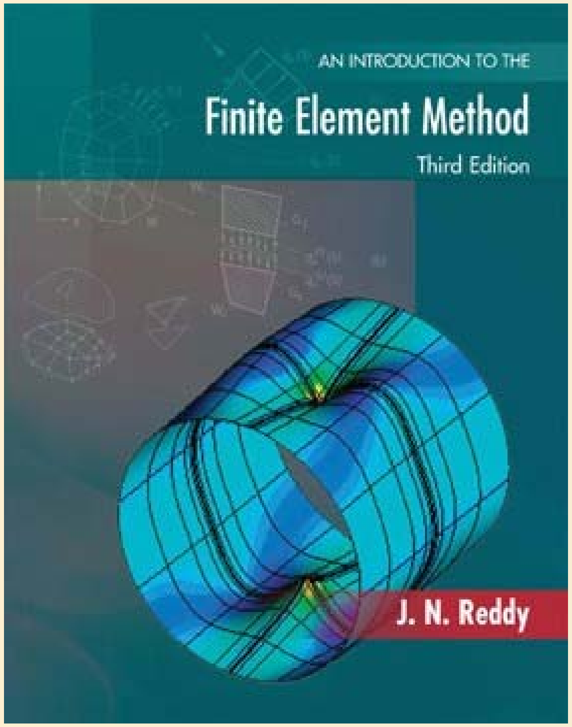
The Finite Element Method

Finite Element Models of Axisymmetric Problems and Numerical Integration

Read: Chapter 9

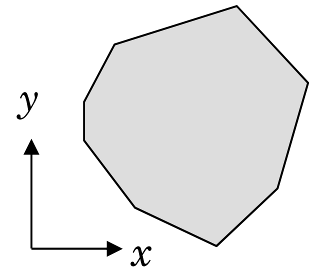


CONTENTS

- Plane 2-D Problem (a review)
- Types of Axisymmetric Problems
- Axisymmetric Problems (2-D)
- Parametric formulations
- Numerical integration and element calculations

PLANE (2-D) PROBLEMS

Review of Plane (2-D) Model



Governing Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f(x,y)$$

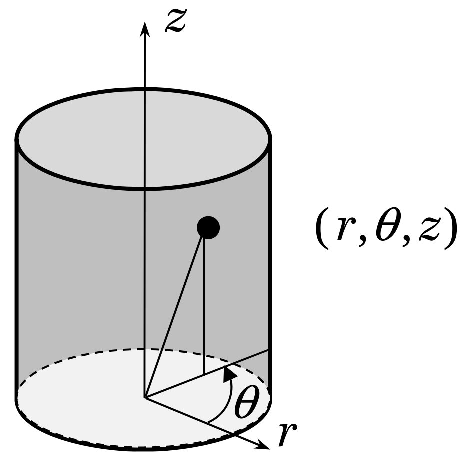
Weak Form

$$0 = \int_{\Omega_e}\left(a_{11}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_{22}\frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_{00}wu - wf\right)dx\,dy - \oint_{\Gamma_e} wq_n\,ds$$

Flux normal to the surface

$$q_n = \left(a_{11}\frac{\partial u}{\partial x}n_x + a_{22}\frac{\partial u}{\partial y}n_y\right)$$

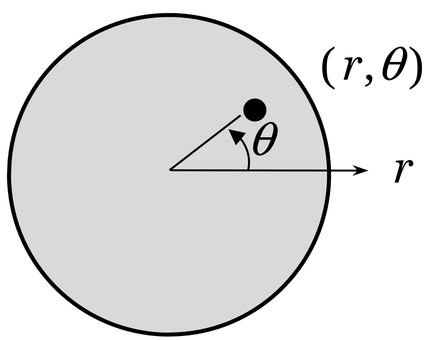
Reduction of Problem Size from 3-D



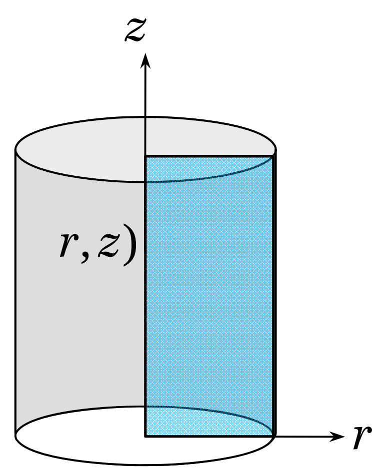
3-D Models

General loading, boundary conditions, and material properties, all of which may change along the length and around the circumference (i.e., with z and θ)

2-D Models



The loading, boundary conditions, and material properties do not change along the length *z*

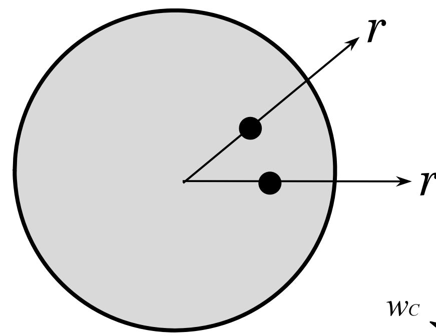


The loading, boundary conditions, and material properties do not change around the circumference

