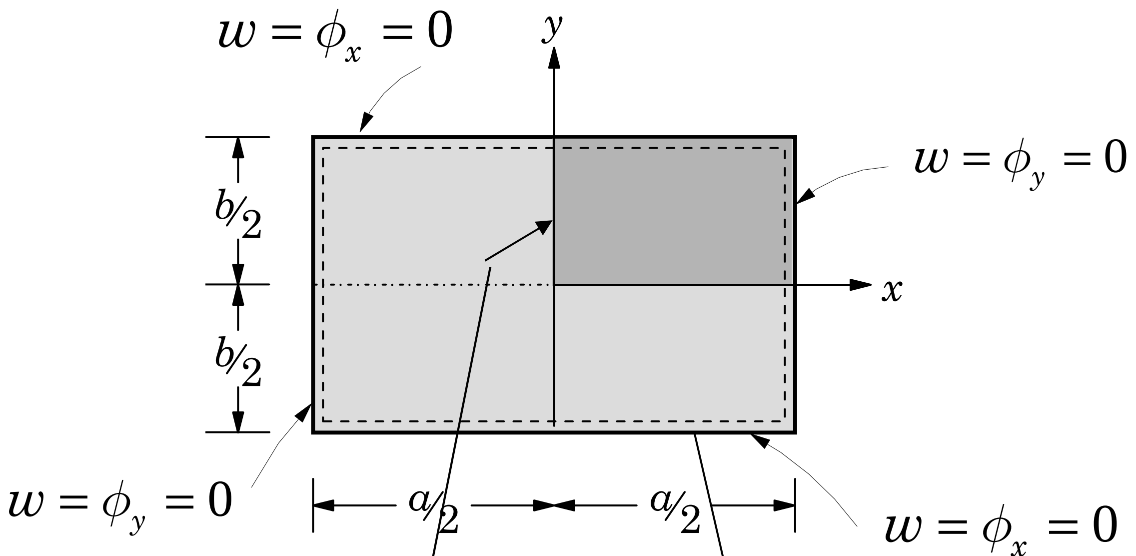
The nine-node element gives virtually the same results for full (3×3 Gauss rule) and mixed (3×3 and 2×2 Gauss rules for bending and shear terms, respectively) integrations. However, the results obtained using the mixed integration are closest to the exact solution.

Full integration gives less accurate results than mixed integration, and the error increases with an increase in side-to-thickness ratio (a/h). This implies that mixed integration is essential for thin plates, especially when modeled by lower-order elements.

Full integration results in smaller errors for quadratic elements and refined meshes than for linear elements and/or coarser meshes.

TYPICAL SUPPORT CONDITIONS: SS-1

Simply Supported Plate (SS1)



Symmetry BCs: $\phi_x = 0$ at $x = 0$; $\phi_y = 0$ at $y = 0$

NUMERICAL EXAMPLES

Non-dimensionalizations and Gauss point locations

$$\bar{w} = w_0(0,0)\frac{E_2 h^3}{a^4 q_0}, \quad \bar{\sigma}_{xx} = \sigma_{xx}(0,0,\frac{h}{2})\frac{h^2}{b^2 q_0}$$

$$\bar{\sigma}_{yy} = \sigma_{yy}(0,0,\frac{h}{4})\frac{h^2}{b^2 q_0}, \quad \bar{\sigma}_{xy} = \sigma_{xy}(\frac{a}{2},\frac{b}{2},-\frac{h}{2})\frac{h^2}{b^2 q_0}$$

$$\bar{\sigma}_{xz} = \sigma_{xz}(\frac{a}{2},0,-\frac{h}{2})\frac{h}{bq_0}\ , \ \bar{\sigma}_{yz} = \sigma_{yz}(0,\frac{b}{2},\frac{h}{2})\frac{h}{bq_0}$$

$$\sigma_{xx}(A,A,\frac{h}{2}), \quad \sigma_{xy}(B,B,-\frac{h}{2}), \quad \sigma_{xz}(B,A,-\frac{h}{2})$$

Table: The Gauss point locations at which the stresses are computed in the finite element analysis of simply supported plates.

Point	2L	4L	8L	1Q9	2Q9	4Q9
A	$0.125a$	$0.0625a$	$0.0312a$	$0.1056a$	$0.0528a$	$0.0264a$
B	$0.375a$	$0.4375a$	$0.4687a$	$0.3943a$	$0.4472a$	$0.4736a$

Simply Supported (SS1) Orthotropic Plate

Table Comparison of the center deflection [$\bar{w} = w \times 10^3(H/q_0 a^4)$] and normal stress [$\bar{\sigma}_{xx} = \sigma_{xx} \times 10(h^2/q_0 a^2)$] of a graphite-epoxy, simply supported square plate under a uniform transverse load (Example 12.5.3).

