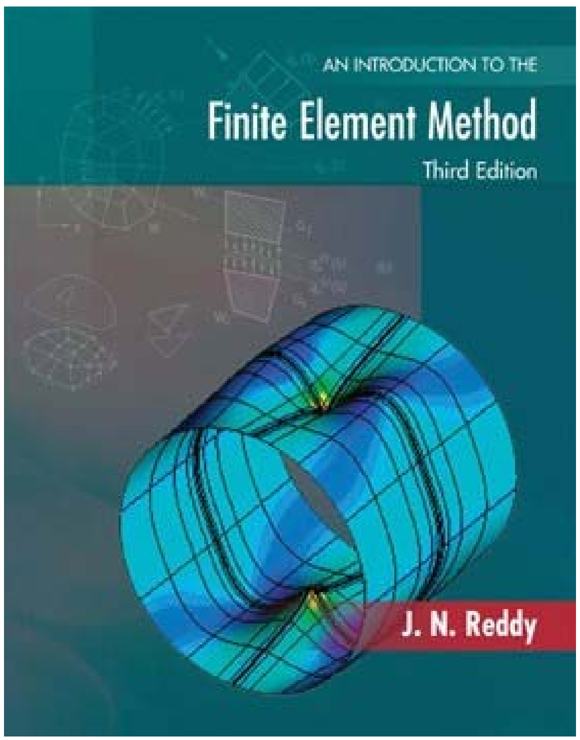
The Finite Element Method

Plane Elasticity

Read: Chapter 11



CONTENTS

- Governing equations
- Weak form formulation
- Finite element models using the weak form
- Triangular and rectangular elements
- Shear locking
- Modeling aspects and discussion

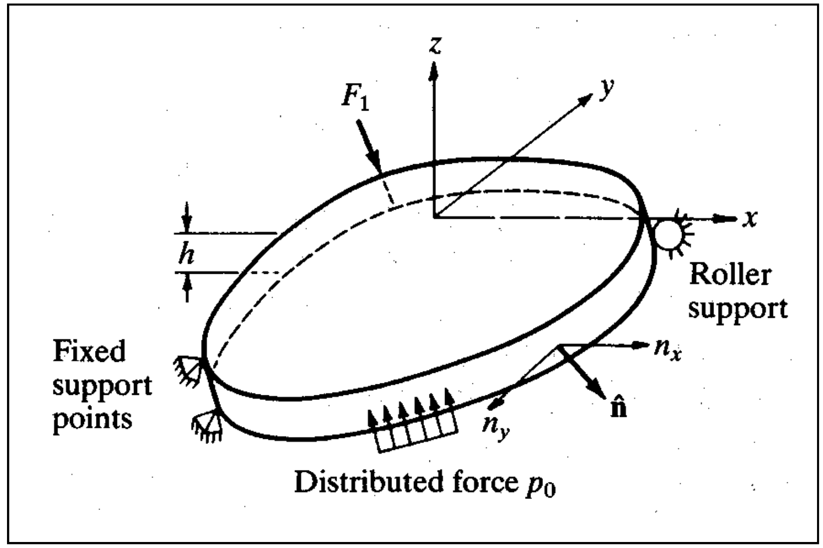
REVIEW OF PLANE ELASTICITY

Plane Stress Problems

$$\sigma_{xx} = \sigma_{xx}(x, y, t), \;\; \sigma_{yy} = \sigma_{yy}(x, y, t),$$

$$\sigma_{xy} = \sigma_{xy}(x, y, t),$$

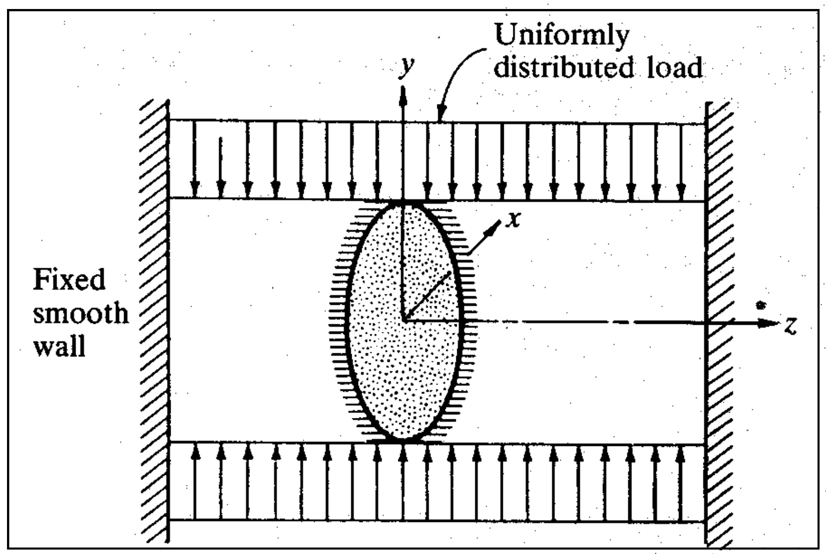
$$\sigma_{xz} = 0, \;\; \sigma_{yz} = 0, \; \sigma_{zz} = 0$$