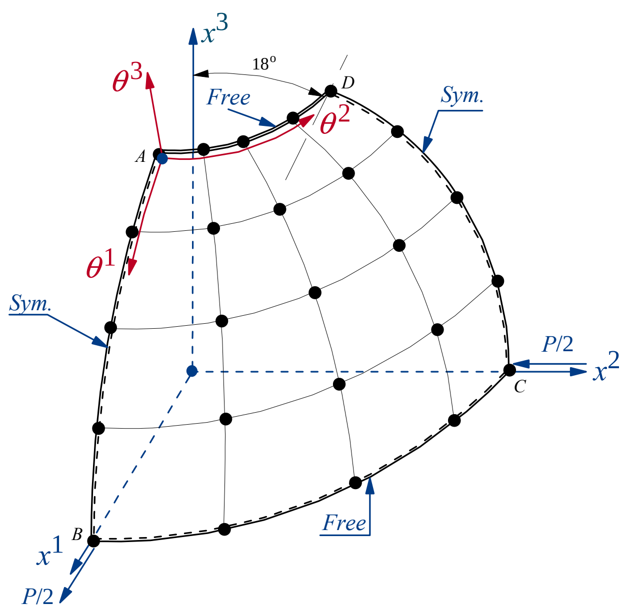
Reduction of Problem Size from 3-D

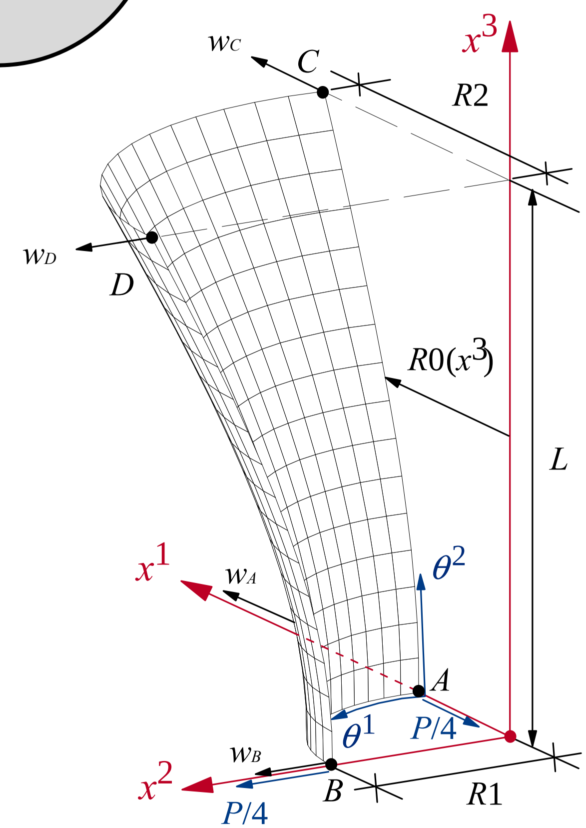
1-D Models

The loading, boundary conditions, and material properties do not change around the circumference as well as the length



Other Axisymmetric Problems



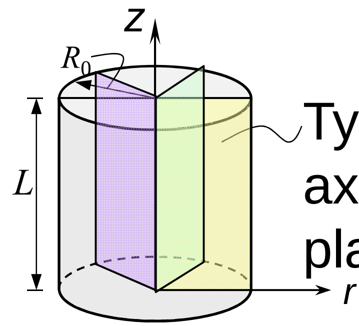


JN Reddy

AXISYMMETRIC PROBLEMS (2-D)

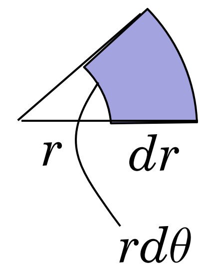
Governing Equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{a}_{11}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(\hat{a}_{22}\frac{\partial u}{\partial z}\right)+\hat{a}_{00}u=\hat{f}(r,z)$$



Weak Form

$$dv = r dr\, d\theta\, dz$$



$$0=\int_{\Omega_e} w\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{a}_{11}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(\hat{a}_{22}\frac{\partial u}{\partial z}\right)+\hat{a}_{00}u-\hat{f}\right] r\,dr\,dz$$

$$0=\int_{\Omega_e}\left(\frac{\partial w}{\partial r}\hat{a}_{11}\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\hat{a}_{22}\frac{\partial u}{\partial z}+w\hat{a}_{00}u-w\hat{f}\right) r\,dr\,dz$$

$$-\oint_{\Gamma_e} w\left(\hat{a}_{11}\frac{\partial u}{\partial r}n_r+\hat{a}_{22}\frac{\partial u}{\partial z}n_z\right)ds$$

$$0=\int_{\Omega_e}\left(\hat{a}_{11}\frac{\partial w}{\partial r}\frac{\partial u}{\partial r}+\hat{a}_{22}\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}+\hat{a}_{00}wu-w\hat{f}\right) r\,dr\,dz-\oint_{\Gamma_e} wq_n\,ds$$

$$q_n=\left(\hat{a}_{11}\frac{\partial u}{\partial r}n_r+\hat{a}_{22}\frac{\partial u}{\partial z}n_z\right)$$

AXISYMMETRIC PROBLEMS (2-D)