$E_1 = 31.8$ Msi, $E_2 = 1.02$ Msi, $\nu_{12} = 0.31$, $G_{12} = G_{23} = G_{13} = 0.96$ Msi

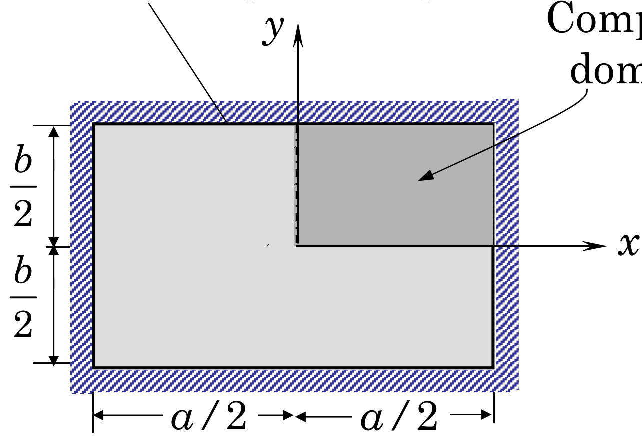
Mesh	Displacement model (CPT)	Displacement model (SDT; $a/h = 10$) Linear	Quadratic
		Center deflection $\bar{w}$	
2×2	0.9220	1.2545	1.2715
4×4	0.9224	1.2186	1.2147
8×8	0.9224	1.2152	1.2147
Exact†	0.9225	1.215	
		Center stress $\bar{\sigma}_{xx}$	
2×2	7.678	6.277	7.192
4×4	7.616	7.256	7.399
8×8	7.600	7.449	7.478
Exact†	7.595	7.512	

† From Reddy (2002); $H = D_{12} + 2D_{66}$.

Clamped Plate

$$\text{CPT}: \; w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0; \;\; \text{SDT}: \; w = \phi_x = \phi_y = 0$$

on all edges of the plate



Symmetry conditions:

$$\begin{cases} \text{CPT}: \dfrac{\partial w}{\partial x} = 0 \text{ at } x = 0; \; \dfrac{\partial w}{\partial y} = 0 \text{ at } y = 0 \\ \text{SDT}: \phi_x = 0 \text{ at } x = 0; \; \phi_y = 0 \text{ at } y = 0 \end{cases}$$

Element type		Location A			
Mesh for CPT & SDT(L)		1×1	2×2	4×4	8×8
Mesh for SDT(Q)			1×1	2×2	4×4
CPT model		$0.05635a$	$0.02817a$	$0.01409a$	$0.03125a$
SDT model	linear	$0.25a$	$0.125a$	$0.0625a$	$0.03125a$
	quadratic	—	$0.1057a$	$0.0528a$	$0.02642a$

Clamped Plate (continued)

Table Comparison of the center deflection and normal stress of a clamped square plate under a uniformly distributed load as obtained using various finite element meshes.

Mesh	Displacement model (CPT)	Displacement model (SDT; $a/h = 10$) Linear	Quadratic
	Center deflection $\bar{w}$		
1×1	0.1943	0.0357	—
2×2	0.1265	0.1459	0.1757
4×4	0.1266	0.1495	0.1586
Analytical	0.1266	—	
	Center stress $\bar{\sigma}_{xx}$		
1×1	2.443	0.000	—
2×2	1.415	1.142	1.321
4×4	1.381	1.333	1.345

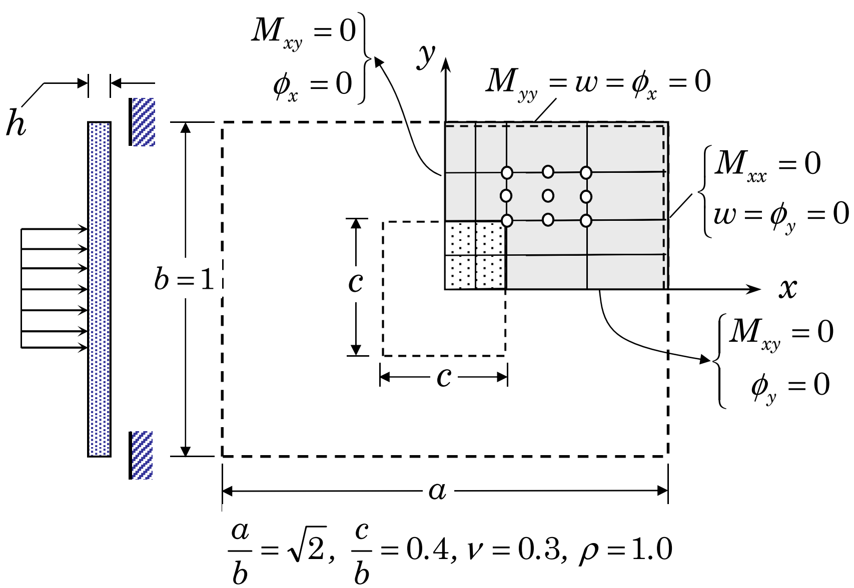
$$\bar{w} = w(0,0)D \times 10^2/q_0 a^4 \;\; [D = Eh^3/12(1-\nu^2)]$$

