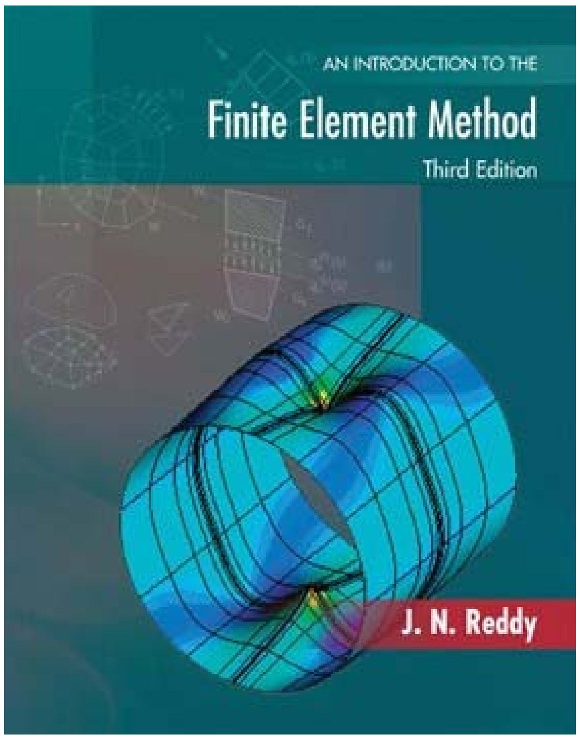
The Finite Element Method

2D Problems involving a single unknown

Read: Chapter 8



CONTENTS

- Model equation Discretization
- Weak form development
- Finite element model
- Approximation functions
- Interpolation functions of higher-order elements
- Post-computation of variables
- Numerical examples
- Transient analysis of 2-D problems

MODEL EQUATION

Model Differential Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)=f \quad \text{in} \quad \Omega$$

a_{ij} – coefficients that describe material behavior

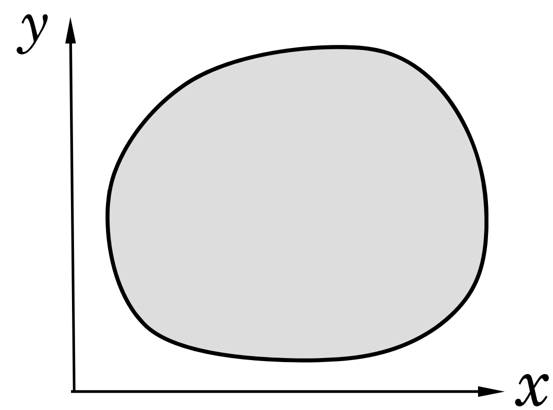
f – source term

Type of boundary conditions

$$u=\hat{u}, \qquad \left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)n_x-\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)n_y-\hat{q}_n=0$$

EXAMPLES OF MODEL EQUATION
(1)– Deflection of a membrane

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0$$

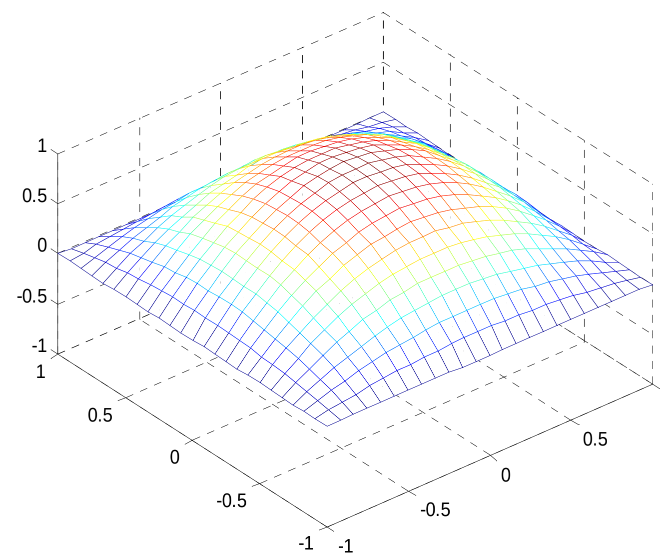


$u(x,y) =$ transverse deflection of a point in the membrane

$f(x,y) =$ applied pressure

$a_{11}(x,y),\ a_{22}(x,y) =$ tensions in the x and y directions, respecti

Transverse deflection of a square membrane fixed on all its sides and subjected to uniform pressure.



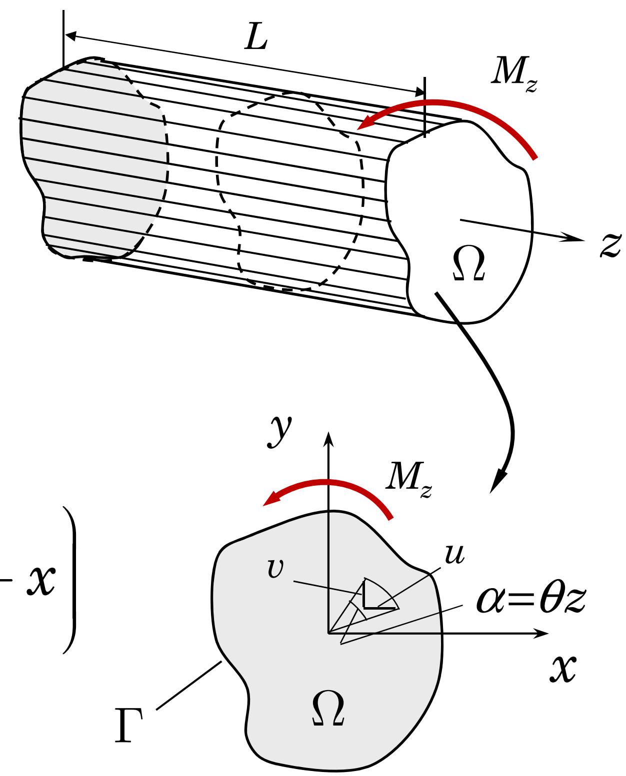
EXAMPLES OF MODEL EQUATION
(2)– Torsion of a cylindrical member

$$-G\theta\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0 \text{ in } \boldsymbol{\Omega}$$

$$u=-\theta zy,\quad v=\theta zx,\quad w=\theta\phi(x,y)$$

$$\left(\frac{\partial\phi}{\partial x}-y\right)n_x+\left(\frac{\partial\phi}{\partial y}-x\right)n_y=0 \text{ on } \Gamma$$

$$\sigma_{xz}(x,y)=G\theta\left(\frac{\partial\phi}{\partial x}-y\right),\quad \sigma_{yz}(x,y)=G\theta\left(\frac{\partial\phi}{\partial y}+x\right)$$



$\phi(x,y)=$ warping function

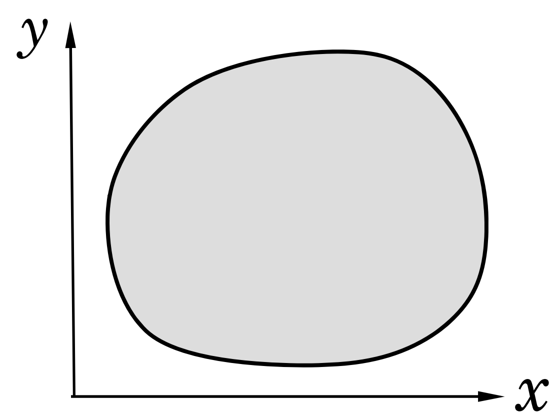
$a_{11}=a_{22}=G\theta$

$G=$ shear modulus

$\theta=$ angle of twist

EXAMPLES OF MODEL EQUATION
(3)– 2D Heat transfer

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)-f=0$$



$$q_x(x,y)=-a_{11}\frac{\partial u}{\partial x},\quad q_y(x,y)=-a_{22}\frac{\partial u}{\partial y}$$

$$-\left(a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y\right)=\hat{q}_n \quad \text{on} \quad \Gamma$$

$u=T$, temperature; $a_{11}=k_x$, $a_{22}=k_y$

k_x, k_y = thermal conductivities in the x and y directions, respectively

f = internal heat generation

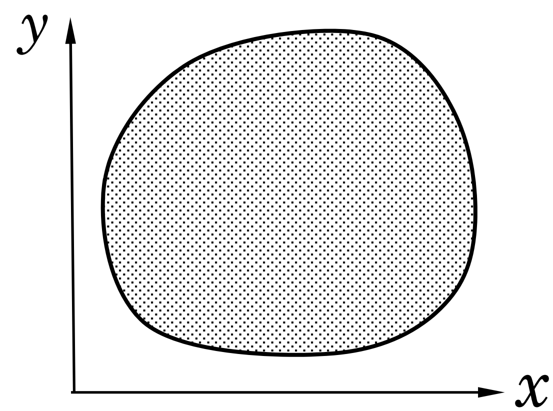
q_n = heat flux normal to the boundary

EXAMPLES OF MODEL EQUATION
(4) – 2D Inviscid flow

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial \phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial \phi}{\partial y}\right)-f=0$$

$$v_x(x,y)=-a_{11}\frac{\partial \phi}{\partial x},\quad v_y(x,y)=-a_{22}\frac{\partial \phi}{\partial y}$$

$$-\left(a_{11}\frac{\partial \phi}{\partial x}n_x+a_{22}\frac{\partial \phi}{\partial y}n_y\right)=\hat{q}_n \text{ on } \Gamma$$



$u=\phi$, water head (velocity potential)

$a_{11},\ a_{22}$ = permeabilities in the x and y directions, respectively

f = internal infiltration

q_n = flow normal to the boundary

WEAK FORM DEVELOPMENT

Approximation $u(x,y) \approx u_h(x,y)$

Weak Form Development

Step 1

$$0 = \int_{\Omega^e} w_i \left[-\underbrace{\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x} \right)}_{F_1} - \underbrace{\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y} \right)}_{F_2} - f \right] dxdy$$

Step 2: Trade differentiations between w and u_h

Consider the identity:

$$\frac{\partial}{\partial x}\left(w_i \cdot F_1\right) = \frac{\partial w_i}{\partial x}F_1 + w_i\frac{\partial F_1}{\partial x} \quad \text{or} \quad -w_i\frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x}\underbrace{\left(w_i \cdot F_1\right)}_{F} + \frac{\partial w_i}{\partial x}F_1$$

Green-Gauss Theorem

$$\int_{\Omega^e} \frac{\partial F}{\partial x} dxdy = \oint_{\Gamma^e} n_x F\, ds, \qquad \int_{\Omega^e} \frac{\partial F}{\partial y} dxdy = \oint_{\Gamma^e} n_y F\, ds$$

WEAK FORM DEVELOPMENT continued

Step 2:

Consider the identity:

$$-w_i \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x}\underbrace{\left(w_i \cdot F_1\right)}_{F} + \frac{\partial w_i}{\partial x} F_1$$

$$\int_{\Omega^e} \frac{\partial F}{\partial x}\, dxdy = \oint_{\Gamma^e} n_x F\, ds \Rightarrow \int_{\Omega^e} \frac{\partial F}{\partial x}\, dxdy = \oint_{\Gamma^e} n_x \left(w_i \cdot F_1\right) ds$$

$$\int_{\Omega^e} \frac{\partial F}{\partial x}\, dxdy = \oint_{\Gamma^e} n_x w_i \left(a_{11} \frac{\partial u_h}{\partial x}\right) ds$$

$$\int_{\Omega^e} \frac{\partial G}{\partial y}\, dxdy = \oint_{\Gamma^e} n_y w_i \left(a_{22} \frac{\partial u_h}{\partial y}\right) ds$$

WEAK FORM DEVELOPMENT (continued)

Step 1

$$0 = \int_{\Omega^e} w_i \left[-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - f \right] dxdy$$

$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - w_i f \right] dxdy$$

$$- \oint_{\Gamma^e} w_i \left[\left(a_{11}\frac{\partial u_h}{\partial x}\right) n_x + \left(a_{22}\frac{\partial u_h}{\partial y}\right) n_y \right] ds$$

Step 2

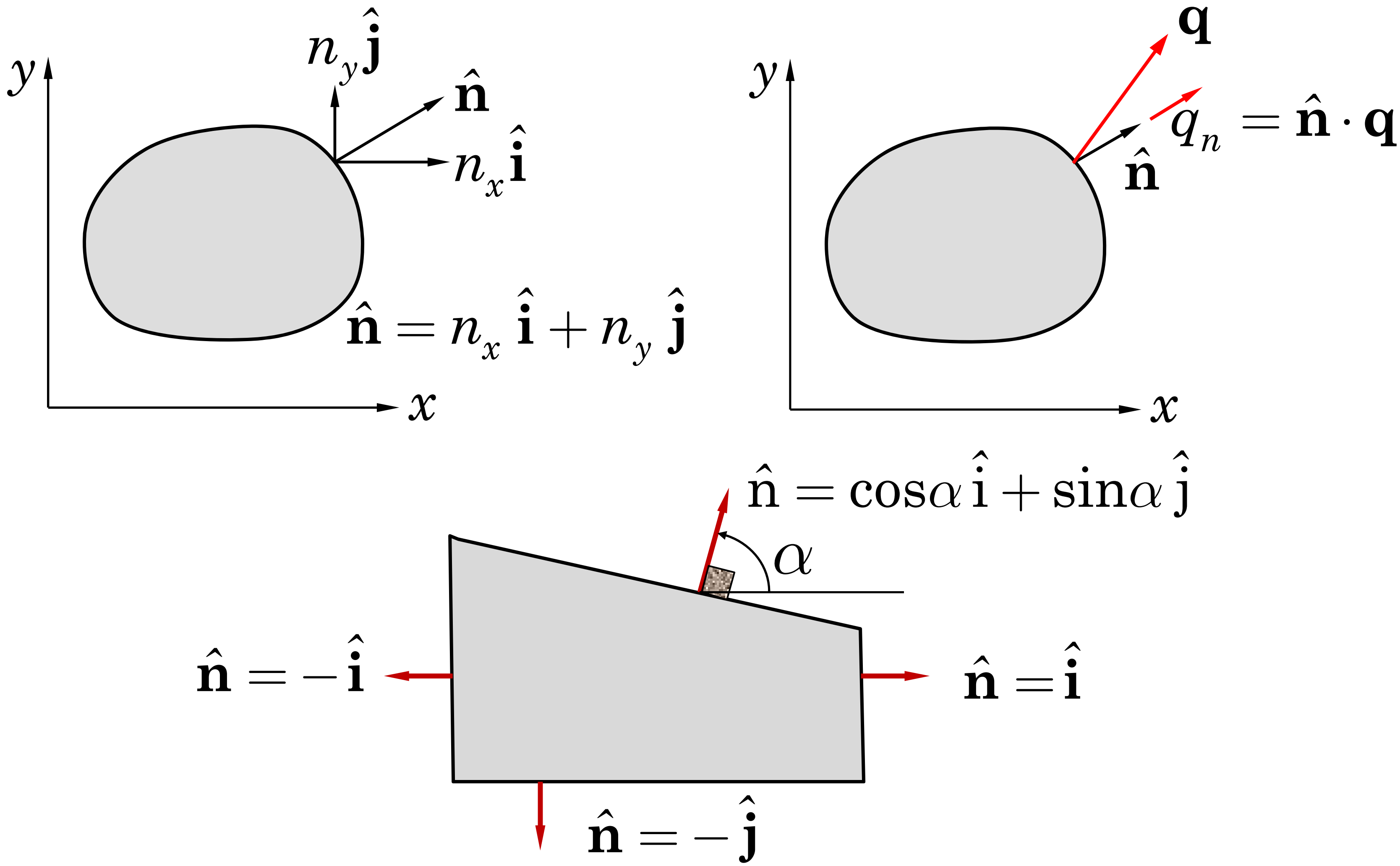
$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - w_i f \right] dxdy$$

$$- \oint_{\Gamma^e} w_i \, q_n \, ds$$

Step 3

$$q_n = \left(a_{11}\frac{\partial u_h}{\partial x}\right) n_x + \left(a_{22}\frac{\partial u_h}{\partial y}\right) n_y \,, \quad \text{flux normal to the boundary}$$