Finite Element Model

$$u(r,z) \approx u_h(r,z) = c_1 + c_2 r + c_3 z + c_4 rz + \cdots \ (n \text{ terms})$$
$$= \sum_{j=1}^{n} u_j \psi_j(r,z)$$

$$[K^e]\{u^e\} = \{F^e\}$$

$$K_{ij}^e = \int_{\Omega^e} \left(\hat{a}_{11} \frac{\partial \psi_i}{\partial r} \frac{\partial \psi_j}{\partial r} + \hat{a}_{22} \frac{\partial \psi_i}{\partial z} \frac{\partial \psi_j}{\partial z} + \hat{a}_{00} \psi_i \psi_j \right) r dr dz$$

$$F_i^e = \int_{\Omega^e} \psi_i f \, r \, dr dz + \oint_{\Gamma_e} \psi_i q_n \, ds$$

PARAMETRIC FORMULATIONS

Geometry: $$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi,\eta), \quad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi,\eta)$$

Solution: $$u(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi,\eta), y(\xi,\eta))$$

Thus, there are two meshes in finite element analysis.

1. ***Superparametric* (*m* > *n*):** The polynomial degree of approximation used for the geometry is of higher order than that used for the dependent variable.

2. ***Isoparametric* (*m* = *n*):** Equal degree of approximation is used for both geometry and dependent variables.

3. ***Subparametric* (*m* < *n*):** Higher-order approximation of the dependent variable is used.

PARAMETRIC FORMULATIONS
(continued)

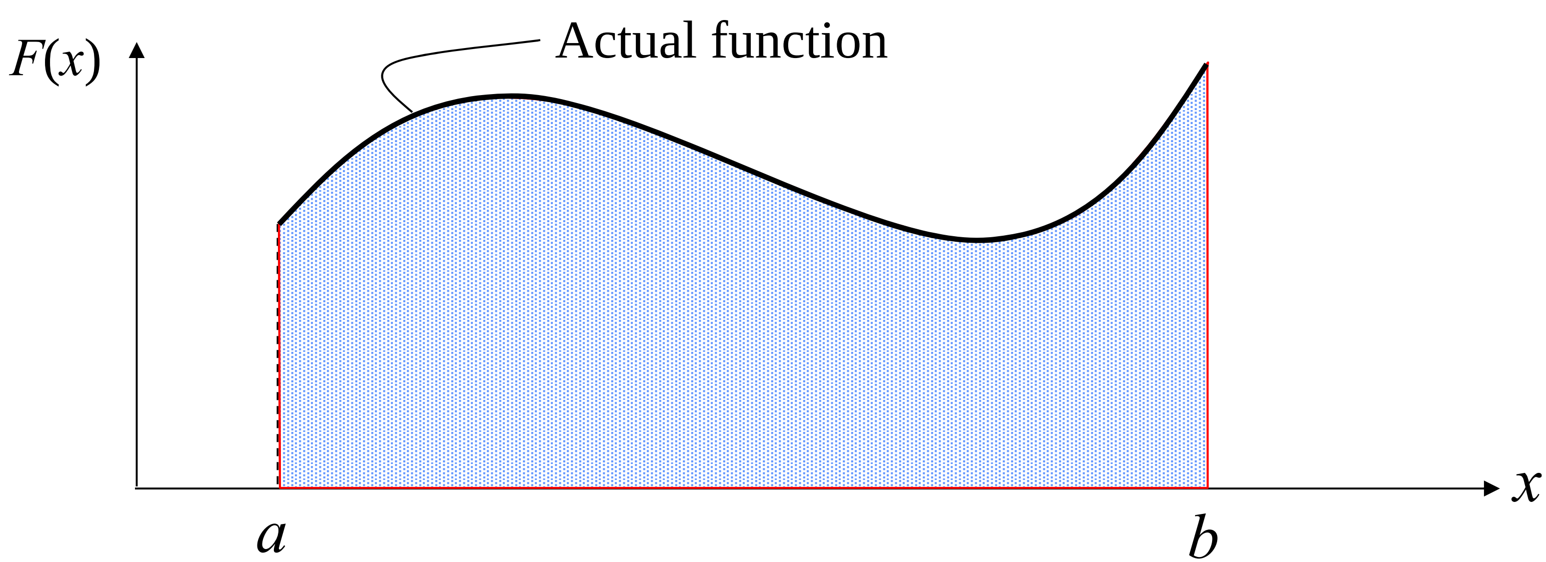
Superparametric formulations are not common, while the isoparametric formulations are the most common. Subparametric formulations arise when the field variables are approximated with C^k (for example, Euler-Bernoulli beam).

Note: It is not correct say 'isoparametric element' because an element is unique in terms of the number of nodes and degrees freedom per node; hence, the approximation order on the element is unique.