$$\bar{\sigma}_x = \sigma_x(A,\ A) \times 10/q_0$$

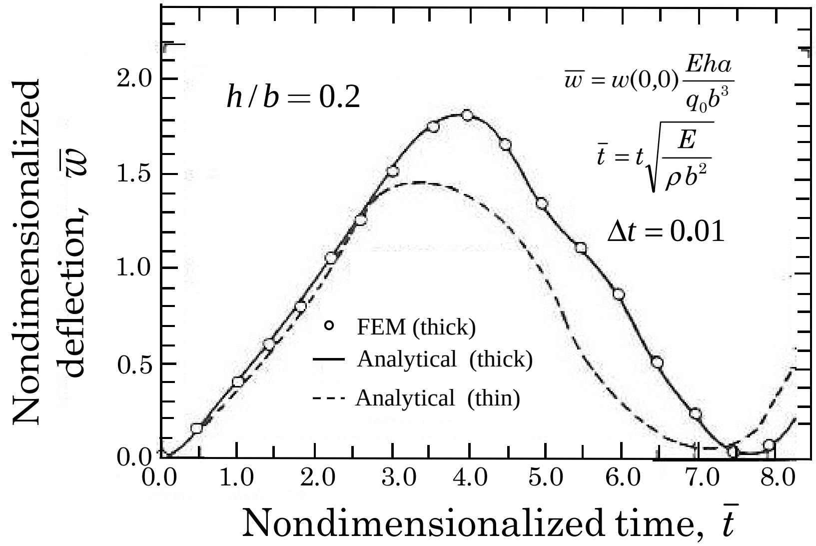
Transient Analysis of Simply Supported (SS1) Isotropic Plate

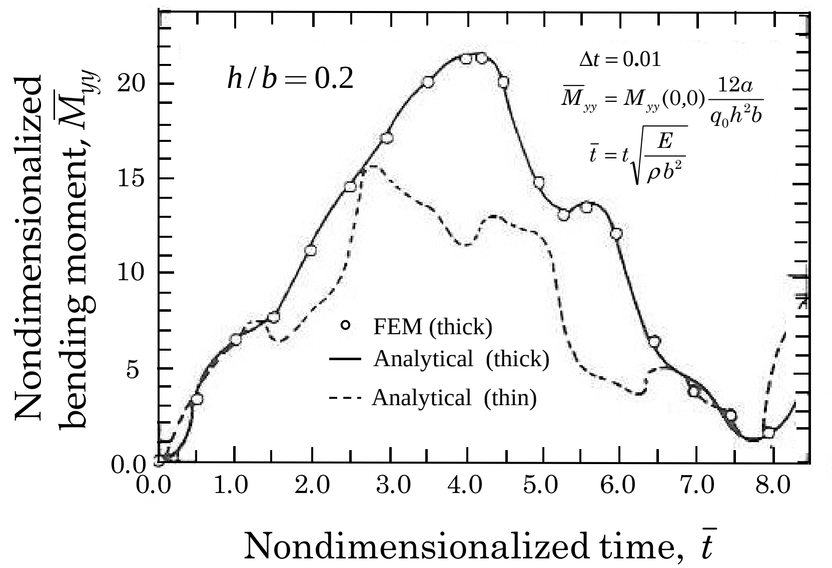
$$h/b = 0.2, \quad a/b = \sqrt{2}, \quad \Delta t = 0.01$$

$$q(x,y,t) = q_0(x,y)H(t) \quad \text{where} \quad q_0(x,y) = \begin{cases} 1 & \text{for} \quad 0 < x,\ y \le 0.2 \\ 0 & \text{for} \quad x,\ y > 0.2 \end{cases}$$



Transient Analysis of Simply Supported Isotropic Plate





SUMMARY

In this lecture we have covered the following topics:

- **Governing Equations of FSDT**
- **Finite element models of FSDT**
- **Shear locking in FSDT**
- **Numerical examples**