FINITE ELEMENT APPROXIMATION



FINITE ELEMENT APPROXIMATION

Weak form

$$0 = \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial w_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - w_i f \right] dxdy$$
$$- \oint_{\Gamma^e} w_i\, q_n\, ds$$

Approximation

$$u(x,y) \approx u_h(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + \cdots \text{ (}n\text{ terms)}$$
$$= \sum_{j=1}^{n} u_j \psi_j(x,y)$$

Finite element model [the ith equation is obtained by replacing the weight function w by ψ_i $(i = 1, 2, \cdots, n)$]

$$0 = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}\right) + \frac{\partial \psi_i}{\partial y}\left(a_{22}\frac{\partial u_h}{\partial y}\right) - \psi_i f \right] dxdy$$
$$- \oint_{\Gamma^e} \psi_i\, q_n\, ds$$

FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$0 = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x}\left(a_{11} \frac{\partial u_h}{\partial x} \right) + \frac{\partial \psi_i}{\partial y}\left(a_{22} \frac{\partial u_h}{\partial y} \right) - \psi_i f \right] dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

$$0 = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \sum_{j=1}^{n} u_j \left(a_{11} \frac{\partial \psi_j}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \sum_{j=1}^{n} u_j \left(a_{22} \frac{\partial \psi_j}{\partial y} \right) \right] dxdy - \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

$$= \sum_{j=1}^{n} u_j \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy - \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$0 = \sum_{j=1}^{n} u_j \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$- \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

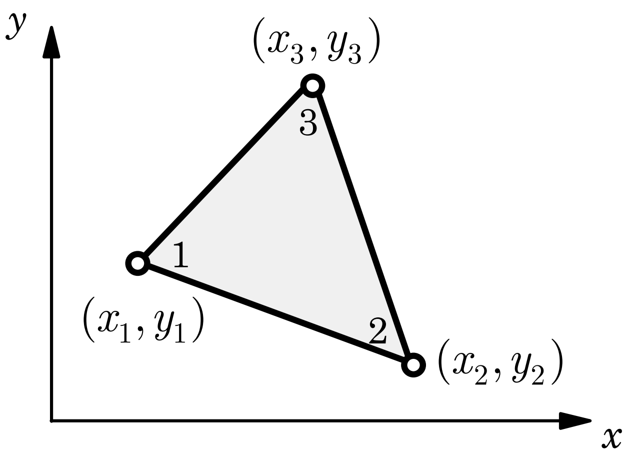
$$= \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e = \sum_{j=1}^{n} K_{ij}^e u_j^e - F_i^e \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e$$

$$K_{ij}^e = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy,$$

$$F_i^e = \int_{\Omega^e} \psi_i f \, dxdy + \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

APPROXIMATION FUNCTIONS

Linear Triangular Element



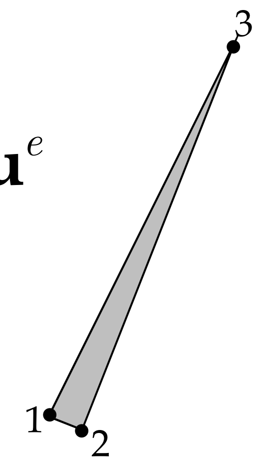
$$u_h^e(x,y) = c_1 + c_2 x + c_3 y$$

$$u_h^e(x_1,y_1) = c_1 + c_2 x_1 + c_3 y_1 \equiv u_1^e$$

$$u_h^e(x_2,y_2) = c_1 + c_2 x_2 + c_3 y_2 \equiv u_2^e$$

$$u_h^e(x_3,y_3) = c_1 + c_2 x_3 + c_3 y_3 \equiv u_3^e$$

$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \end{Bmatrix} \Rightarrow \mathbf{u}^e = \mathbf{A}^e \mathbf{c}^e \text{ or } \mathbf{c}^e = \mathbf{A}^{-1} \mathbf{u}^e$$



APPROXIMATION FUNCTIONS
Linear Triangular Element (continued)

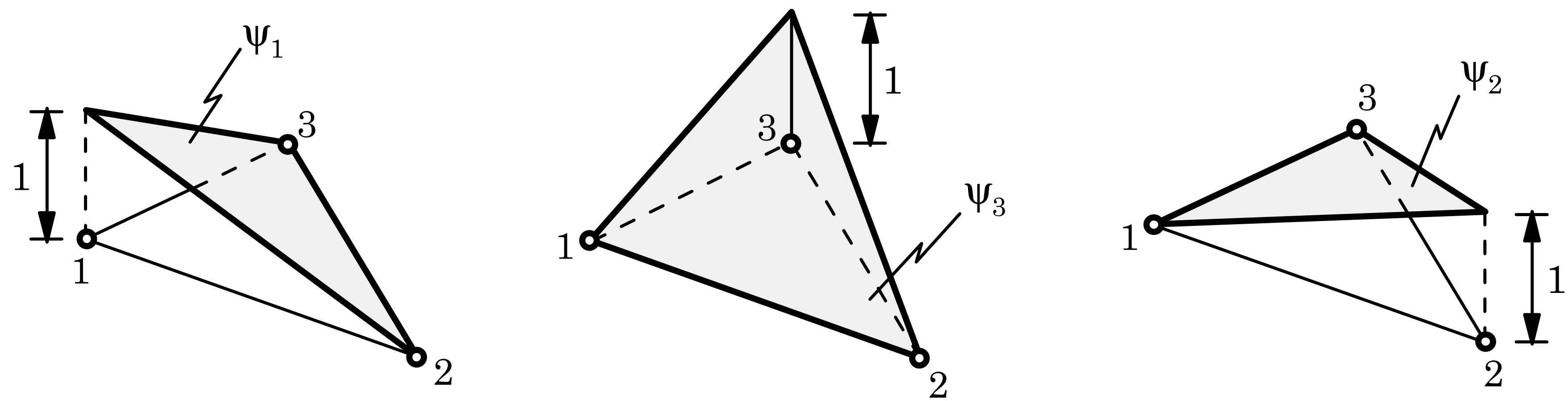
$$u_h^e(x,y) = c_1 + c_2 x + c_3 y = \begin{Bmatrix}1 & x & y\end{Bmatrix}\begin{Bmatrix}c^e\end{Bmatrix} = \begin{Bmatrix}1 & x & y\end{Bmatrix}\left[A\right]^{-1}\begin{Bmatrix}u^e\end{Bmatrix}$$

$$= \psi_1^e(x,y)u_1^e + \psi_2^e(x,y)u_2^e + \psi_3^e(x,y)u_3^e = \sum_{j=1}^{3} u_j^e \psi_j^e(x,y)$$

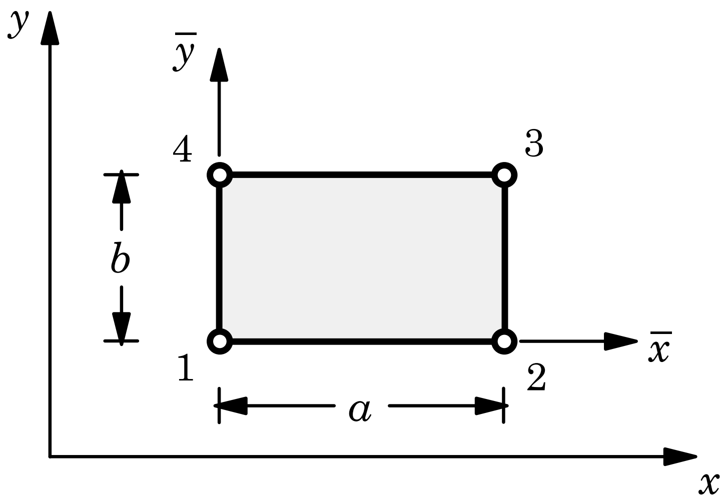
$$\psi_i^e(x,y) = \frac{1}{2\Delta^e}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

$$\alpha_i = x_j y_k - y_j x_k,\ \beta_i = y_j - y_k,\ \gamma_i = -(x_j - x_k)$$

$$\Delta^e = \text{Area of the triangle}, \quad 2\Delta^e = \text{Determinant of } [A]$$



APPROXIMATION FUNCTIONS
Linear Rectangular Element



$$u_h^e(\bar{x},\bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\,\bar{y}$$

$$u_h^e(\bar{x}_1,\bar{y}_1) = c_1 = u_1^e$$

$$u_h^e(\bar{x}_2,\bar{y}_2) = c_1 + c_2 a \equiv u_2^e$$

$$u_h^e(\bar{x}_3,\bar{y}_3) = c_1 + c_2 a + c_3 b + c_4 ab \equiv u_3^e$$

$$u_h^e(\bar{x}_4,\bar{y}_4) = c_1 + c_3 b \equiv u_4^e$$

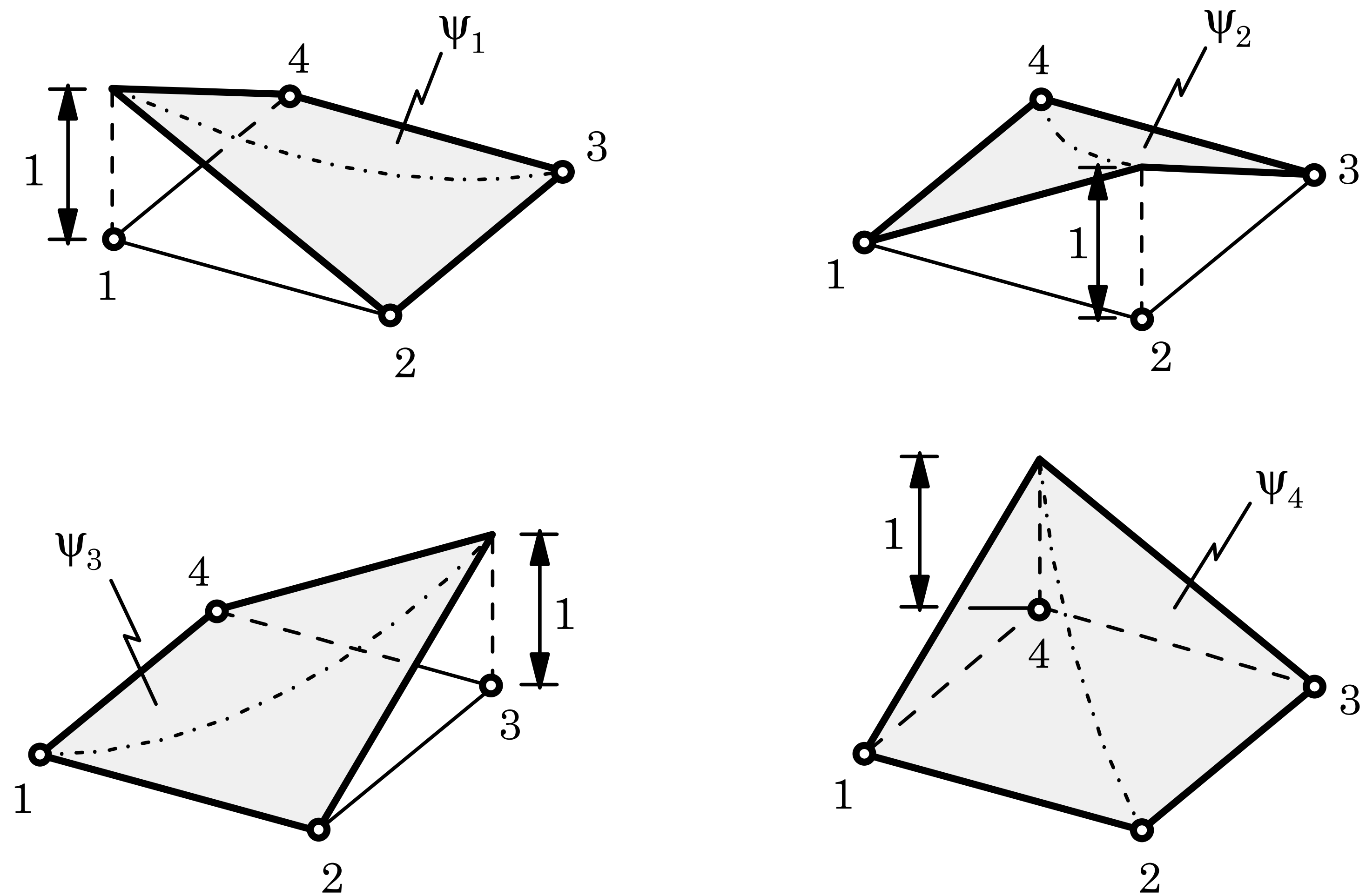
$$\psi_1^e(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right), \quad \psi_2^e(\bar{x},\bar{y}) = \frac{\bar{x}}{a}\left(1-\frac{\bar{y}}{b}\right),$$

$$\psi_3^e(\bar{x},\bar{y}) = \frac{\bar{x}}{a}\frac{\bar{y}}{b}, \quad \psi_4^e(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b},$$

$$\psi_j^e(x_i,y_i) = \delta_{ij}$$

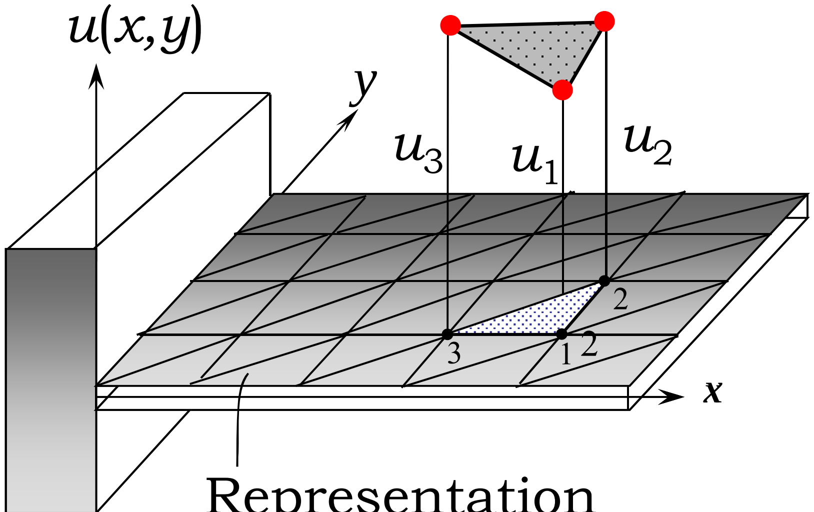
APPROXIMATION FUNCTIONS

Linear Rectangular Element (continued)