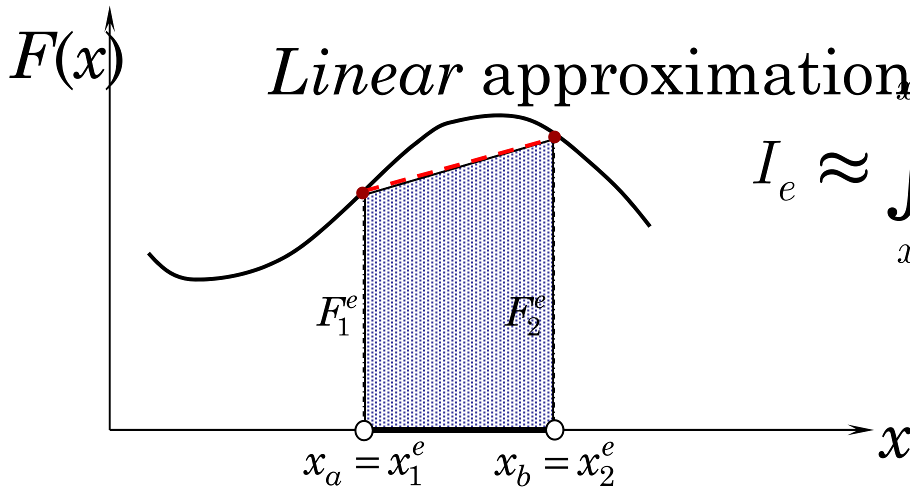
NUMERICAL INTEGRATION IN 1-D

Determine the integral of a function

$$I = \int_a^b F(x)\,dx$$

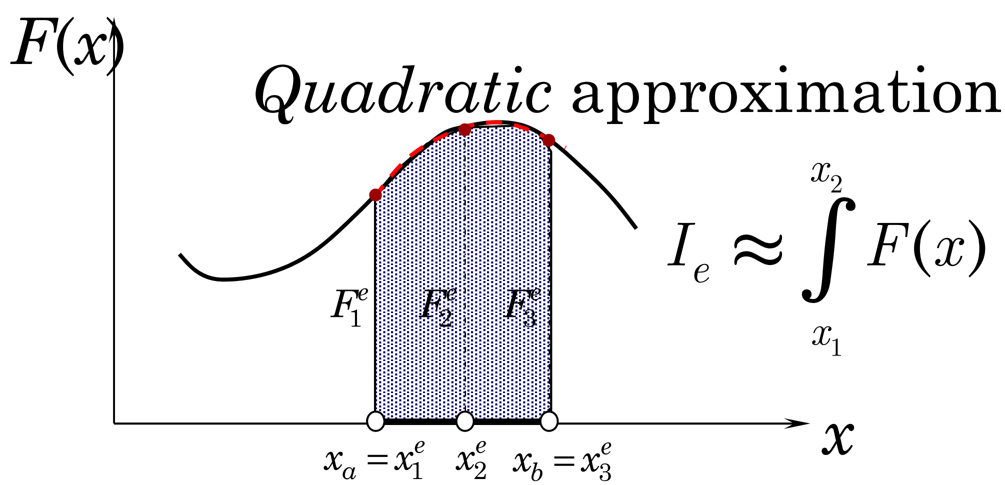


NUMERICAL INTEGRATION (continued)



Trapezoidal Rule

$$I_e \approx \int_{x_1}^{x_2} F(x)\, dx = \frac{h}{2}[F_1^e + F_2^e]$$



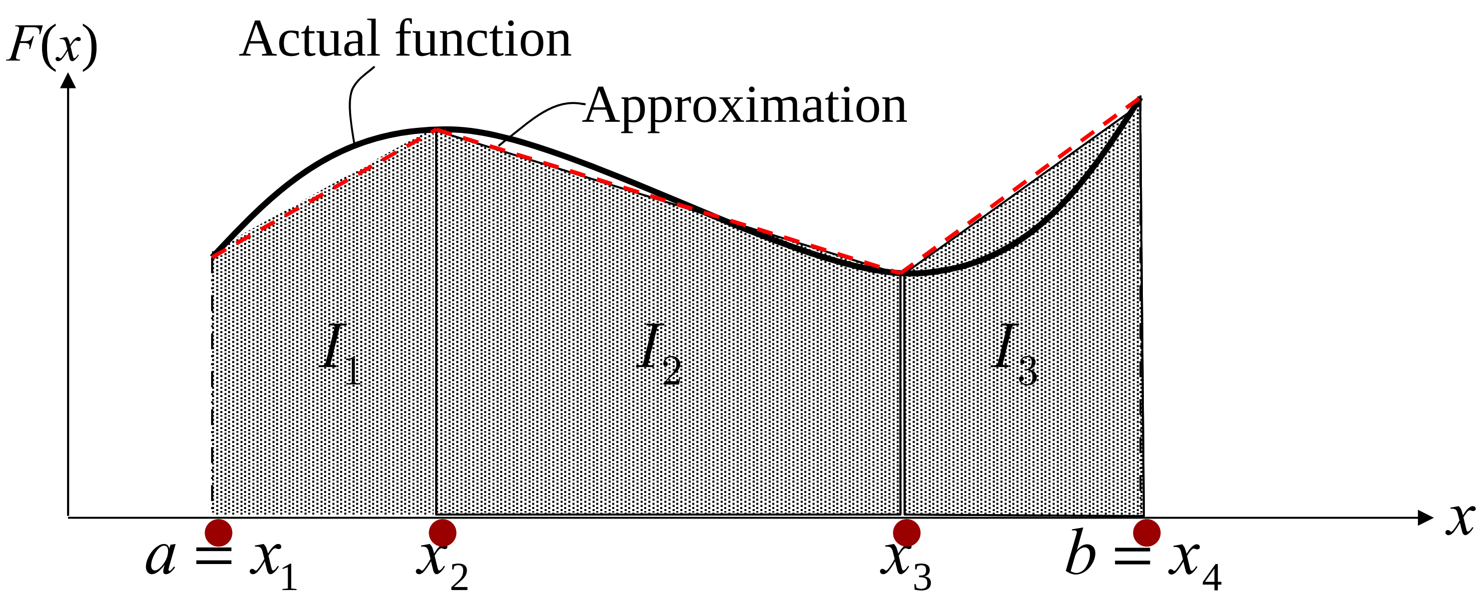
Simpson's Rule

$$I_e \approx \int_{x_1}^{x_2} F(x)\, dx = \frac{h}{6}[F_1^e + 4F_2^e + F_3^e]$$

NUMERICAL INTEGRATION (continued)