Plane Strain Problems

$$u = u(x, y, t), \;\; v = v(x, y, t), \;\; w = 0$$

$$\Rightarrow \varepsilon_{xz} = 0, \;\; \varepsilon_{yz} = 0, \; \varepsilon_{zz} = 0$$



GOVERNING EQUATIONS

Stress equations of motion

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

Stress-strain relations (orthotropic)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

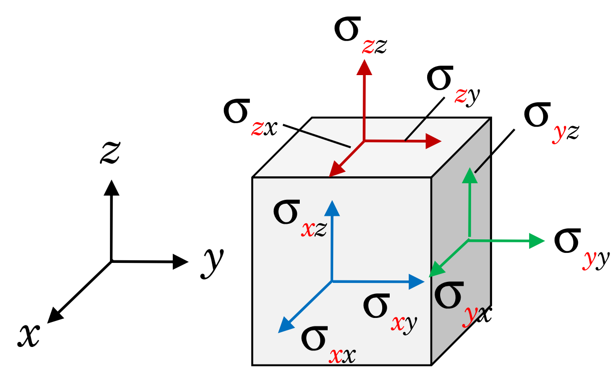
Cauchy's formula

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y$$

$$t_y = \sigma_{xy} n_x + \sigma_{yy} n_y$$

Strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



GOVERNING EQUATIONS (continued)

Plane-strain constitutive relations

$$C_{11} = \frac{E_1(1-\nu_{12}\nu_{21})}{(1+\nu_{12})(1-\nu_{12}-2\nu_{12}\nu_{21})}, \quad C_{22} = \frac{E_2(1-\nu_{12}\nu_{21})}{(1+\nu_{12})(1-\nu_{12}-2\nu_{12}\nu_{21})}$$

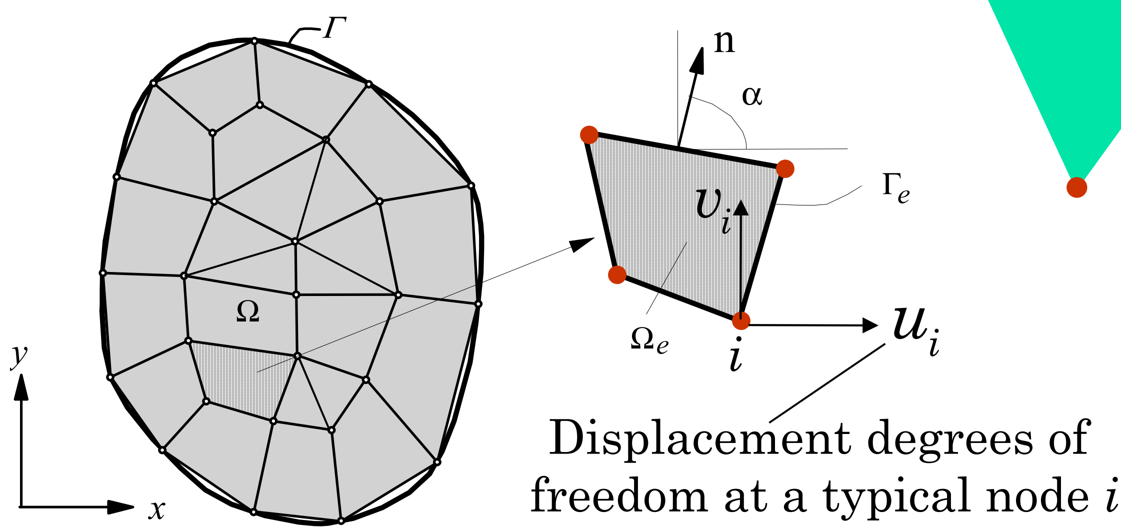
$$C_{12} = \frac{\nu_{12}E_2}{(1-\nu_{12}-2\nu_{12}\nu_{21})}, \qquad C_{66} = G_{12}, \qquad \nu_{12}E_2 = \nu_{21}E_1$$

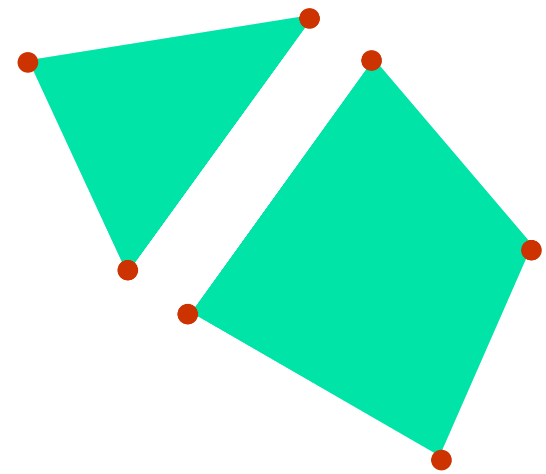
Plane-stress constitutive relations

$$C_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})}, \quad C_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}, \quad C_{12} = \frac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})}$$

$$C_{66} = G_{12}, \qquad \nu_{12}E_2 = \nu_{21}E_1$$

DOMAIN DISCRETIZATION (MESH)