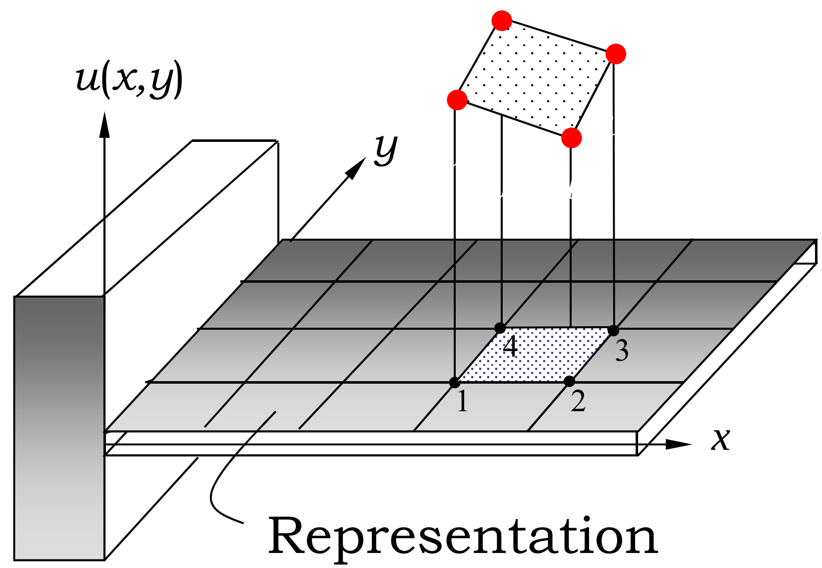


Interpolation Property of the Approximation Functions

$$\psi_j^e(x_i, y_i) = \begin{Bmatrix} 1, & i = j, & \text{a fixed value} \\ 0, & i \neq j, & \text{a fixed value} \end{Bmatrix} = \delta_{ij}; \quad \sum_{j=1}^{3} \psi_j^e(x, y) = 1$$



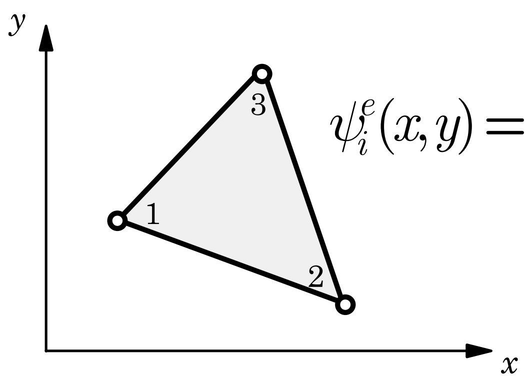
Representation of the *domain* by 3-node triangles



Representation of the *domain* by 4-node rectangles

NUMERICAL EVALUATION OF COEFFICIENT MATRICES

Linear Triangular Element



$$\psi_i^e(x,y) = \frac{1}{2\Delta^e}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

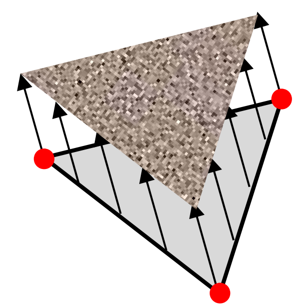
$\Delta^e =$ Area of the triangle

$$K_{ij}^e = \int_{\Delta^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$a_{00} = 0$$

$$= a_{11}^e \int_{\Delta^e} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} dxdy + a_{22}^e \int_{\Delta^e} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} dxdy$$

$$= \frac{1}{4\Delta^e}\left(a_{11}^e \beta_i \beta_j + a_{22}^e \gamma_i \gamma_j \right)$$



$$f_i^e = f_e \int_{\Omega^e} \psi_i \, dxdy = \frac{f_e \Delta^e}{3}$$

NUMERICAL EVALUATION OF COEFFICIENT MATRICES

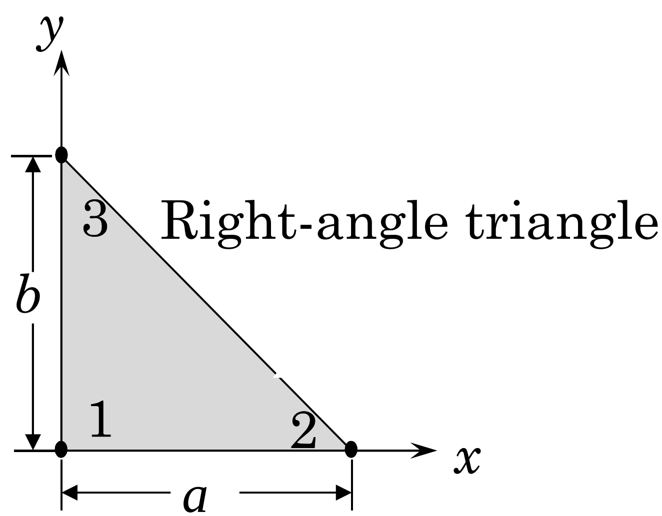
Linear Right-Angled Triangular Element

$$[K^e] = a_{11}^e \frac{b}{2a}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22}^e \frac{a}{2b}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

When $a_{11}^e = a_{22}^e = k_e$, we have

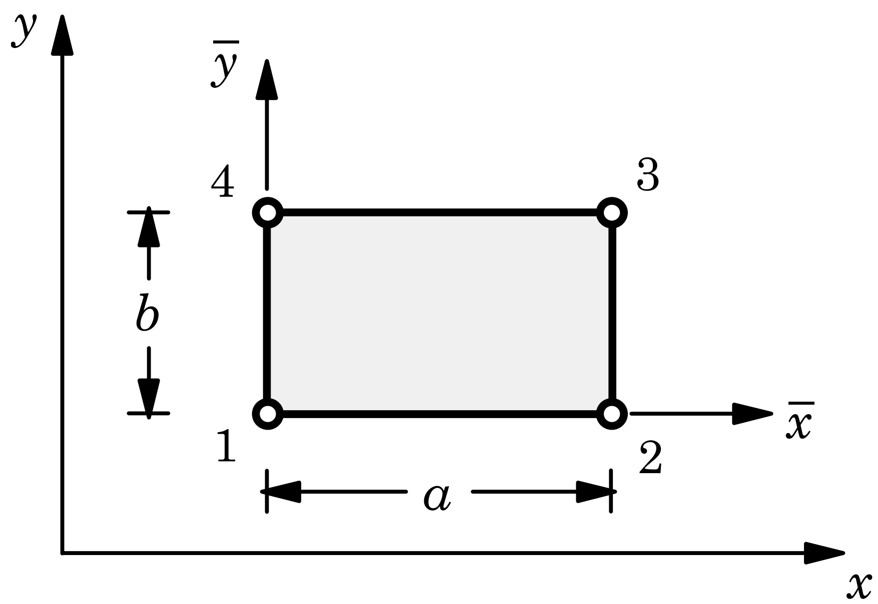
$$[K^e] = \frac{k_e}{2ab}\begin{bmatrix} a^2 + b^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix}$$

$$\{f^e\} = \frac{f_e \Delta^e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



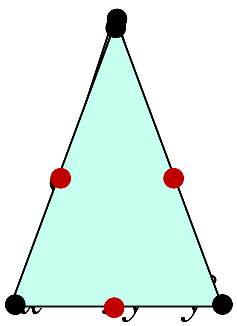
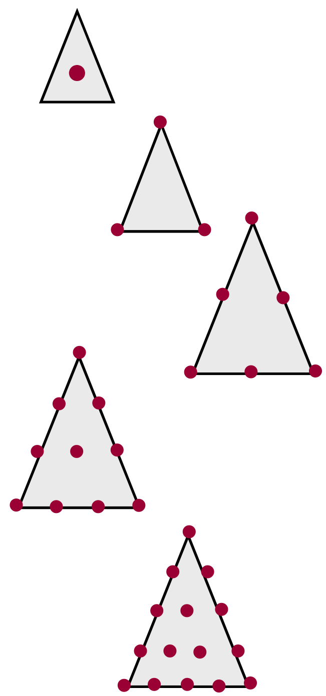
NUMERICAL EVALUATION OF COEFFICIENT MATRICES

Linear Rectangular Element

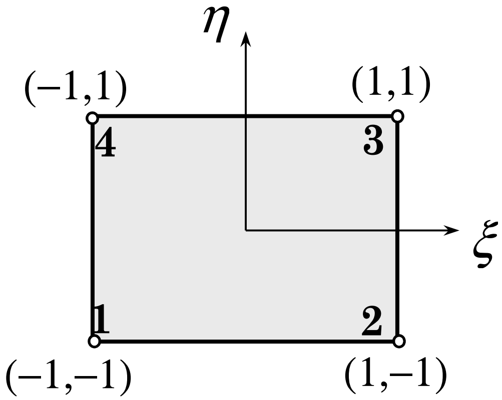


$$[K^e] = a_{11}^e \frac{b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + a_{22}^e \frac{a}{6b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}, \quad \{f^e\} = \frac{f_e ab}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Higher-Order Approximations Functions: Pascal's Triangle

Pascal's triangle	Deg. of polynom.	No. of terms	Element
	0	1	
	1	3	
	2	6	
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10	
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15	
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	5	21	(Figure not shown)

LINEAR AND QUADRATIC *MASTER* RECTANGULAR ELEMENTS

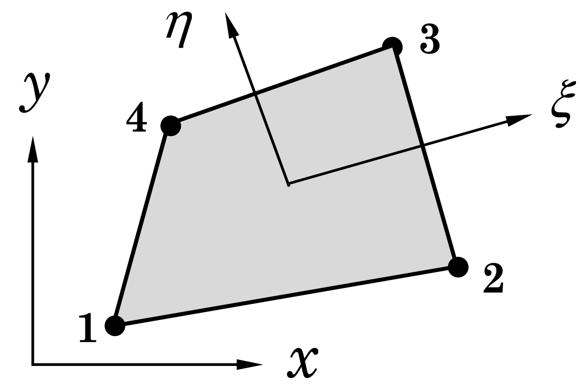


$$\psi_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$

$$\psi_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$

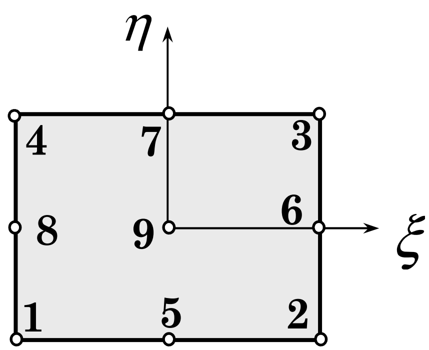
$$\psi_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$

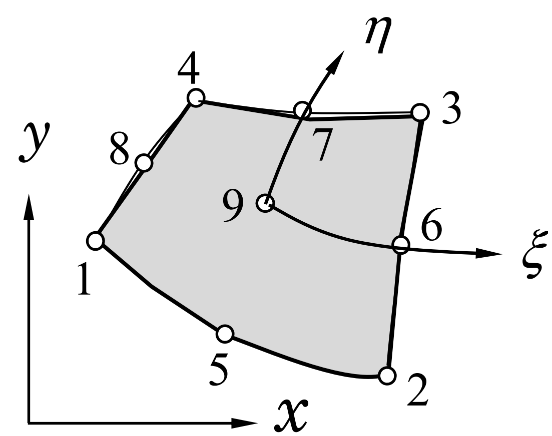
$$\psi_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$



Master elements

Physical elements





$$\psi_1 = \tfrac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$$

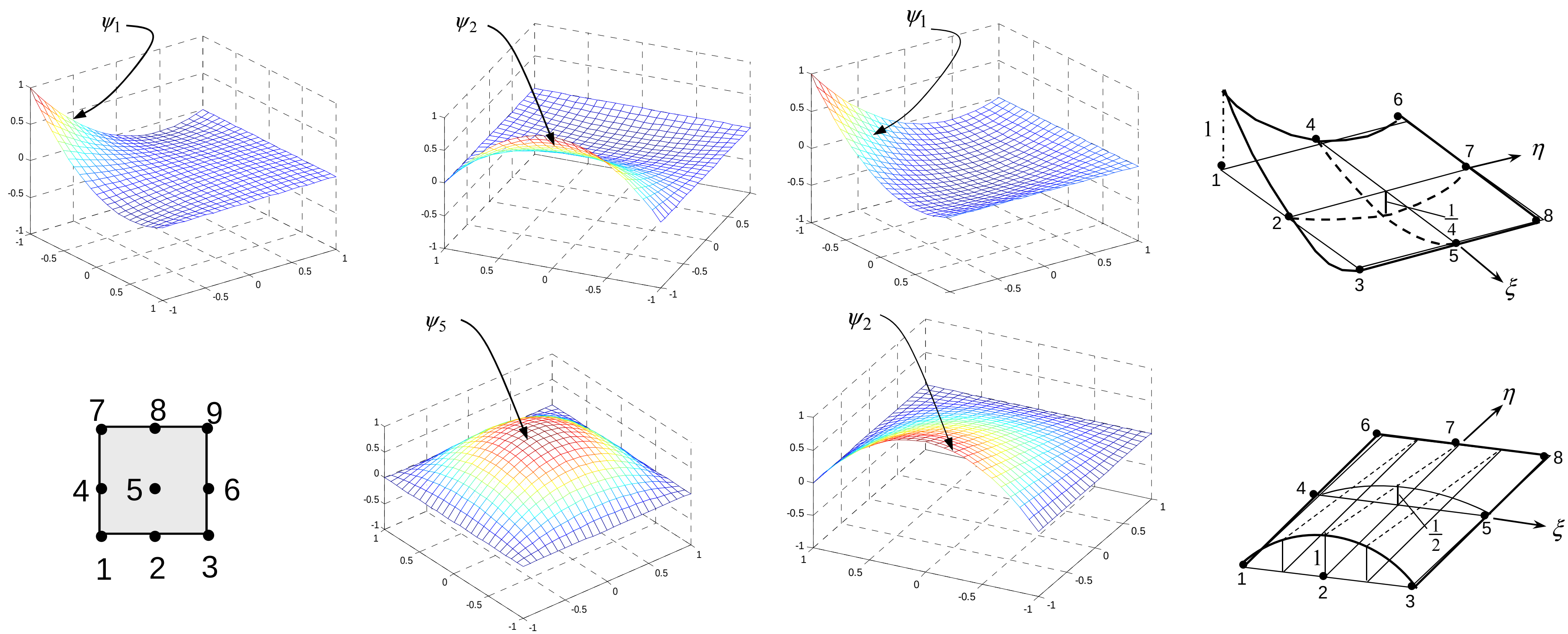
$$\psi_2 = \tfrac{1}{4}(\xi^2+\xi)(\eta^2-\eta)$$