$$F(x) \approx \begin{cases} c_0^1 + c_1^1 x, & a \le x \le x_2 \\ c_0^2 + c_1^2 x, & x_2 \le x \le x_3 \\ c_0^3 + c_1^3 x, & x_3 \le x \le b \end{cases}$$

$$I_i = \int_{x_i}^{x_{i+1}} (c_0^i + c_1^i x)\, dx$$



NUMERICAL INTEGRATION (continued)

$$F(x) \approx \begin{cases} F_1^1\psi_1^1(x) + F_2^1\psi_2^1(x) \\ F_1^2\psi_1^2(x) + F_2^2\psi_2^2(x) \\ F_1^3\psi_1^3(x) + F_2^3\psi_2^3(x) \end{cases}$$

$$I_i = \int_{x_i}^{x_{i+1}} [F_1^i\psi_1^i(x) + F_2^i\psi_2^i(x)]\,dx$$

$$= \frac{x_{i+1} - x_i}{2}[F_1^i + F_2^i]$$

$$I \approx I_1 + I_2 + I_3$$

$$= \frac{h_1}{2}[F_1^1 + F_2^1] + \frac{h_2}{2}[F_1^2 + F_2^2] + \frac{h_3}{2}[F_1^3 + F_2^3]$$

$$= F_1\frac{h_1}{2} + F_2\left(\frac{h_1}{2} + \frac{h_2}{2}\right) + F_3\left(\frac{h_2}{2} + \frac{h_3}{2}\right) + F_4\frac{h_3}{2}$$

$$= \sum_{i=1}^{N} F(x_i)W_i, \;\; F_i = F(x_i), \;\; W_i - \text{Weights}$$

NUMERICAL INTEGRATION (continued)

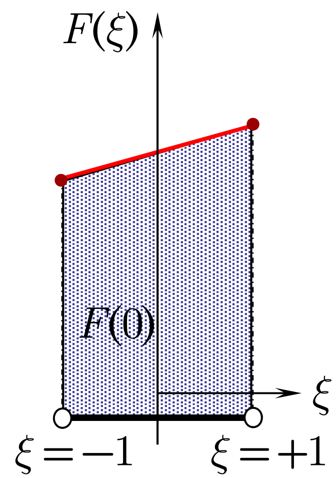
$$G_{ij} = \int_{x_a}^{x_b} F_{ij}(x)\,dx \approx \sum_{I=1}^{NPT} F_{ij}(x_I)W_I \quad \text{– General}$$

$$K_{ij} = \int_{-1}^{+1} F_{ij}(\xi)\,d\xi \approx \sum_{I=1}^{NGP} F_{ij}(\xi_I)W_I \quad \text{– Gauss rule}$$

$$NGP = \left[\frac{p+1}{2}\right],$$ Nearest larger integer equal to $(p+1)/2$

$$p = 1 \Rightarrow \text{NGP} = 1$$
$$p = 2 \text{ or } 3 \Rightarrow \text{NGP} = 2$$
$$p = 4 \text{ or } 5 \Rightarrow \text{NGP} = 3$$