Weak Forms of the Equations

$$0 = h_e \int_{\Omega^e} w_1 \left(\rho \ddot{u} - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - f_x \right) dA$$

$$= h_e \int_{\Omega^e} \left(\rho w_1 \ddot{u} + \frac{\partial w_1}{\partial x} \sigma_{xx} + \frac{\partial w_1}{\partial y} \sigma_{xy} - w_1 f_x \right) dA$$

$$- h_e \oint_{S^e} w_1 t_x \, dS$$

$$0 = h_e \int_{\Omega^e} w_2 \left(\rho \ddot{v} - \frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} - f_y \right) dA$$

$$= h_e \int_{\Omega^e} \left(\rho w_2 \ddot{v} + \frac{\partial w_2}{\partial x} \sigma_{xy} + \frac{\partial w_2}{\partial y} \sigma_{yy} - w_2 f_y \right) dA$$

$$- h_e \oint_{S^e} w_2 t_y \, dS$$

$$\sigma_{xx} = c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y}$$

$$\sigma_{yy} = c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y}$$

$$\sigma_{xy} = c_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$t_x = \left(\sigma_{xx} n_x + \sigma_{xy} n_y \right), \quad t_y = \left(\sigma_{xy} n_x + \sigma_{yy} n_y \right)$$

$$w_1 \sim \delta u, \quad w_2 \sim \delta v$$

FINITE ELEMENT MODEL
using weak form

Finite element approximation

$$u \approx \sum_{j=1}^{n} u_j^e(t)\,\psi_j^e(x,y), \quad v \approx \sum_{j=1}^{n} v_j^e(t)\,\psi_j^e(x,y)$$

Finite element model

$$\begin{bmatrix} [M^{11}] & [0] \\ [0] & [M^{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}\} \\ \{\ddot{v}\} \end{Bmatrix} + \begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

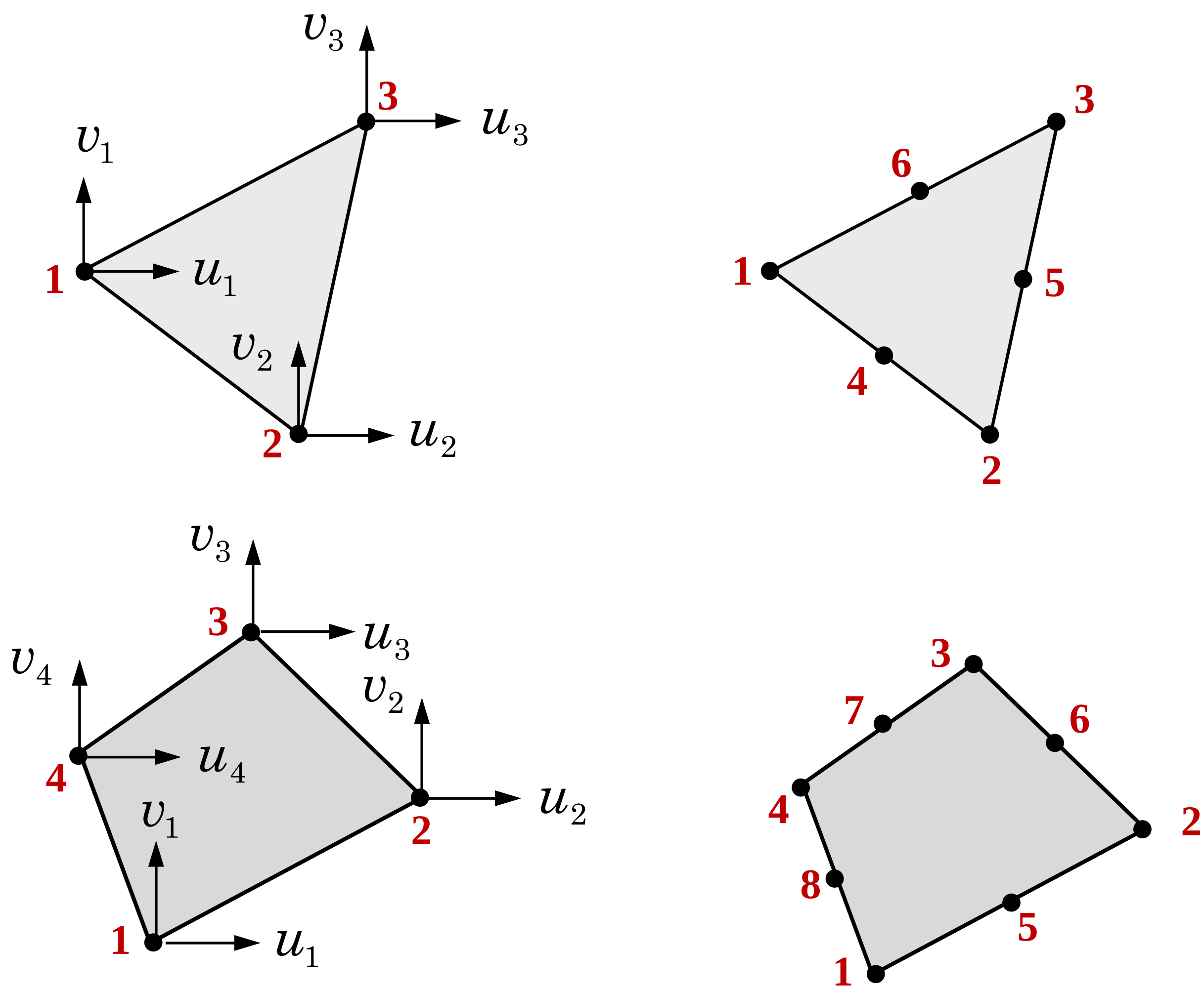
$$[\mathbf{M}]\{\ddot{\boldsymbol{\Delta}}\} + [\mathbf{K}]\{\boldsymbol{\Delta}\} = \{\mathbf{F}\}$$