$$\psi_3 = \tfrac{1}{4}(\xi^2+\xi)(\eta^2+\eta)$$

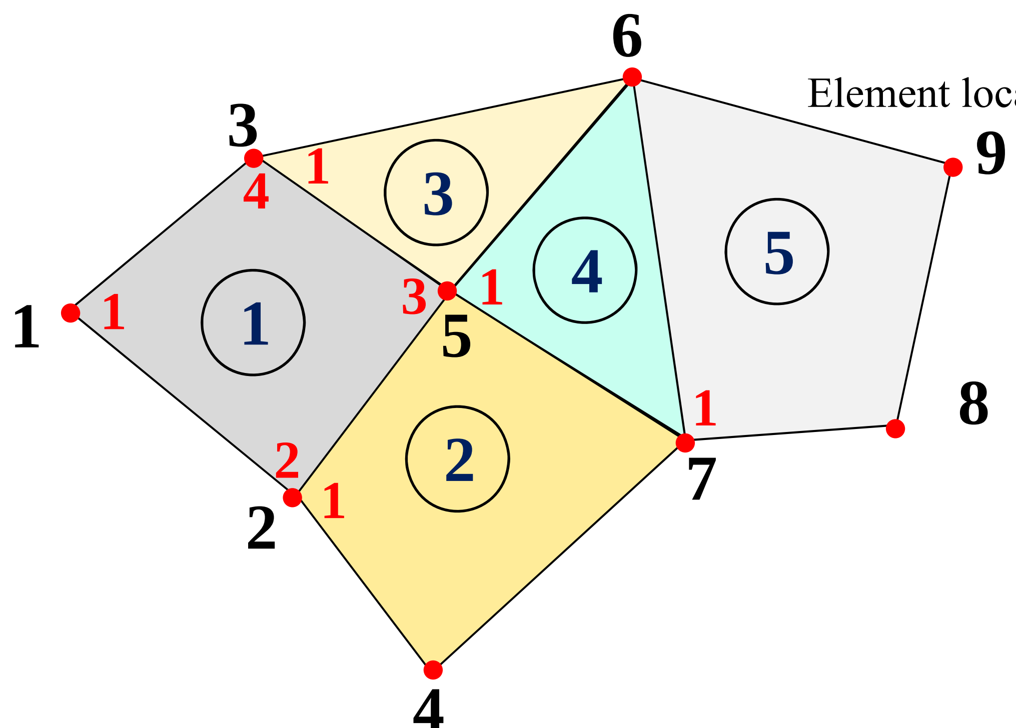
$$\psi_4 = \tfrac{1}{4}(\xi^2-\xi)(\eta^2+\eta),\quad \psi_5 = \tfrac{1}{2}(1-\xi^2)(\eta^2-\eta),\quad \psi_6 = \tfrac{1}{2}(\xi^2+\xi)(1-\eta^2)$$

$$\psi_7 = \tfrac{1}{2}(1-\xi^2)(\eta^2+\eta),\quad \psi_8 = \tfrac{1}{2}(\xi^2-\xi)(1-\eta^2),\quad \psi_9 = (1-\xi^2)(1-\eta^2)$$

PLOTS OF SELECTIVE QUADRATIC INTERPOLATION FUNCTIONS OF RECTANGULAR ELEMENTS



ASSEMBLY OF ELEMENTS/EQUATIONS



Connectivity matrix

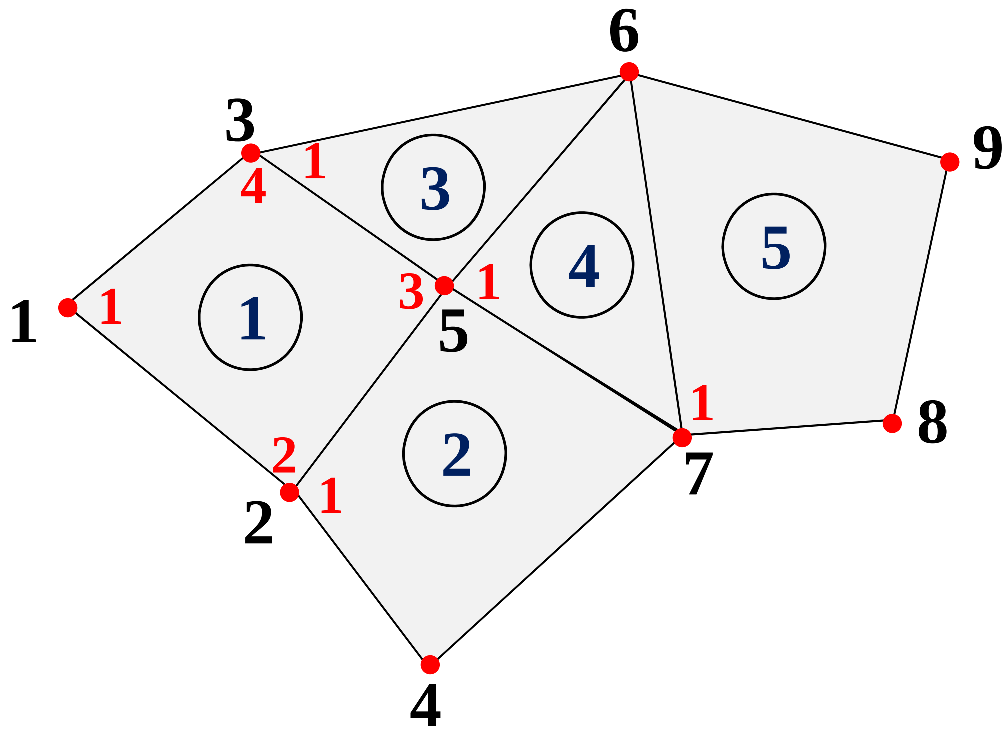
Element local node numbers$\rightarrow$ $\boxed{1 \quad 2 \quad 3 \quad 4}$

$$[B] = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 2 & 4 & 7 & 5 \\ 3 & 5 & 6 & \times \\ 5 & 7 & 6 & \times \\ 7 & 8 & 9 & 6 \end{bmatrix}$$

$K_{IJ} =$ a relationship between certain property of global node I and global node J.
$= 0$, if node I and J do not belong to the same element

$$K_{11} = K_{11}^{(1)},\ K_{13} = K_{14}^{(1)},\ K_{15} = K_{13}^{(1)},\ K_{14} = 0,$$
$$K_{22} = K_{22}^{(1)} + K_{11}^{(2)},\ K_{25} = K_{23}^{(1)} + K_{14}^{(2)},\ K_{26} = 0,$$
$$K_{55} = K_{33}^{(1)} + K_{44}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)},$$
$$K_{56} = K_{23}^{(3)} + K_{13}^{(4)},\ F_1 = F_1^{(1)},\ F_2 = F_2^{(1)} + F_1^{(2)},$$
$$F_5 = F_3^{(1)} + F_4^{(2)} + F_2^{(3)} + F_1^{(4)}$$

POST-COMPUTATION OF VARIABLES



$$u_h^e(x,y) = \sum_{j=1}^{n} u_j^{(e)} \psi_j^{(e)}(x,y), \quad (x,y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x}, \quad \frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y}, \quad (x,y) \in \Omega^e$$

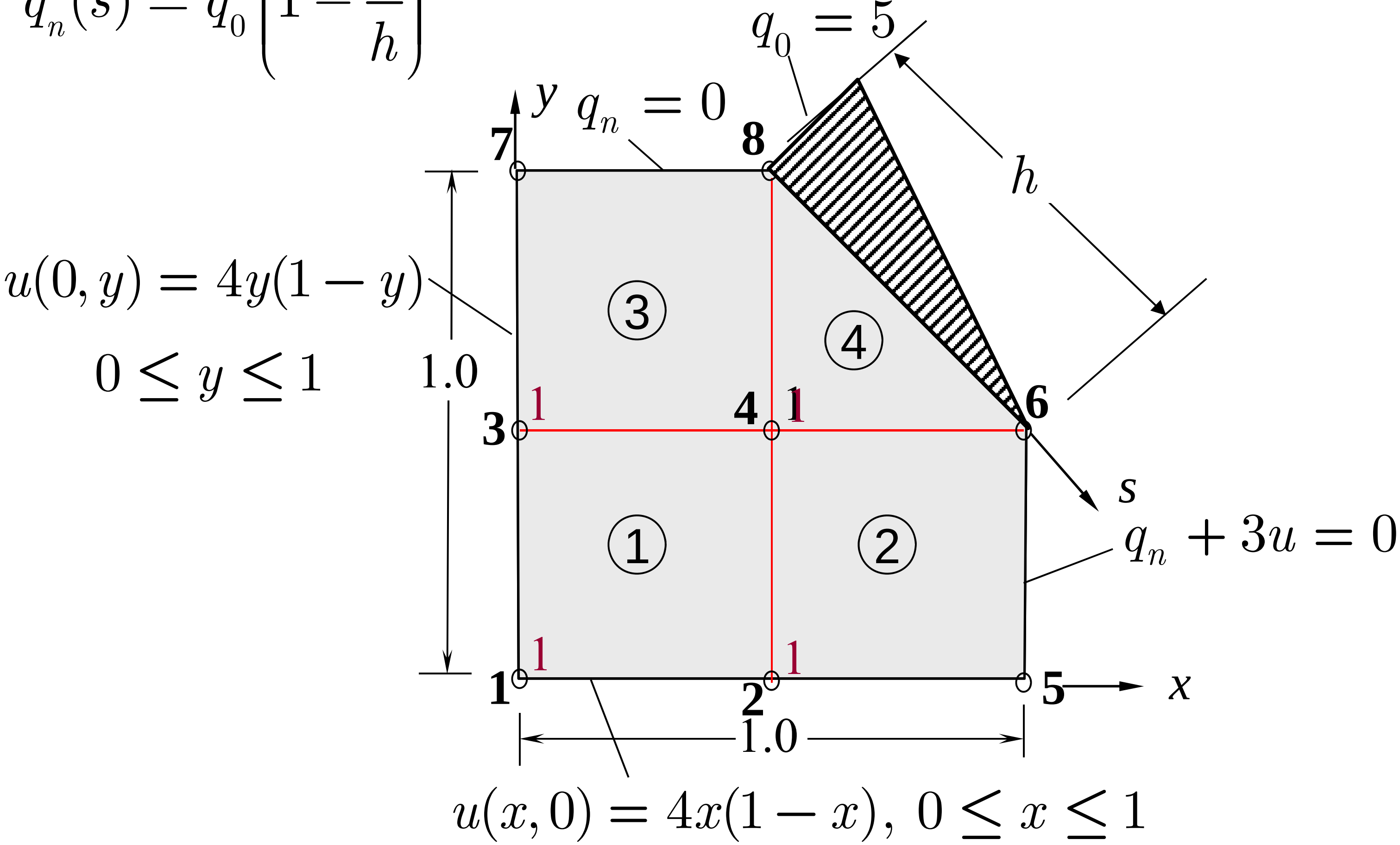
DISCUSSION (distributed boundary source and convection type boundary condition)

$$Q_i = \int_0^h q_n(s)\,\psi_i(s)\,ds$$

$$q_n(s) = q_0\left(1 - \frac{s}{h}\right)$$

For linear elements

$$Q_8 = \frac{2}{3}\left(\frac{q_n h}{2}\right),\quad Q_6 = \frac{1}{3}\left(\frac{q_n h}{2}\right)$$

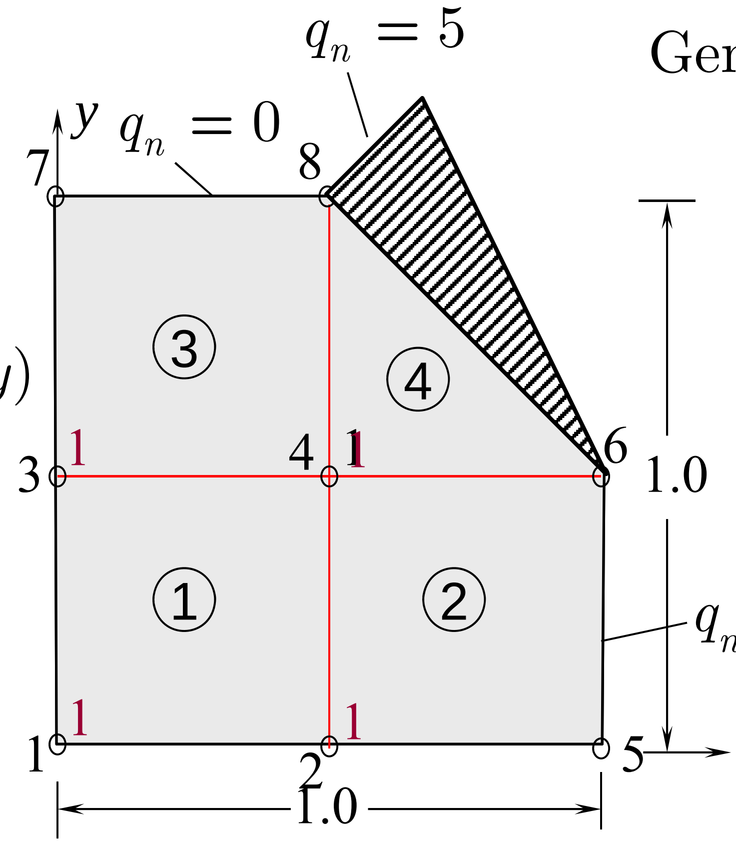


DISCUSSION (computation of convection type BC)

$$Q_i = \int_0^h q_n(s)\,\psi_i(s)\,ds = \int_0^h -\beta(u-u_0)\,\psi_i(s)\,ds$$
$$= \int_0^h -\beta\Big(\sum_{j=1}^n u_j\psi_j - u_0\Big)\,\psi_i(s)\,ds$$
$$= -\beta\sum_{j=1}^n u_j\int_0^h \psi_i\psi_j\,ds + \beta\int_0^h u_0\psi_i\,ds$$

General form of the BC:

$$q_n + \beta(u-u_0) = 0$$

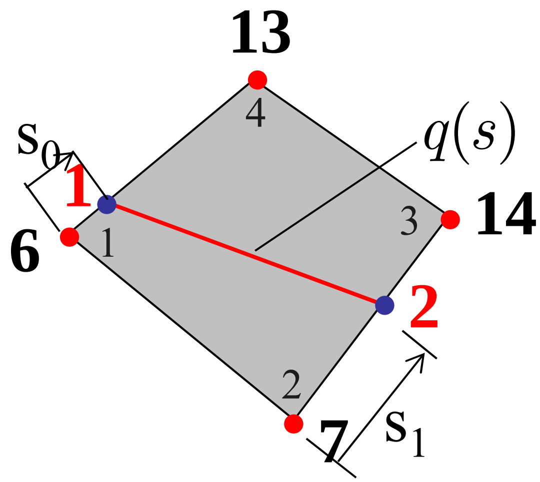


$$q_n + 3u = 0\ (\beta = 3,\ u_0 = 0)$$

$$u(0,y) = 4y(1-y)$$
$$0 \le y \le 1$$

$$u(x,0) = 4x(1-x),\ 0 \le x \le 1$$

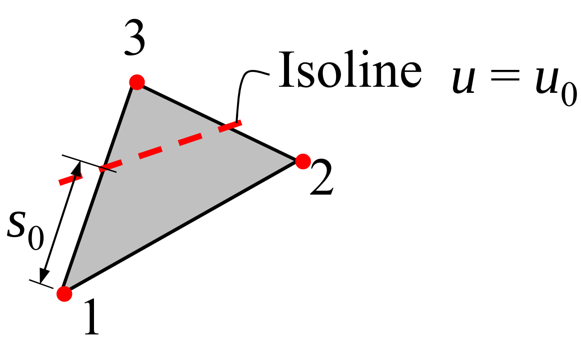
DISCUSSION
(line source contribution to nodes)