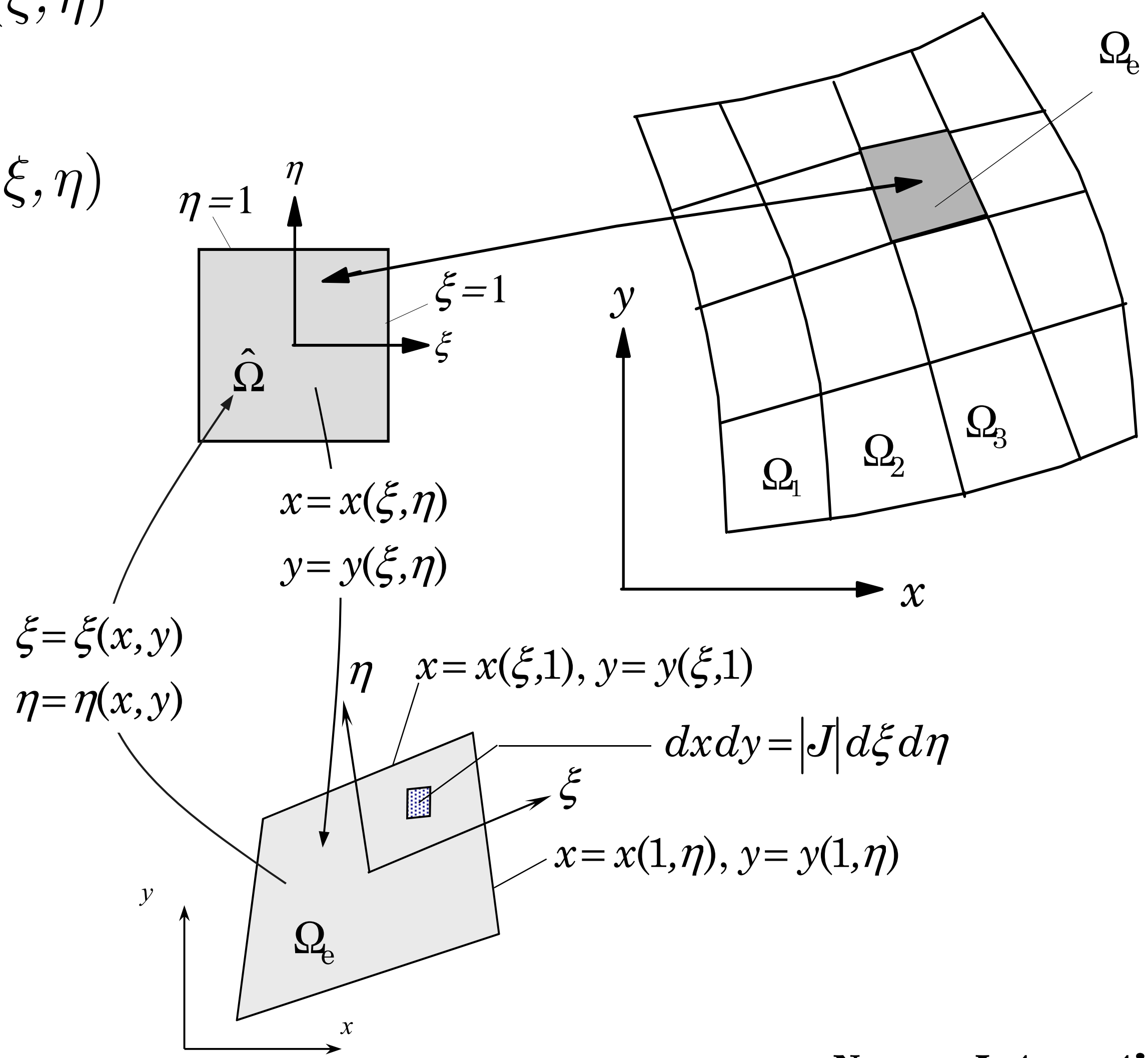
$$I_i = \int_{-1}^{+1} F(\xi)\,d\xi = \frac{h}{2}[F_1 + F_2] = 2\frac{F_1 + F_2}{2}$$
$$= 2 \times F(0) = F(\xi_1)W_1$$



2D NUMERICAL INTEGRATION

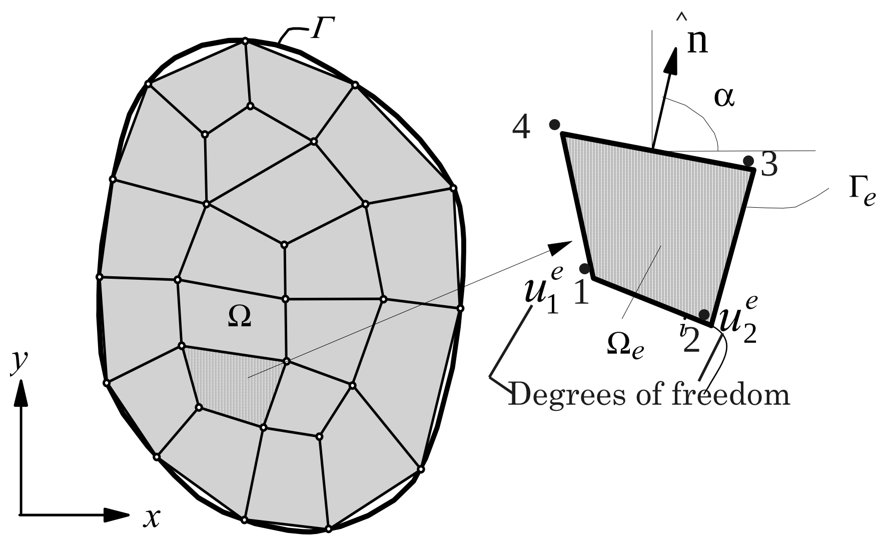
$$x(\xi,\eta)=\sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi,\eta)$$

$$y(\xi,\eta)=\sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi,\eta)$$



NUMERICAL EVALUATION OF INTEGRAL COEFFICIENTS

- Transformation of the integrals posed on arbitrary-shaped element to the master element domain so that evaluation of the integrals is made easy.
- The Gauss integration rule that evaluates an integral expression as a linear sum of the integrand evaluated at certain points (Gauss points) and weights (Gauss weights) is used.



$$u(x,y) \approx u_h^e(x,y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x,y)$$

$$u(x,y) \approx u_h^e(x(\xi,\eta), y(\xi,\eta))$$

$$= \sum_{j=1}^{n} u_j^e \psi_j^e(\xi,\eta)$$

TRANSFORMATION OF THE INTEGRAL

$$K_{ij}^{e} = \int_{\Omega_e} \left[a_{11} \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right] dxdy$$