$$K_{ij}^{11} = h_e \int_{\Omega^e} \left(c_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + c_{66}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right) dA \qquad F_i^1 = h_e \int_{\Omega^e} \psi_i f_x \, dA + h_e \oint_{S^e} \psi_i t_x \, dS$$

$$K_{ij}^{12} = K_{ji}^{21} = h_e \int_{\Omega^e} \left(c_{12}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y} + c_{66}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x}\right) dA \qquad F_i^2 = h_e \int_{\Omega^e} \psi_i f_y dx\, dy + h_e \oint_{S^e} \psi_i t_y \, dS$$

$$K_{ij}^{22} = h_e \int_{\Omega^e} \left(c_{66}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + c_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right) dA$$

FINITE ELEMENTS



Numerical Values of Typical Element Matrices

$$K_{ij}^{11} = h_e \int_{\Omega^e} (c_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + c_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y}) dA$$

$$[K^{11}] = \frac{c_{11}h}{6a} \begin{bmatrix} 2b & -2b & -b & b \\ -2b & 2b & b & -b \\ -b & b & 2b & -2b \\ b & -b & -2b & 2b \end{bmatrix} + \frac{c_{66}h}{6b} \begin{bmatrix} 2a & -2a & -a & a \\ -2a & 2a & a & -a \\ -a & a & 2a & -2a \\ a & -a & -2a & 2a \end{bmatrix}$$

$$K_{ij}^{22} = h_e \int_{\Omega^e} (c_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + c_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y}) dA$$

$$[K^{22}] = \frac{c_{66}h}{6a} \begin{bmatrix} 2b & -2b & -b & b \\ -2b & 2b & b & -b \\ -b & b & 2b & -2b \\ b & -b & -2b & 2b \end{bmatrix} + \frac{c_{22}h}{6b} \begin{bmatrix} 2a & -2a & -a & a \\ -2a & 2a & a & -a \\ -a & a & 2a & -2a \\ a & -a & -2a & 2a \end{bmatrix}$$

Eigenvalue Problems

$$\mathbf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F}$$

Assume periodic solution

$$\Delta = \Delta_0 \; e^{i\omega t}$$

$$\left(-\lambda \mathbf{M}^e + \mathbf{K}^e \right) \Delta_0^e = \mathbf{Q}_0^e, \quad \lambda = \omega^2$$

Assembled equations

$$\left(-\lambda \mathbf{M} + \mathbf{K} \right) \mathbf{U}_0 = \mathbf{Q}_0, \quad \lambda = \omega^2$$

TIME APPROXIMATIONS

Semidiscrete FE model

$$\mathbf{M}^e \ddot{\Delta}^e + \mathbf{K}^e \Delta^e = \mathbf{F}^e$$

Newmark scheme (second-order equations)

$$\Delta^{s+1} = \Delta^s + \delta t\,\dot{\Delta}^s + \tfrac{1}{2}\left(\delta t\right)^2 \ddot{\Delta}^{s,\gamma}$$

$$\dot{\Delta}^{s+1} = \dot{\Delta}^s + \delta t\,\ddot{\Delta}^{s,\alpha}, \quad \ddot{\Delta}^{s,\alpha} = (1-\alpha)\ddot{\Delta}^s + \alpha\,\ddot{\Delta}^{s+1}$$

Fully discretized model

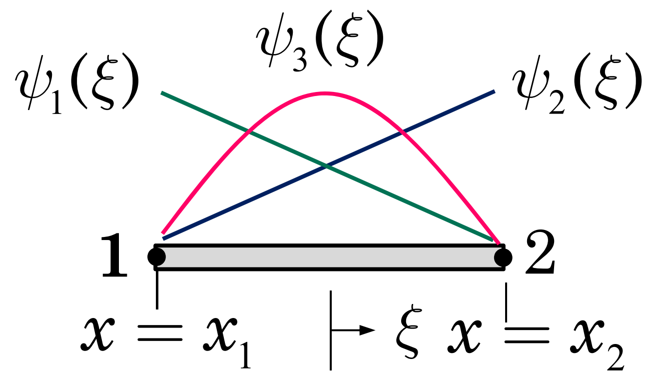
$$\hat{\mathbf{K}}^{s+1}\Delta^{s+1} = \hat{\mathbf{F}}^{s+1}, \qquad \hat{\mathbf{K}}^{s+1} = \mathbf{K}^{s+1} + a_3\mathbf{M}^{s+1}$$

$$\hat{\mathbf{F}}^{s+1} = \mathbf{F}^{s+1} + \mathbf{M}^{s+1}\left(a_3\Delta^s + a_4\dot{\Delta}^s + a_5\ddot{\Delta}^s\right)$$