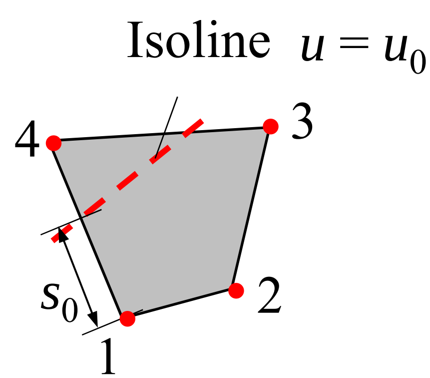


$$q_i = \int_0^h q(s)\,\psi_i(s)\,ds$$

$$Q_6 = q_1\psi_1^{(e)}(s_0),\quad Q_{13} = q_1\psi_4^{(e)}(s_0)$$

$$Q_7 = q_2\psi_2^{(e)}(s_1),\quad Q_{14} = q_2\psi_3^{(e)}(s_1)$$





Derivatives of the Solution

Linear triangular element

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e \beta_j^e, \quad (x, y) \in \Omega^e$$

$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e \gamma_j^e, \quad (x, y) \in \Omega^e$$

Linear Rectangular element

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial x} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e (\beta_j^e + \mu_j^e y), \quad (x, y) \in \Omega^e$$

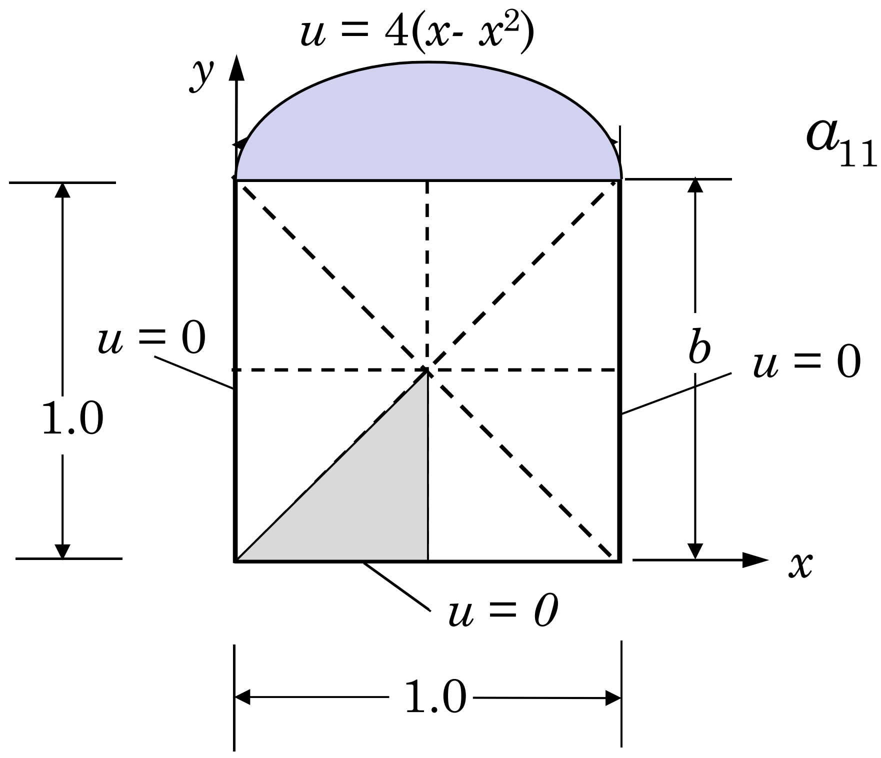
$$\frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^{(e)} \frac{\partial \psi_j^{(e)}}{\partial y} = \frac{1}{2\Delta^e} \sum_{j=1}^{n} u_j^e (\gamma_j^e + \mu_j^e x), \quad (x, y) \in \Omega^e$$

A NUMERICAL EXAMPLE

Given the following differential equation

$$-k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f_0$$

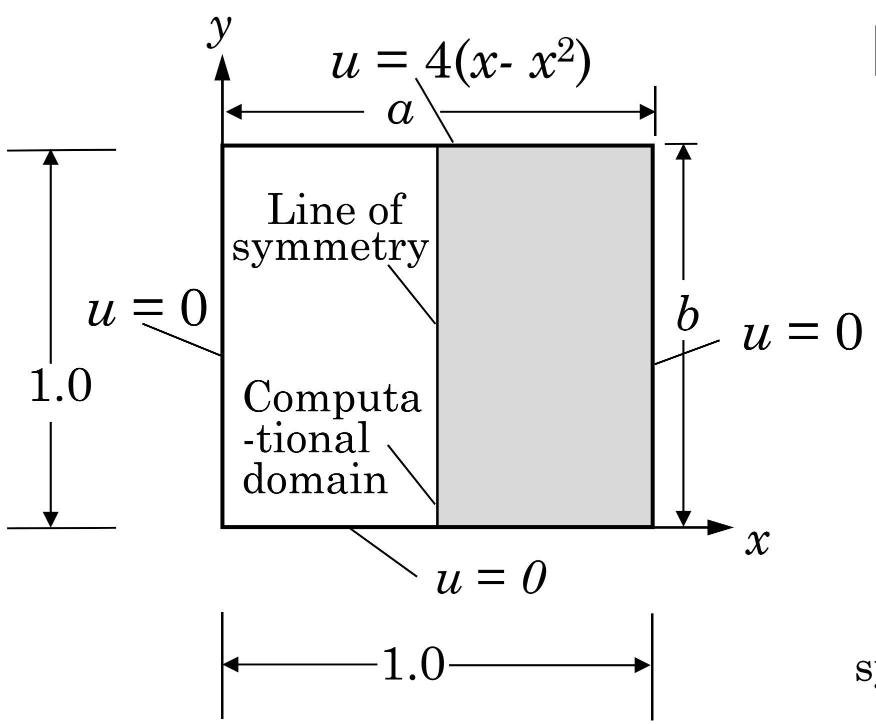
on the square domain and boundary conditions shown in the figure below, *determine the values of the unknown u at nodes.*



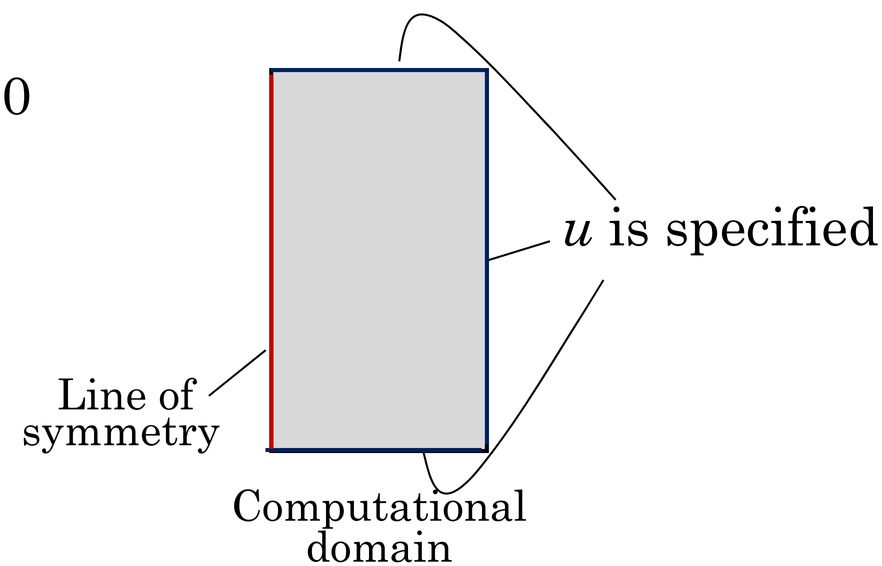
$$a_{11}=a_{11}^e,\ \ a_{22}=a_{22}^e,\ f=f_0^e$$

A NUMERICAL EXAMPLE (continued)

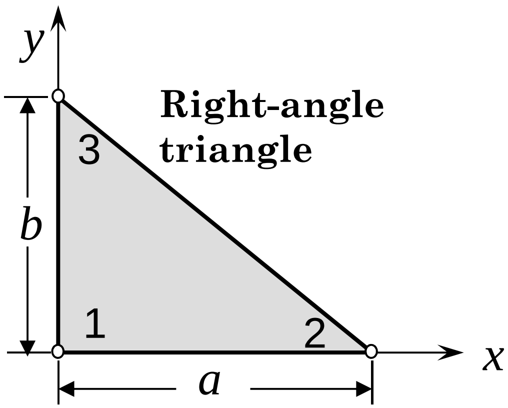
$$-k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f_0$$



$$\begin{aligned} u(0,1) &= 0.0,\\ u(0.25,1) &= 0.\,75,\\ u(0.5,1) &= 1.00,\\ u(0.75,1) &= 0.75,\\ u(1,1) &= 0.0 \end{aligned}$$

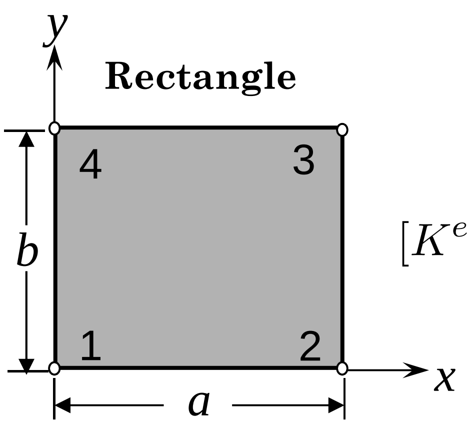


EXAMPLE (continued): Element Matrices



$$[K^e] = \frac{a_{11}^e}{2a}\begin{bmatrix} b & -b & 0 \\ -b & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{a_{22}^e}{2b}\begin{bmatrix} a & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & a \end{bmatrix}$$

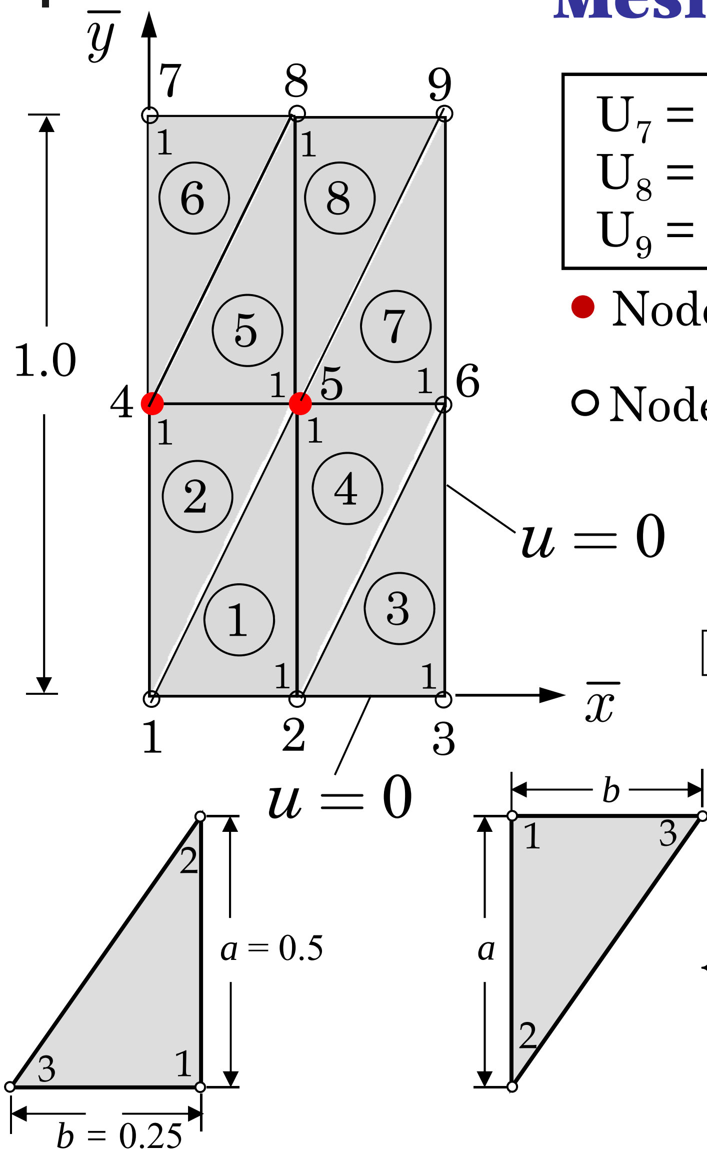
$$\{f^e\} = \frac{f_0^e A_T^e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad A_T^e = \text{ area of the triangle}$$



$$[K^e] = \frac{a_{11}}{6a}\begin{bmatrix} 2b & -2b & -b & b \\ -2b & 2b & b & -b \\ -b & b & 2b & -2b \\ b & -b & -2b & 2b \end{bmatrix} + \frac{a_{22}}{6b}\begin{bmatrix} 2a & a & -a & -2a \\ a & 2a & -2a & -a \\ -a & -2a & 2a & a \\ -2a & -a & a & 2a \end{bmatrix}$$

$$\{f^e\} = \frac{f_0^e A_R^e}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \quad A_R^e = \text{ area of the rectangle}$$

A NUMERICAL EXAMPLE (continued):
Mesh of triangles



$U_7 = 1.00$, $U_8 = 0.75$, $U_9 = 0.00$

$$u = u_0(x) = 4x(1-x)$$
$$u_0(\bar{x}) = 4(0.25 - \bar{x}^2)$$

● Nodes at which solution is the *unknown*

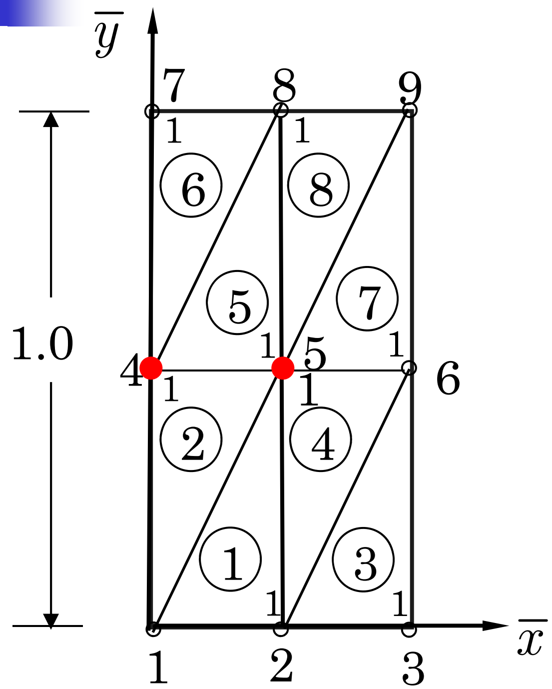
○ Nodes at which solution is the *known*

$$[K^e] = \frac{k}{2}\begin{bmatrix} \frac{b}{a}+\frac{a}{b} & -\frac{b}{a} & -\frac{a}{b} \\ -\frac{b}{a} & \frac{b}{a} & 0 \\ -\frac{a}{b} & 0 & \frac{a}{b} \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 2.5 & -0.5 & -2.0 \\ -0.5 & 0.5 & 0.0 \\ -2.0 & 0.0 & 2.0 \end{bmatrix}$$

$$\{f^e\} = \frac{f_0}{6ab}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{0.125 f_0}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$a = 0.5,\ b = 0.25$