W_I – Gauss weights

ξ_I, η_J – Gauss locations

$$= \int_{\Omega_e} F_{ij}(x,y)\, dxdy = \int_{\hat{\Omega}} F_{ij}(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi d\eta$$

$$= \int_{\hat{\Omega}} F_{ij}(\xi,\eta)\, J\, d\xi d\eta \approx \sum_{I=1}^{NGP}\sum_{J=1}^{NGP} W_I\ W_J\ \hat{F}_{ij}(\xi_I, \eta_J)$$

$$\frac{\partial \psi_i}{\partial \xi} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_i}{\partial \eta} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial \xi} \\[2ex] \dfrac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\[2ex] \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\[2ex] \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix}$$

NUMERICAL INTEGRATION

Jacobian matrix $[J]=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \xi} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \xi}\\ \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \eta} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \eta}\end{bmatrix}$

$$=\begin{bmatrix}\frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \xi}\\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \eta}\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ \vdots & \vdots\\ x_m & y_m\end{bmatrix},\quad dxdy = J\, d\xi\, d\eta$$

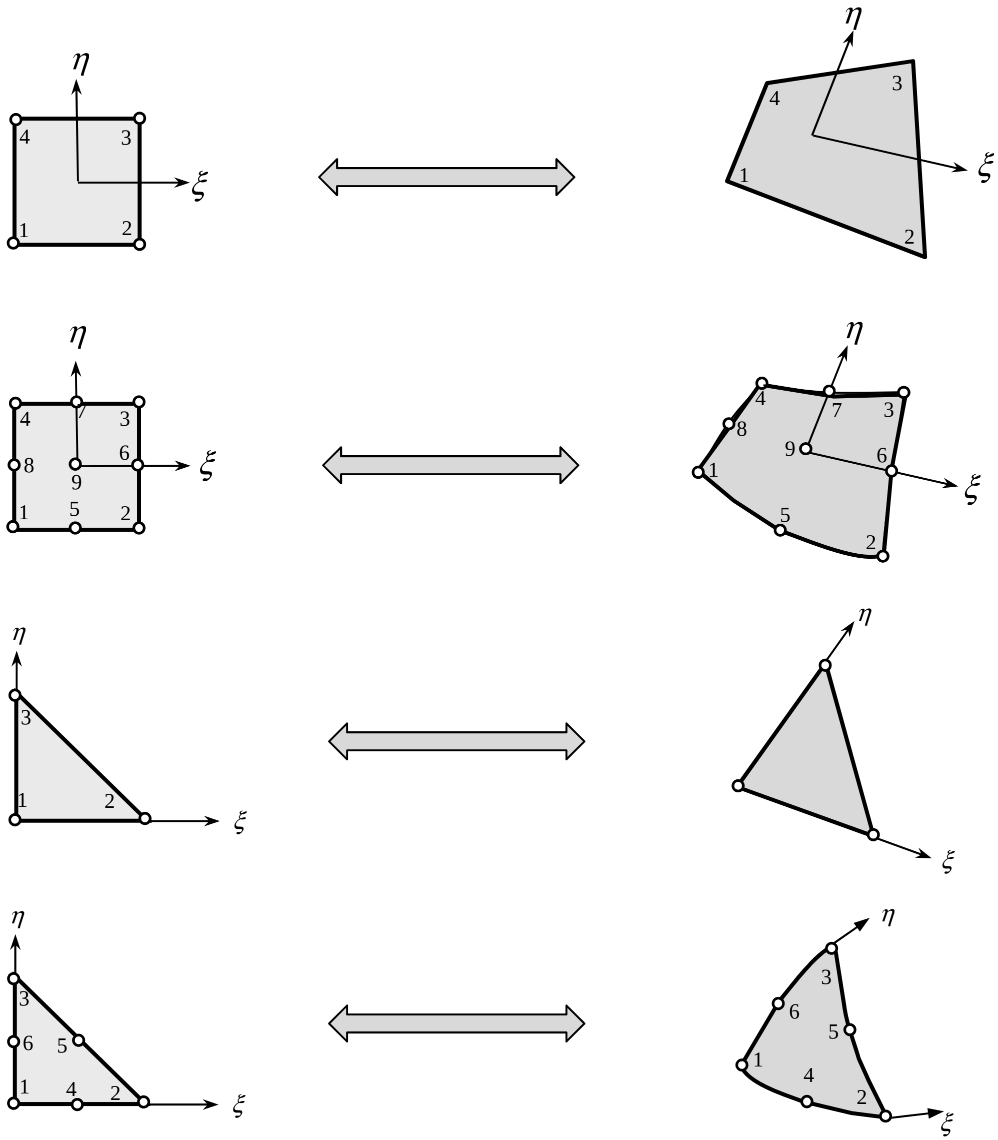
Global derivatives in terms of the local derivatives

$$\begin{Bmatrix}\frac{\partial \psi_i}{\partial x}\\ \frac{\partial \psi_i}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial \psi_i}{\partial \xi}\\ \frac{\partial \psi_i}{\partial \eta}\end{Bmatrix}=[J^*]\begin{Bmatrix}\frac{\partial \psi_i}{\partial \xi}\\ \frac{\partial \psi_i}{\partial \eta}\end{Bmatrix}$$