Static Condensation Procedure

$$\psi_1(\xi) = \tfrac{1}{2}(1-\xi)$$
$$\psi_2(\xi) = \tfrac{1}{2}(1+\xi)$$
$$\psi_3(\xi) = 1-\xi^2$$

$$x = x_1\psi_1(\xi) + x_2\psi_2(\xi)$$
$$= \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi$$



$$u(x) = u_1\psi_1(\xi) + u_2\psi_2(\xi) + \alpha\psi_3(\xi)$$

$$0 = \int_{x_1}^{x_2}\left(EA\frac{d\delta u}{dx}\frac{du}{dx} - f\,\delta u\right)dx - \delta u(x_1)P_1 - \delta u(x_2)P_2$$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ \alpha \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ 0 \end{Bmatrix}$$

Static Condensation Procedure

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \alpha \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ 0 \end{Bmatrix}$$

$$K_{33}\alpha = f_3 - K_{31}u_1 - K_{32}u_2 \Rightarrow \alpha = \left(K_{33}\right)^{-1}\left(f_3 - K_{31}u_1 - K_{32}u_2\right)$$

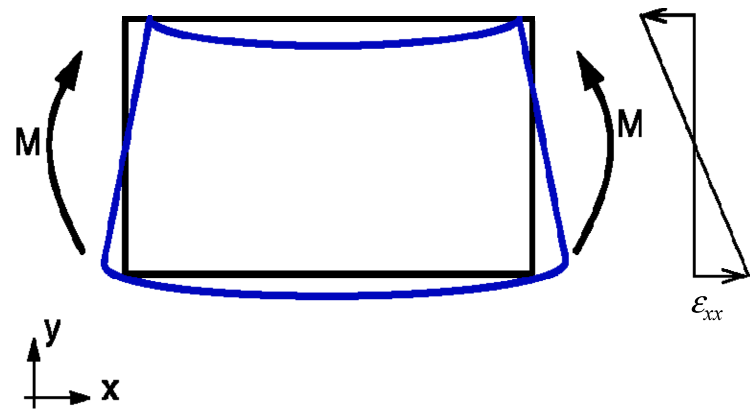
$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\alpha} \\ \mathbf{K}_{\alpha u} & K_{\alpha\alpha} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \alpha \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ f_3 \end{Bmatrix}, \qquad \alpha = \left(K_{\alpha\alpha}\right)^{-1}\left(f_3 - \mathbf{K}_{\alpha u}\mathbf{u}\right)$$

$$\mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\alpha}\alpha = \mathbf{F}_u \Rightarrow \mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\alpha}\left(K_{\alpha\alpha}\right)^{-1}\left(f_3 - \mathbf{K}_{\alpha u}\mathbf{u}\right) = \mathbf{F}_u$$

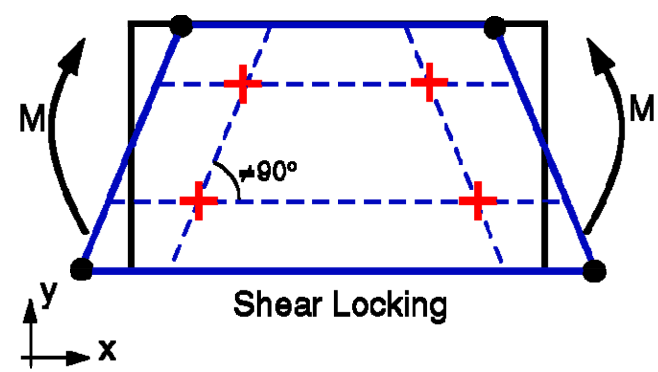
$$\left[\mathbf{K}_{uu} - \mathbf{K}_{u\alpha}\left(K_{\alpha\alpha}\right)^{-1}\mathbf{K}_{\alpha u}\right]\mathbf{u} = \mathbf{F}_u - \mathbf{K}_{u\alpha}\left(K_{\alpha\alpha}\right)^{-1} f_3 \text{ or } \hat{\mathbf{K}}_{uu}\mathbf{u} = \hat{\mathbf{F}}_u$$

BEHAVIOR OF THE LINEAR PLANE ELASTICITY ELEMENT

Pure bending deformation



Linear element Deformation (incorrect)



Elimination of Shear Locking in Linear Elements

Basic idea: Start with the standard expansion in terms of bilinear shape functions for a typical linear quadrilateral element, quadratic modes are added as follows:

$$\mathbf{u}_h(\xi,\eta)=\sum_{j=1}^{4}\psi_j(\xi,\eta)\mathbf{\Delta}_j^e+\sum_{i=5}^{6}\psi_i(\xi,\eta)\boldsymbol{\alpha}_i^e$$

where $\psi_i \ (i=1,2,3,4)$ are the standard bilinear interpolation functions and

$$\psi_5(\xi,\eta)=1-\xi^2, \qquad \psi_6(\xi,\eta)=1-\eta^2$$

are the incompatible modes; $\boldsymbol{\alpha}_i^e \ (i=5,6)$ are generalized displacements (node-less variables).