A NUMERICAL EXAMPLE (continued)



Equations for nodes 4 and 5:

$$K_{41}U_1 + K_{42}U_2 + \cdots + K_{49}U_9 = F_4$$
$$K_{51}U_1 + K_{52}U_2 + \cdots + K_{59}U_9 = F_5$$

Known nodal values:

$$U_1 = U_2 = U_3 = U_6 = U_9 = 0, U_7 = 1.0, U_8 = 0.75$$

Simplified equations for nodes 4 and 5:

$$K_{44}U_4 + K_{45}U_5 + K_{47}U_7 + K_{48}U_8 = F_4$$
$$K_{54}U_4 + K_{55}U_5 + K_{58}U_8 = F_5$$

Global coefficients:

$$K_{44} = K_{11}^2 + K_{33}^5 + K_{22}^6 = 1.25k + k + 0.25k = 2.5k$$

$$[K^e] = \frac{k}{2}\begin{bmatrix} 2.5 & -0.5 & -2.0 \\ -0.5 & 0.5 & 0.0 \\ -2.0 & 0.0 & 2.0 \end{bmatrix}, \quad \{f^e\} = \frac{0.125 f_0}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

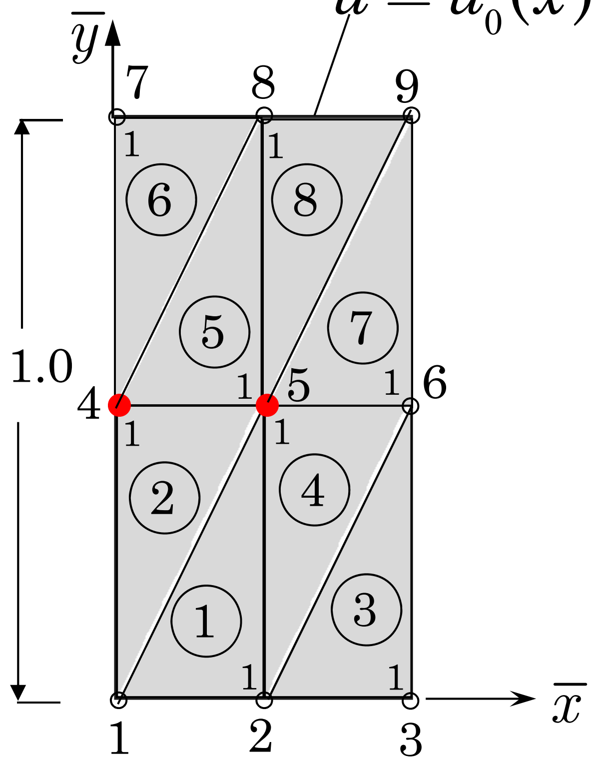
$$K_{45} = K_{13}^2 + K_{31}^5 = -k - k = -2k$$

$$K_{47} = K_{21}^6 = -0.25k; \; K_{48} = K_{23}^6 + K_{32}^5 = 0$$

$$K_{55} = K_{22}^1 + K_{33}^2 + K_{11}^4 + K_{11}^5 + K_{33}^7 + K_{22}^8 = 2.5k + 0.5k + 2k = 5k$$

$$K_{57} = 0, \; K_{58} = K_{12}^5 + K_{21}^8 = -0.5k$$

EXAMPLE (continued)



Final condensed equations for the 2×2 mesh of triangles are

$$k\begin{bmatrix} 2.5 & -2.0 \\ -2.0 & 5.0 \end{bmatrix}\begin{Bmatrix} U_4 \\ U_5 \end{Bmatrix} = f_0\begin{Bmatrix} 0.25 \\ 0.50 \end{Bmatrix} - k\begin{Bmatrix} (-0.25)\times 1 \\ (-0.5)\times 0.75 \end{Bmatrix}$$

and the solution is

$$U_4 = 0.2353, \quad U_5 = 0.1691$$

Final condensed equations for the 2×2 mesh of rectangles are

$$\frac{1}{6}\begin{bmatrix} 10 & -7 \\ -7 & 20 \end{bmatrix}\begin{Bmatrix} U_4 \\ U_5 \end{Bmatrix} = -\frac{1}{6}\begin{Bmatrix} 1\times 1 - 2.5\times 0.75 \\ -2.5\times 1 + 2\times 1 + 2\times 0.75 \end{Bmatrix} \quad U_4 = 0.1623, \quad U_5 = 0.1068$$

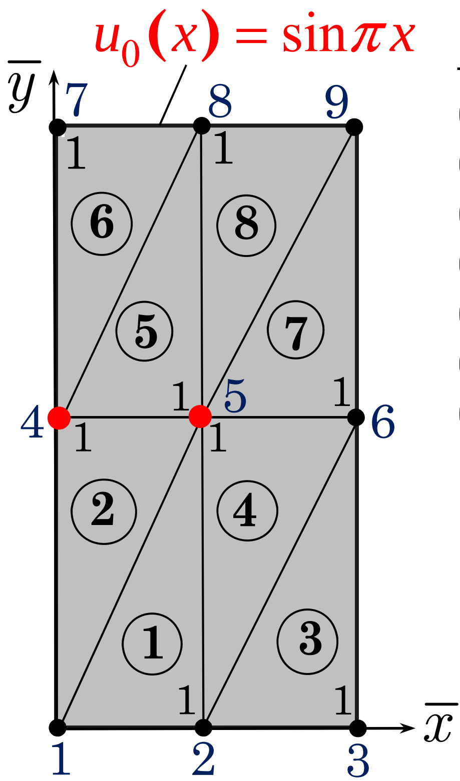


Table: Comparison of FEM solution with the exact solution (meshes of linear triangular elements).

x	y	2×2	4×4	8×8	Exact
0.5000	0.125	-	-	0.0355	0.0349
0.5000	0.250	-	0.0797	0.0764	0.0752
0.5000	0.375	-	-	0.1291	0.1273
0.5000	0.500	0.2303	0.2080	0.2015	0.1993
0.5000	0.625	-	-	0.3050	0.3024
0.5000	0.750	-	0.4630	0.4554	0.4527
0.5000	0.875	-	-	0.6758	0.6737

Table: Comparison of FEM solution with the exact solution (meshes of linear rectangular elements).

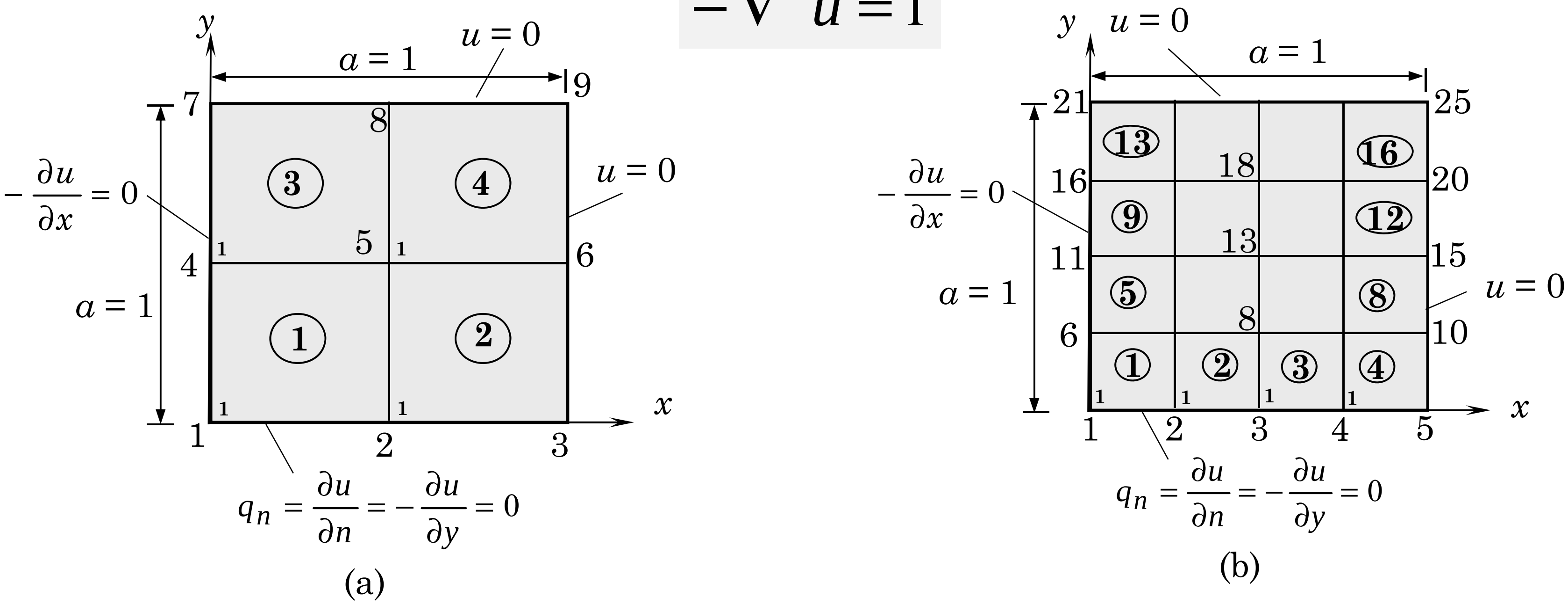
x	y	2×2	4×4	8×8	Exact
0.5000	0.125	-	-	0.0343	0.0349
0.5000	0.250	-	0.0703	0.0740	0.0752
0.5000	0.375	-	-	0.1255	0.1273
0.5000	0.500	0.1520	0.1895	0.1969	0.1993
0.5000	0.625	-	-	0.2996	0.3024
0.5000	0.750	-	0.4410	0.4499	0.4527
0.5000	0.875	-	-	0.6716	0.6737

Example 8.3.1 from the book

2×2 mesh

$$-\nabla^2 u = 1$$

4×4 mesh



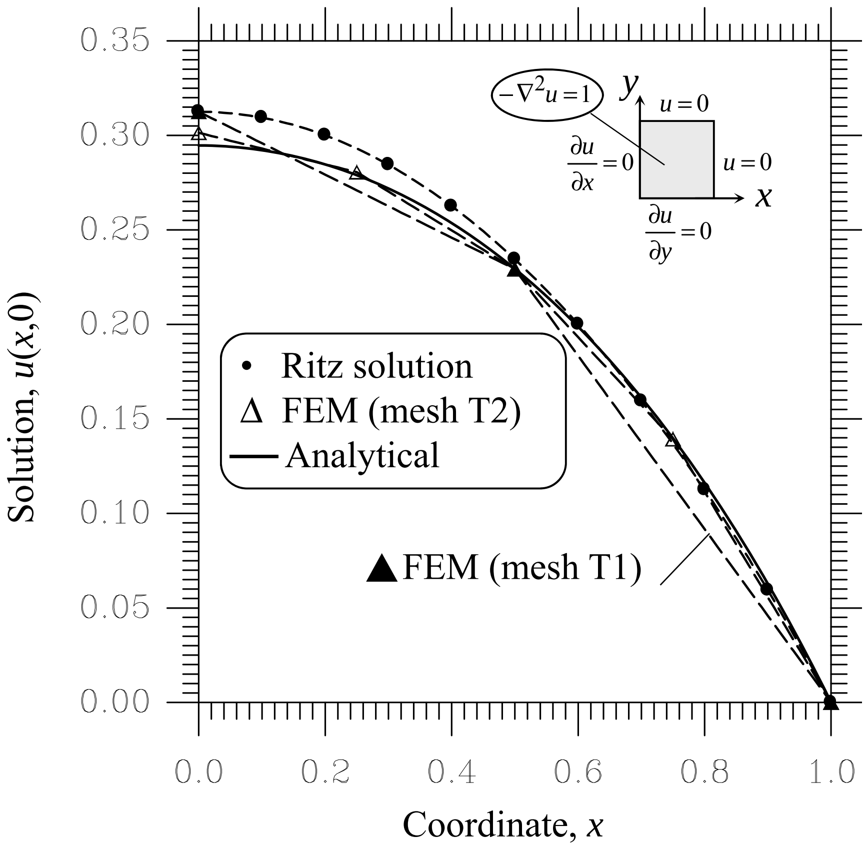
For example, equation for node 1 is:

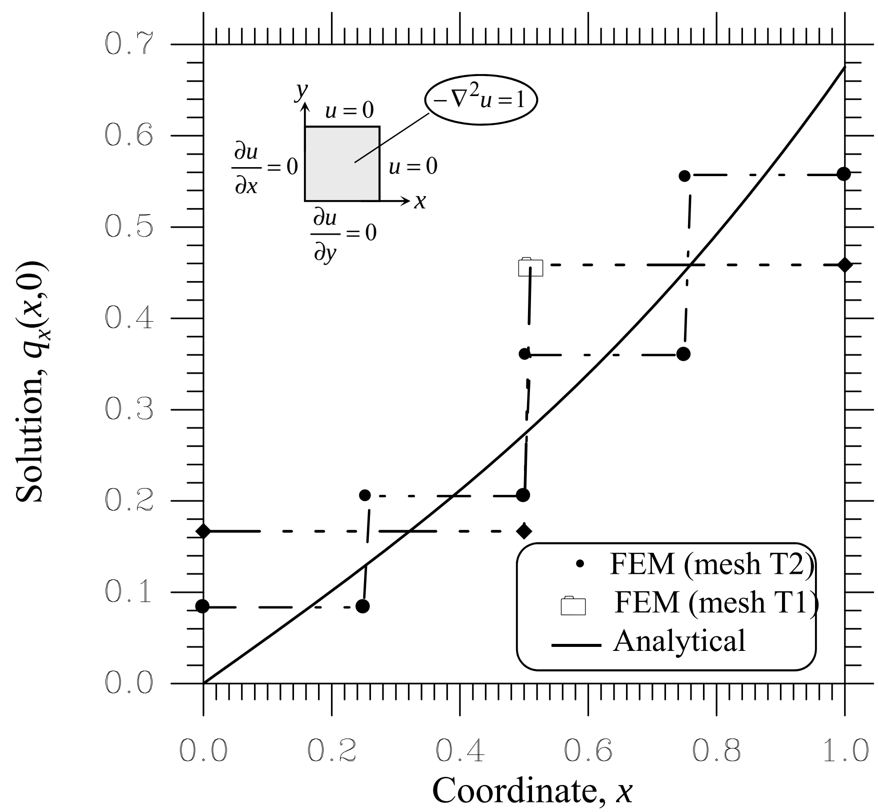
$$K_{11}U_1 + K_{12}U_2 + K_{14}U_4 + K_{15}U_5 = F_1$$