ELEMENT CALCULATIONS

$$
\begin{aligned}
K_{ij}^{e} &= \int_{\Omega_e} \left[a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dxdy \\
&= \int_{\hat{\Omega}} \left\{ a_{11}(\xi,\eta) \left(J_{11}^{*} \frac{\partial \psi_i}{\partial \xi} + J_{12}^{*} \frac{\partial \psi_i}{\partial \eta} \right) \left(J_{11}^{*} \frac{\partial \psi_j}{\partial \xi} + J_{12}^{*} \frac{\partial \psi_j}{\partial \eta} \right) \right. \\
&\qquad \left. + a_{22}(\xi,\eta) \left(J_{21}^{*} \frac{\partial \psi_i}{\partial \xi} + J_{22}^{*} \frac{\partial \psi_i}{\partial \eta} \right) \left(J_{21}^{*} \frac{\partial \psi_j}{\partial \xi} + J_{22}^{*} \frac{\partial \psi_j}{\partial \eta} \right) \right\} J \, d\xi d\eta \\
&= \int_{\hat{\Omega}} \hat{F}_{ij}^{e}(\xi,\eta)\, d\xi\, d\eta = \int_{-1}^{1} \int_{-1}^{1} \hat{F}_{ij}^{e}(\xi,\eta)\, d\xi\, d\eta \\
&\approx \sum_{I=1}^{NGP_\xi} \sum_{J=1}^{NGP_\eta} \hat{F}_{ij}^{e}(\xi_I,\eta_J) W_I W_J
\end{aligned}
$$

Master element Actual element



GAUSS QUADRATURE

$$\int_{\Omega_e} F_{ij}(x,y)\,dxdy = \int_{\hat{\Omega}_e} \hat{F}_{ij}(\xi,\eta)\,d\xi\,d\eta$$

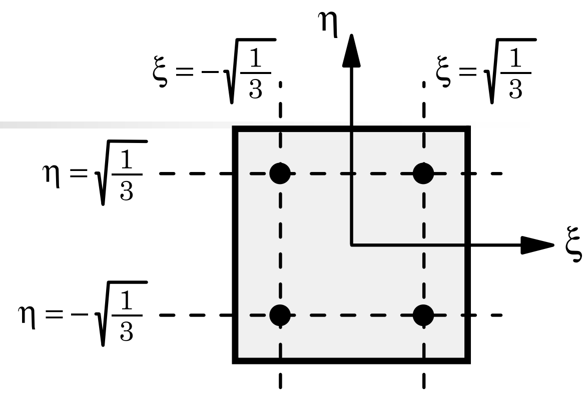
$$\approx \sum_{I,J=1}^{N} W_I W_J \hat{F}_{ij}(\xi_I,\eta_J)$$

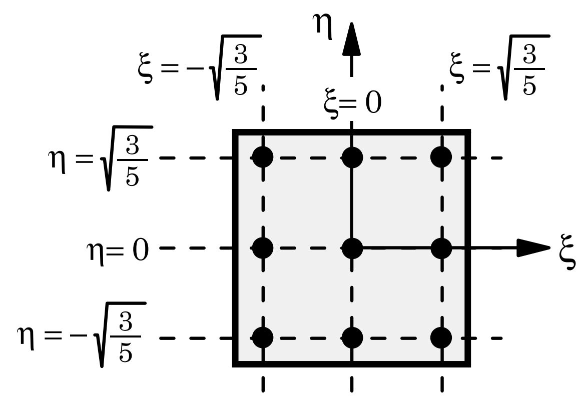
Domain of the *master* element

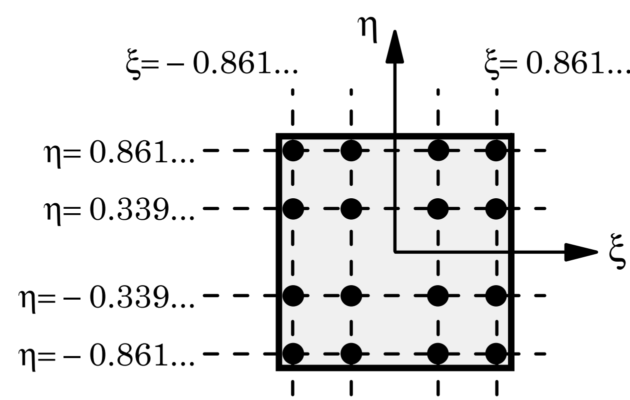
Domain of the *physical* element

Gauss *points*

Gauss *weights*







$$\begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\ \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial \xi} \\ \dfrac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = [J^*] \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial \xi} \\ \dfrac{\partial \psi_i}{\partial \eta} \end{Bmatrix}$$

SUMMARY

- Types of Axisymmetric Problems
- 2-D Axisymmetric Problem in a single variable
- There are two different, in general, meshes in finite element analysis: one to represent the geometry and the other to approximate the solution. Based on the elements (i.e., approximations) used for the geometry and solutions, various formulations are defined.
- The integrals posed on arbitrary-shaped element are expressed as those on a master element domain so that evaluation of the integrals is made easy.
- The Gauss integration rule evaluates an integral expression as the sum of the integrand evaluated at certain points (Gauss points) and multiplied with suitable weights (Gauss weights).