The generalized displacements associated with the incompatible modes are unique to each element and must be eliminated via "static condensation".

Elimination of Shear Locking in Linear Elements (continued)

We begin with the standard definition of the stiffness matrix:

$$\mathbf{K} = \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\, d\mathbf{x}$$

where $\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \mathbf{B}_4 \quad | \quad \mathbf{B}_5 \quad \mathbf{B}_6]$ and $\boldsymbol{\alpha} = \begin{Bmatrix} \alpha_5 \\ \alpha_6 \end{Bmatrix}$

$$\begin{bmatrix} \mathbf{K}^{\Delta\Delta} & \mathbf{K}^{\Delta\alpha} \\ \mathbf{K}^{\alpha\Delta} & \mathbf{K}^{\alpha\alpha} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Delta} \\ \boldsymbol{\alpha} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \end{Bmatrix} \quad \text{or} \quad \mathbf{K}^{\Delta\Delta}\Delta + \mathbf{K}^{\Delta\alpha}\boldsymbol{\alpha} = \mathbf{F}, \quad \mathbf{K}^{\alpha\Delta}\Delta + \mathbf{K}^{\alpha\alpha}\boldsymbol{\alpha} = 0$$

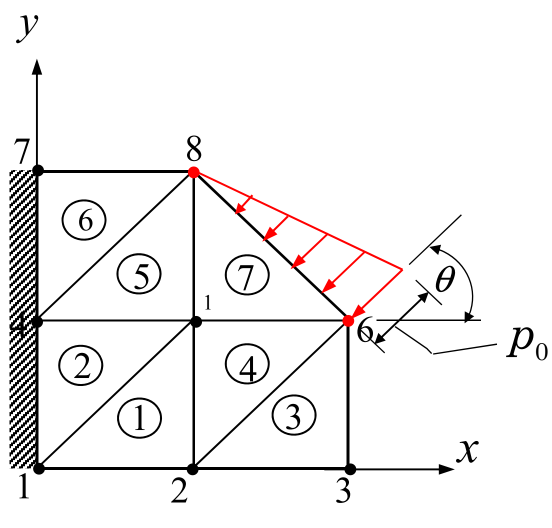
Static condensation: Solving the second equation for $\boldsymbol{\alpha}$, we obtain

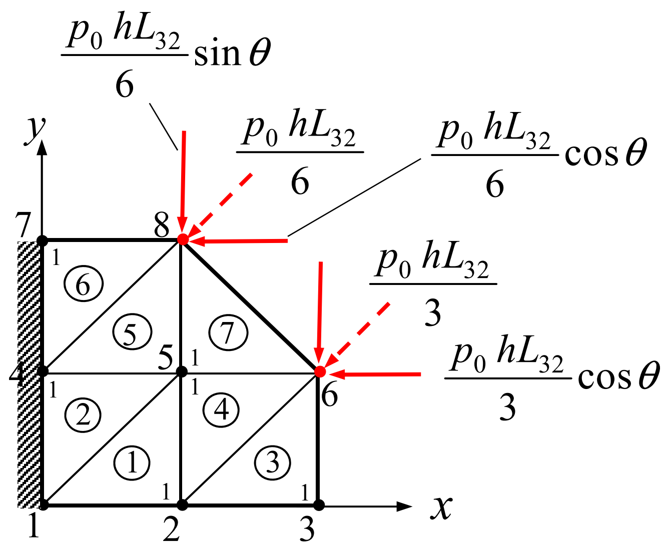
$$\alpha = -\left(\mathbf{K}^{\alpha\alpha}\right)^{-1}\mathbf{K}^{\alpha\Delta}\boldsymbol{\Delta}$$

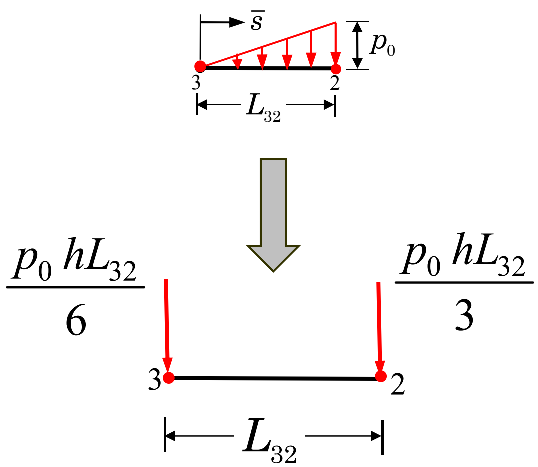
Substituting into the first equation, we arrive at

$$\hat{\mathbf{K}}\boldsymbol{\Delta} = \mathbf{F} \quad \text{where} \quad \hat{\mathbf{K}} = \mathbf{K}^{\Delta\Delta} - \mathbf{K}^{\Delta\alpha}(\mathbf{K}^{\alpha\alpha})^{-1}\mathbf{K}^{\alpha\Delta}$$