or

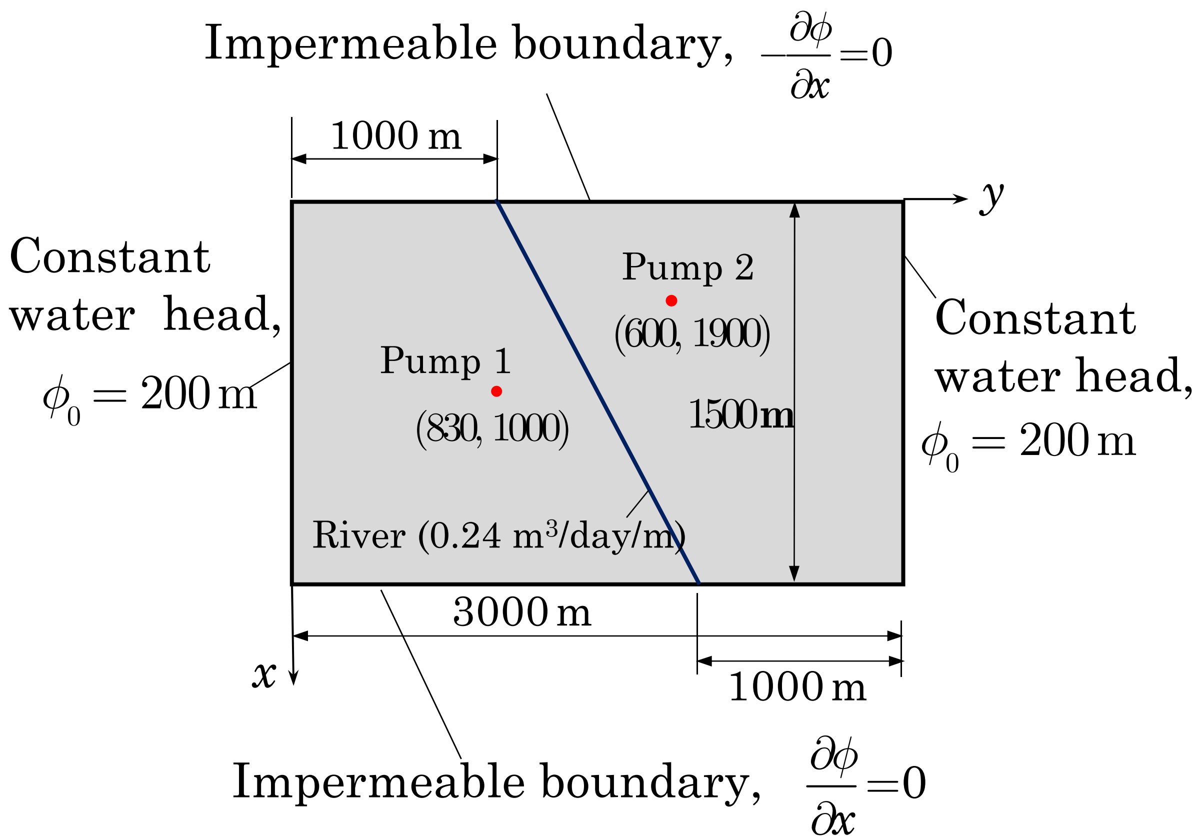
$$K_{11}^{(1)}U_1 + K_{12}^{(1)}U_2 + K_{14}^{(1)}U_4 + K_{13}^{(1)}U_5 = f_1^{(1)} + Q_1^{(1)} = f_1^{(1)}$$

Finite Element, Ritz, and Exact Solutions





Example: Ground-water Flow



Governing Equation

$$-\left(a_{11}\frac{\partial^2\phi}{\partial x^2} + a_{22}\frac{\partial^2\phi}{\partial y^2}\right) = 0$$

$$a_{11} = 20 \text{ m}^3/\text{day}/\text{m}^2$$

$$a_{22} = 40 \text{ m}^3/\text{day}/\text{m}^2$$

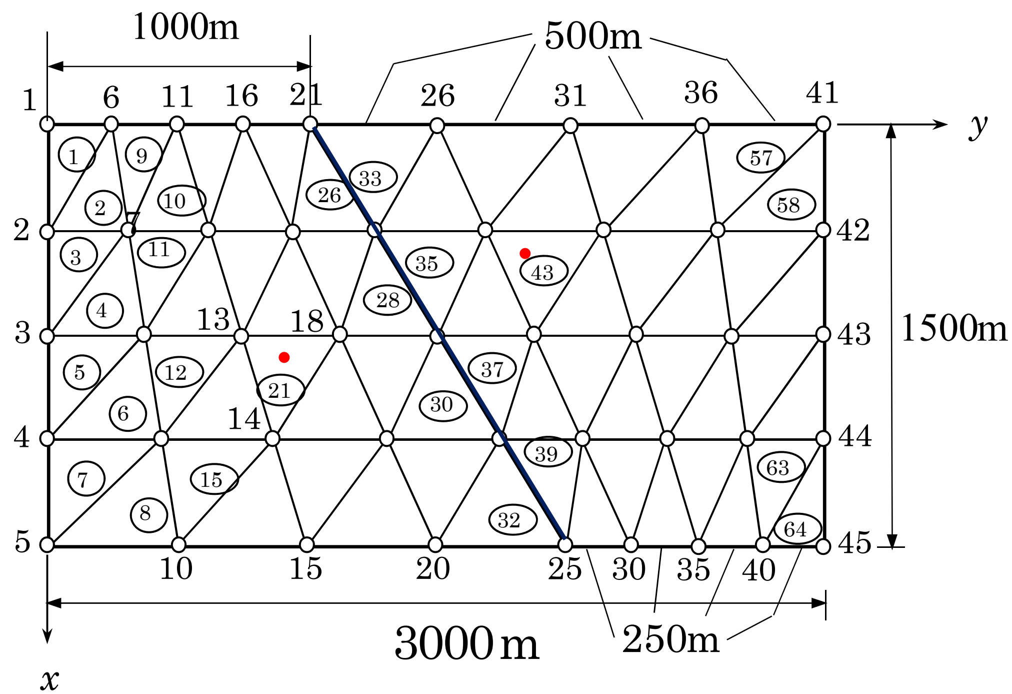
$$-\frac{\partial\phi}{\partial x} = v_x, \quad -\frac{\partial\phi}{\partial y} = v_y$$

Discussion of the modeling issues:

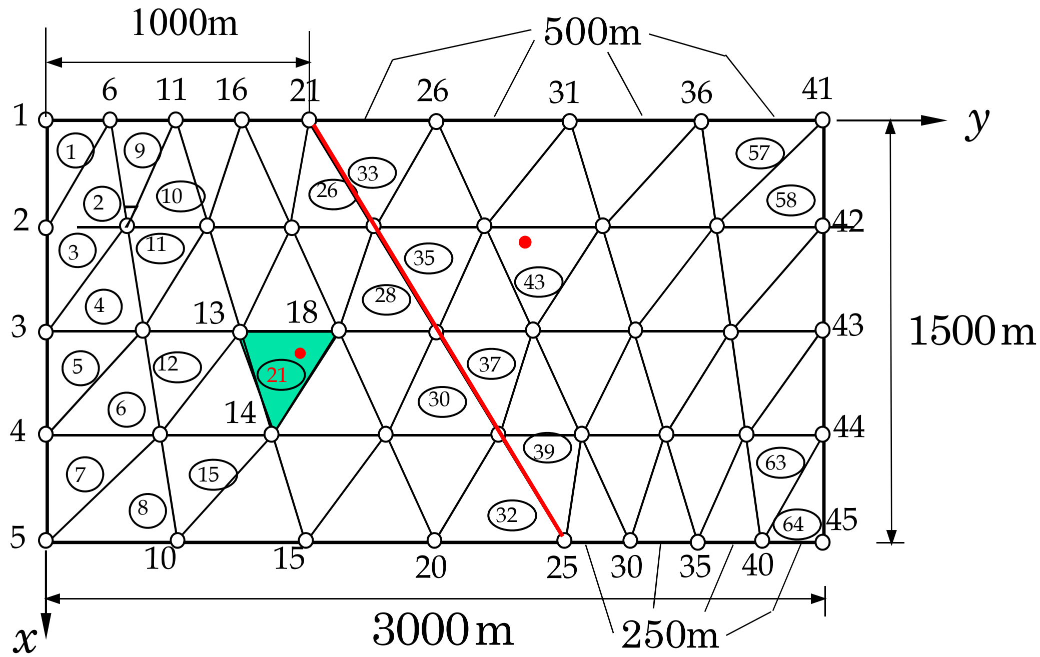
- **Problem data**
- **Boundary conditions**

Example: Ground-water Flow

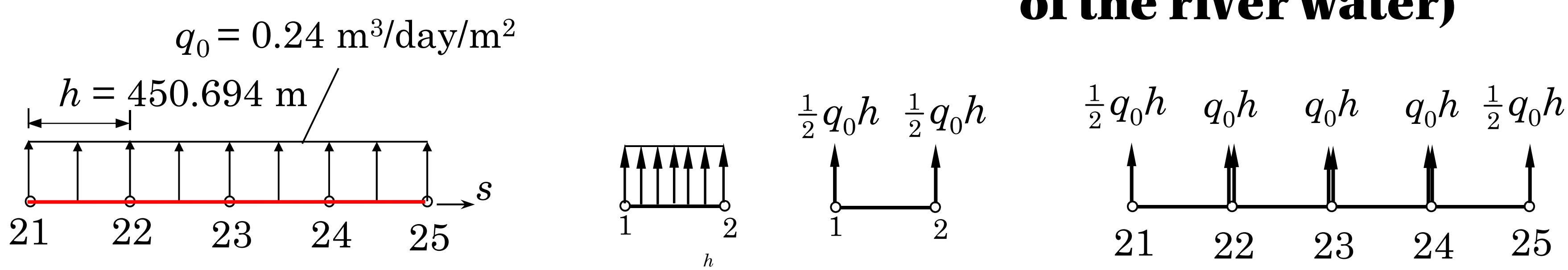
- **Discussion of the modeling issues:**
- **Mesh (not so refined)**



Example: Ground-water flow (continued)



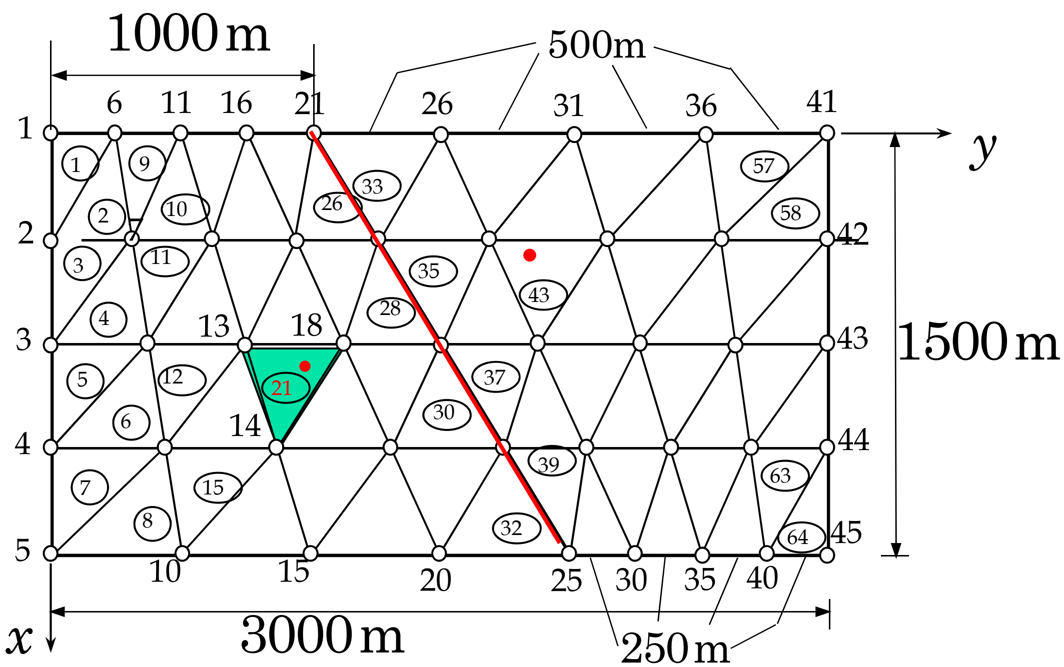
Computation of the line source of water (due to infiltration of the river water)



$$q_i = \int_0^h q_0 \psi_i(s)ds = \frac{1}{2}q_0h$$

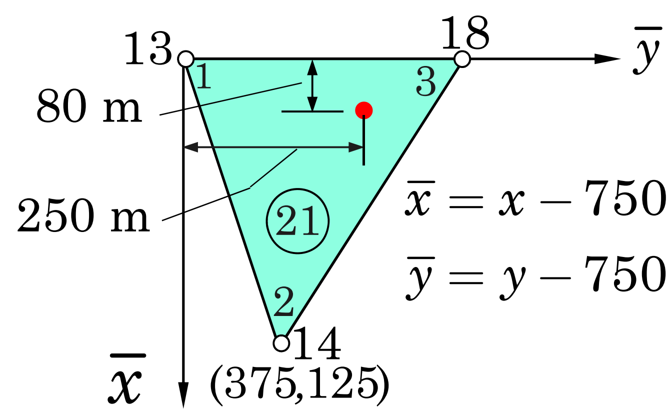
Example: Ground-water flow (continued)

Computation of the point source of water (due to pumping)



Q_0 Rate of pumping

(negative source)



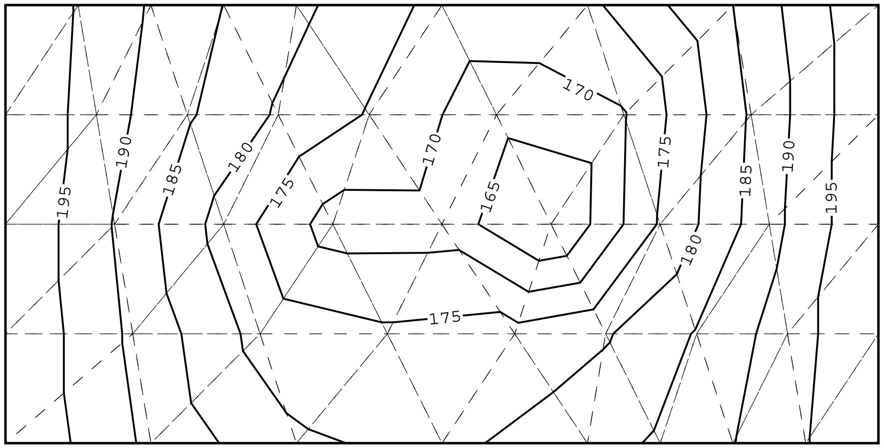
$$Q_i = \int_{\Omega^e} Q_0 \delta(x - x_0, y - y_0)\psi_i(x, y) \; dxdy$$

$$= Q_0 \psi_i(x_0, y_0)$$

Example: Ground-water flow (continued)

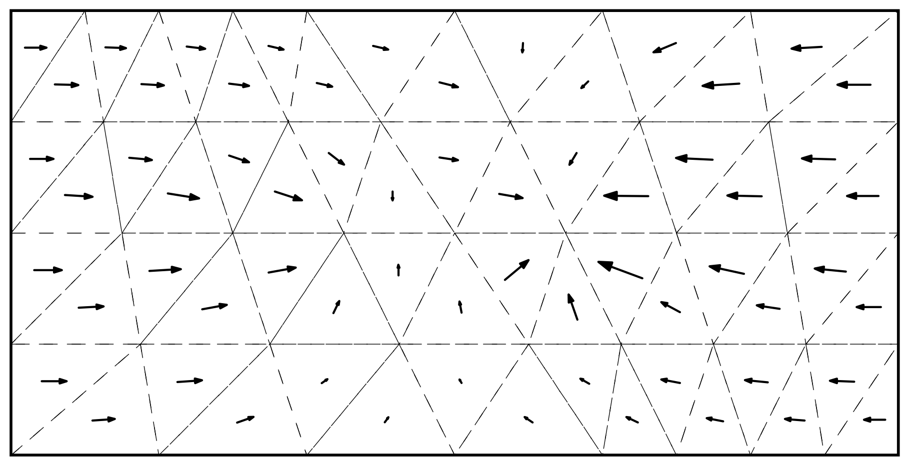
Discussion of the results

Lines of constant ϕ (water head)