MODELING ASPECTS: Load Calculation







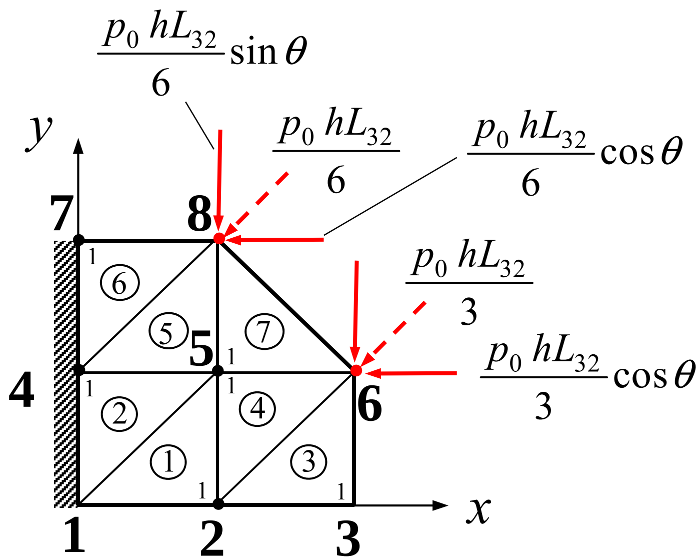
$$F_8^x = -\frac{p_0 hL_{32}}{6}\cos\theta \;\Rightarrow F_{15} = -\frac{p_0 hL_{32}}{6}\cos\theta$$

$$F_8^y = -\frac{p_0 hL_{32}}{6}\sin\theta \;\Rightarrow F_{16} = -\frac{p_0 hL_{32}}{6}\sin\theta$$

$$F_6^x = -\frac{p_0 hL_{32}}{3}\cos\theta \;\Rightarrow F_{11} = -\frac{p_0 hL_{32}}{3}\cos\theta$$

$$F_6^y = -\frac{p_0 hL_{32}}{3}\sin\theta \;\Rightarrow F_{12} = -\frac{p_0 hL_{32}}{3}\sin\theta$$

MODELING ASPECTS: Assembly of Element coefficients

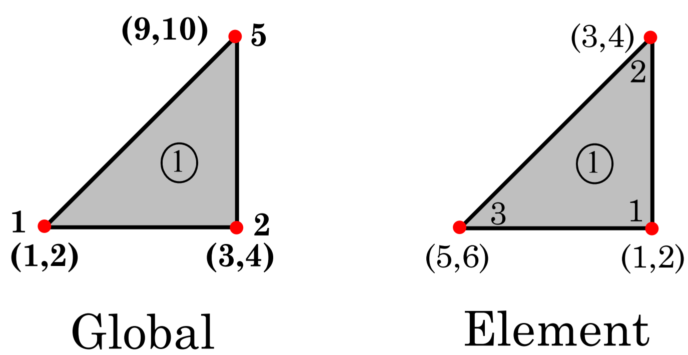


$$K_{11} = K_{55}^1 + K_{33}^2,\quad K_{12} = K_{56}^1 + K_{34}^2$$

$$K_{13} = K_{51}^1,\quad K_{15} = 0,\quad K_{23} = K_{61}^1$$

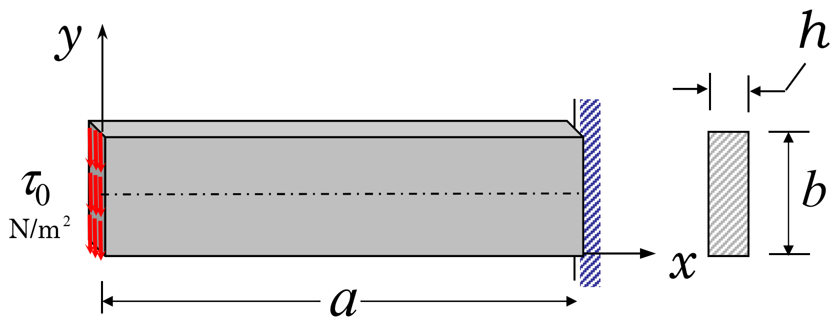
$$K_{19} = K_{53}^1 + K_{35}^2,\quad K_{88} = K_{22}^2 + K_{66}^5 + K_{44}^6$$

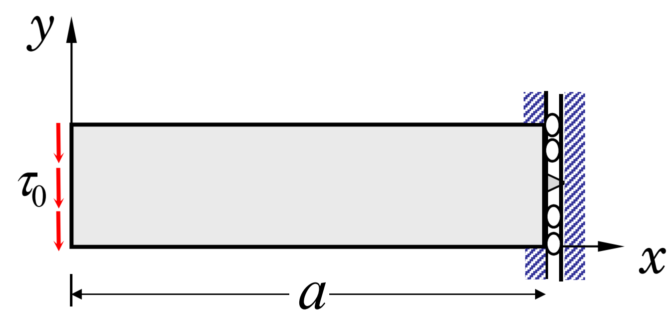
$$K_{77} = K_{11}^2 + K_{55}^5 + K_{33}^6,\ F_8 = F_2^2 + F_6^5 + F_4^6$$



MODELING ASPECTS:
Bending of a cantilever beam

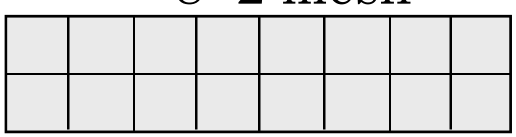
$E = 30\,\text{msi},\;\; \nu = 0.25,\;\; a = 10\text{ in.},\; b = 2\text{ in.},\; \tau_0 = 150\,\text{psi}$





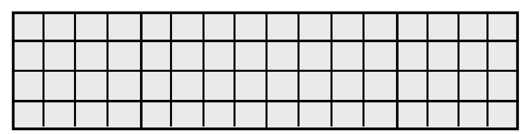
Beam versus elasticity model

8×2 mesh



2-D Meshes

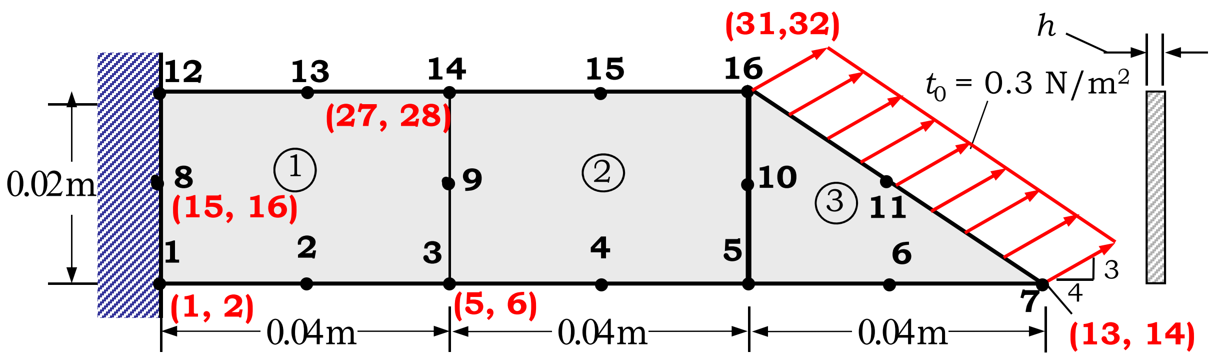
16×4 mesh

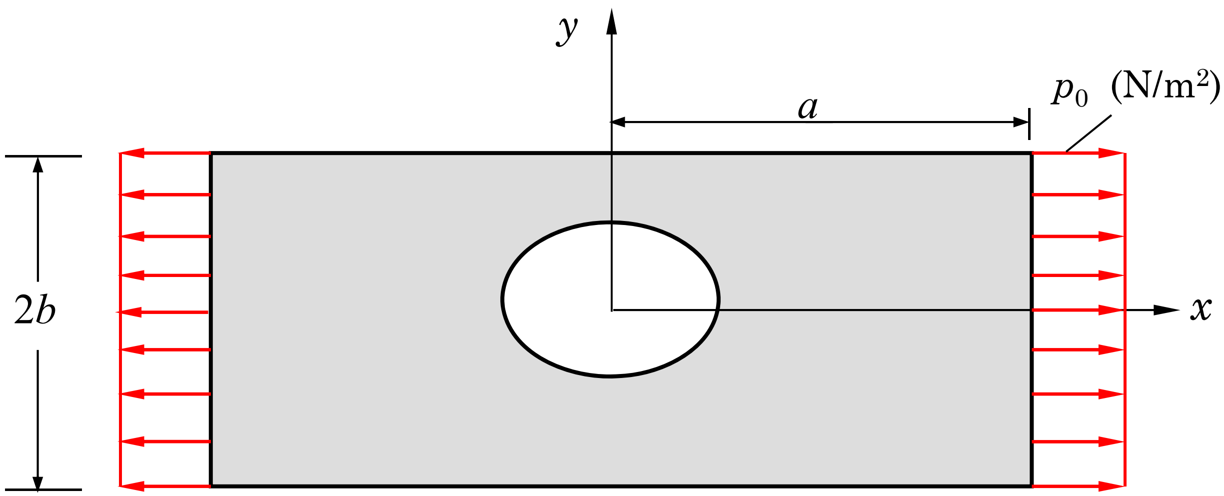


$$t_y = -\tau_0 h \;\text{ N/m at } x = 0$$

DISCUSSION PROBLEMS

Plane stress $E_1 = E_2 = 69$ GPa, $\nu = 0.333$, $G = 26$ GPa, $h =$ 0.01 m





One quadrant of the domain is used in the finite element analysis (isotropic plate of thickness h)

SUMMARY

In this lecture we have covered the following topics:

- Review of plane stress and plane strain
- Governing equations of plane elasticity
- Finite element models using the weak form
- Static condensation
- Incompatible elements and shear locking
- Discussion problems