- **Equipotential lines should be straight lines because of linear approximation used.**

Plots of velocity vectors



- **Velocity vectors should not cross impermeable boundary**
- **Velocity vectors should head towards the wells, where water is being drawn out**

2-D Problems: 44

Transient Analysis of 2-D Problems

Model Governing Differential Equation

$$c_1 \frac{\partial u}{\partial t} + c_2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a_{11} \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22} \frac{\partial u}{\partial y}\right) = f(x, y, t)$$

Weak formulation

$$0 = \int_{\Omega_e} w_i \left[c_1 \frac{\partial u_h}{\partial t} + c_2 \frac{\partial^2 u_h}{\partial t^2} - \frac{\partial}{\partial x}\left(a_{11} \frac{\partial u_h}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22} \frac{\partial u_h}{\partial y}\right) - f(x, y, t) \right] dx dy$$

$$= \int_{\Omega_e} \left[w_i c_1 \frac{\partial u_h}{\partial t} + w_i c_2 \frac{\partial^2 u_h}{\partial t^2} + a_{11} \frac{\partial w_i}{\partial x} \frac{\partial u_h}{\partial x} + a_{22} \frac{\partial w_i}{\partial y} \frac{\partial u_h}{\partial y} - w_i f \right] dx dy - \oint_{\Gamma_e} q w_i \, ds$$

SPATIAL DISCRETIZATION: Finite Element Model

Approximation

$$u(x,y,t) \approx u_h^e(x,y,t) = \sum_{j=1}^{n} u_j^e(t)\psi_j^e(x,y)$$

Finite element model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

$$C_{ij}^e = \int_{\Omega_e} c_1 \psi_i \psi_j \; dxdy, \qquad M_{ij}^e = \int_{\Omega_e} c_2 \psi_i \psi_j \; dxdy$$

$$K_{ij}^e = \int_{\Omega_e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$F_i^e = \int_{\Omega_e} f \, \psi_i \; dxdy + \oint_{\Gamma_e} q \, \psi_i \; ds$$

TIME APPROXIMATIONS (Parabolic)

Semidiscrete FE model

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}, \quad 0 < t < T$$

NOTE: From here, the procedure is the same as in 1-D

Alfa-family of approximation

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \Delta t_{s+1}\left[\alpha\dot{\mathbf{u}}_{s+1} + (1-\alpha)\dot{\mathbf{u}}_s\right]$$

$$\mathbf{C}\mathbf{u}_{s+1} = \mathbf{C}\mathbf{u}_s + \Delta t_{s+1}\left[\alpha\mathbf{C}\dot{\mathbf{u}}_{s+1} + (1-\alpha)\mathbf{C}\dot{\mathbf{u}}_s\right]$$

$$\mathbf{C}\dot{\mathbf{u}}_{s+1} = \mathbf{F}_{s+1} - \mathbf{K}_{s+1}\mathbf{u}_{s+1} \qquad \mathbf{C}\dot{\mathbf{u}}_s = \mathbf{F}_s - \mathbf{K}_s\mathbf{u}_s$$

$$\left(\mathbf{C} + \alpha\Delta t_{s+1}\mathbf{K}_{s+1}\right)\mathbf{u}_{s+1} = \left(\mathbf{C} - (1-\alpha)\Delta t_s\mathbf{K}_s\right)\mathbf{u}_s + \Delta t_{s+1}\left[\alpha\mathbf{F}_{s+1} + (1-\alpha)\mathbf{F}_s\right]$$

Fully discretized model

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1}, \qquad \hat{\mathbf{K}}_{s+1} = \alpha\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C},$$

$$\hat{\mathbf{F}}_{s+1} = \left[(1-\alpha)\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C}\right]\mathbf{u}_s + \Delta t_{s+1}\left[\alpha\mathbf{F}_{s+1} + (1-\alpha)\mathbf{F}_s\right]$$

TIME APPROXIMATIONS (Hyperbolic)

Semidiscrete FE model

$$\mathbf{C}^e\dot{\mathbf{u}}^e + \mathbf{M}^e\ddot{\mathbf{u}}^e + \mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e$$

Newmark scheme (second-order equations)

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \Delta t\,\dot{\mathbf{u}}_s + \tfrac{1}{2}(\Delta t)^2\,\ddot{\mathbf{u}}_{s,\gamma}$$

$$\dot{\mathbf{u}}_{s+1} = \dot{\mathbf{u}}_s + \Delta t\,\ddot{\mathbf{u}}_{s,\alpha},\quad \ddot{\mathbf{u}}_{s,\alpha} = (1-\alpha)\ddot{\mathbf{u}}_s + \alpha\,\ddot{\mathbf{u}}_{s+1}$$

Fully discretized model

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}_{s+1} = \hat{\mathbf{F}}_{s+1}, \qquad \hat{\mathbf{K}}_{s+1} = \mathbf{K}_{s+1} + a_3\mathbf{M}_{s+1} + a_5\mathbf{C}_{s+1}$$

$$\hat{\mathbf{F}}_{s+1} = \mathbf{F}_{s+1} + \mathbf{M}_{s+1}\left(a_3\mathbf{u}_s + a_4\dot{\mathbf{u}}_s + a_5\ddot{\mathbf{u}}_s\right) + \mathbf{C}_{s+1}\left(a_5\mathbf{u}_s + a_6\dot{\mathbf{u}}_s + a_7\ddot{\mathbf{u}}_s\right)$$

An Example: Transient Heat Conduction Problem

Governing equation

$$c_1\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=f(x,y,t)$$

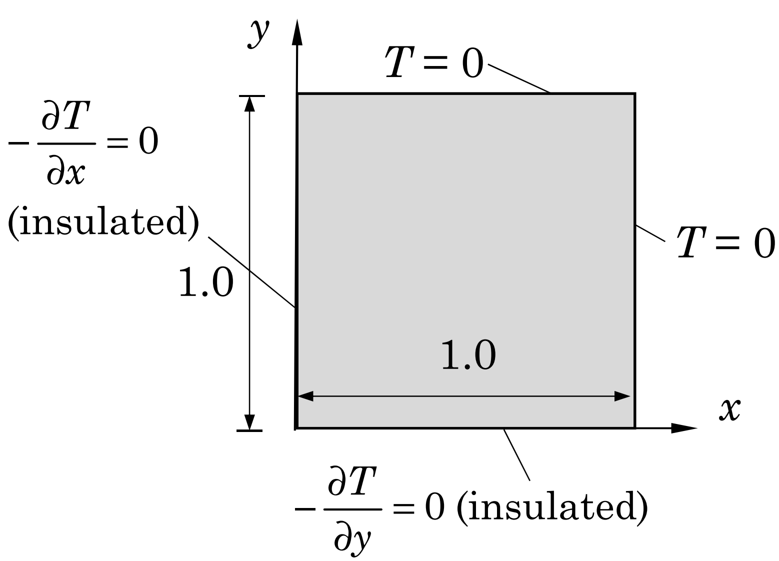
Boundary conditions

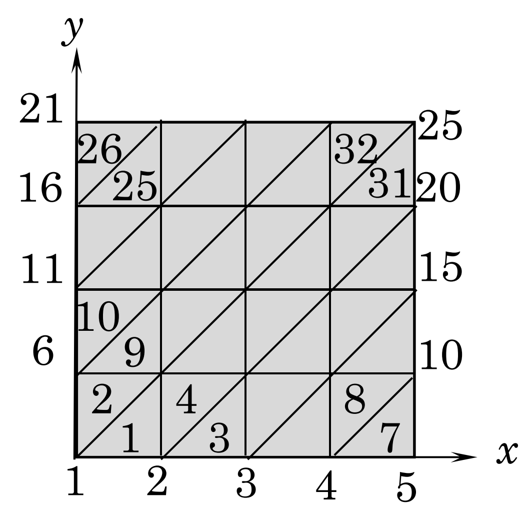
$$k\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right)=0 \text{ on } x=0 \text{ and } y=0$$

$$T=0 \text{ on } y=1 \text{ and } x=1$$

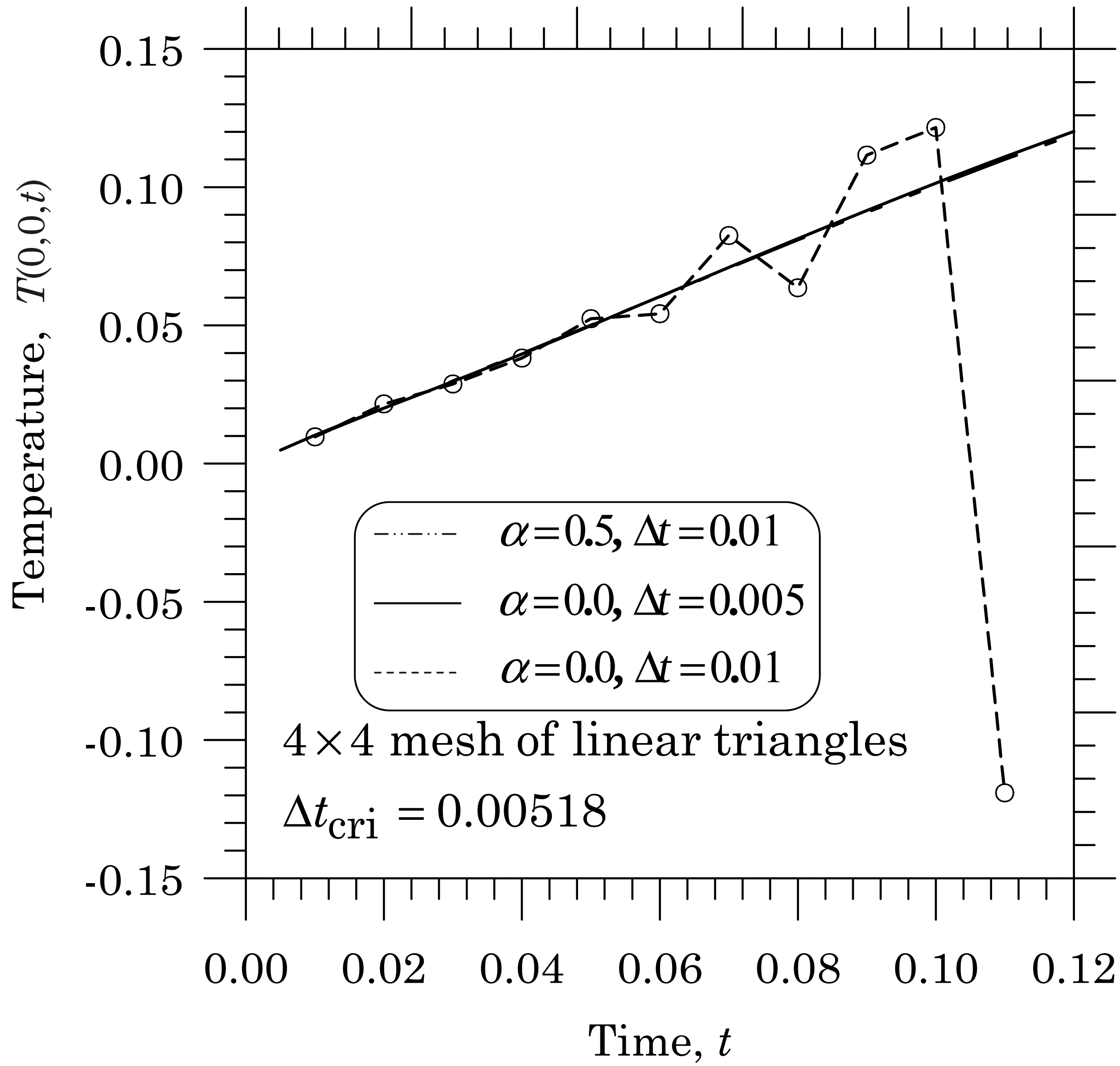
Initial condition

$$T(x,y,0)=1$$

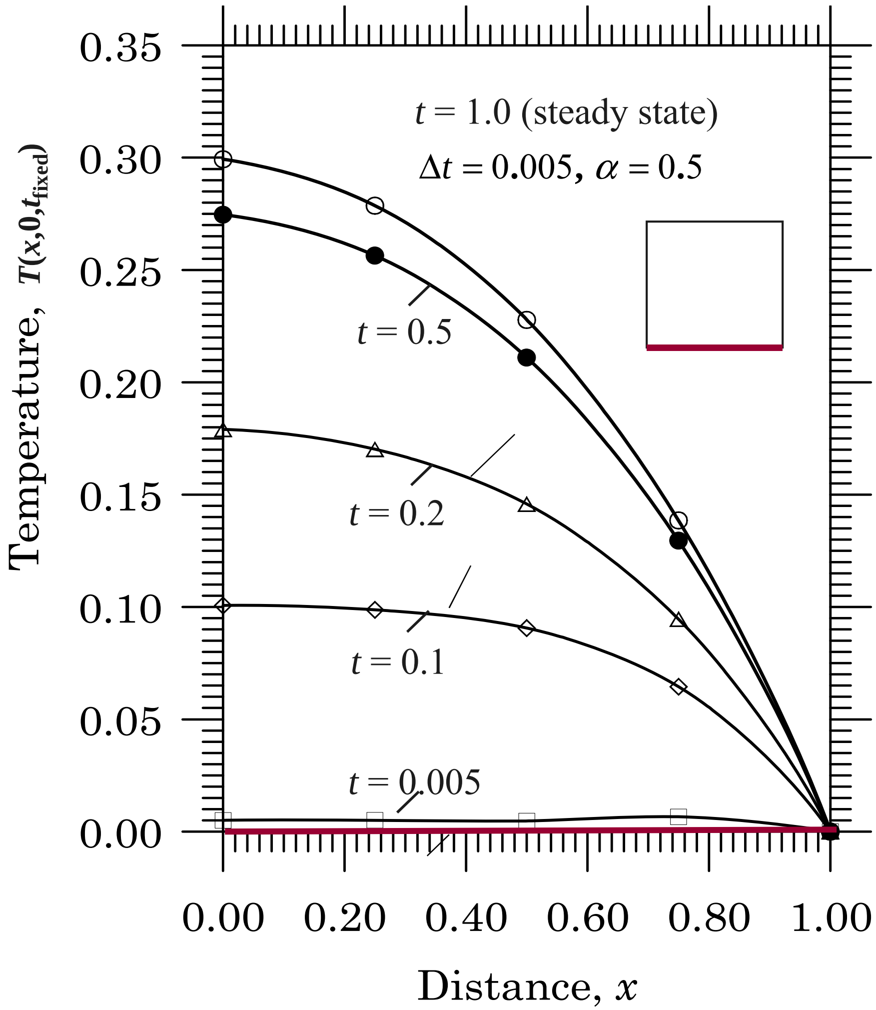


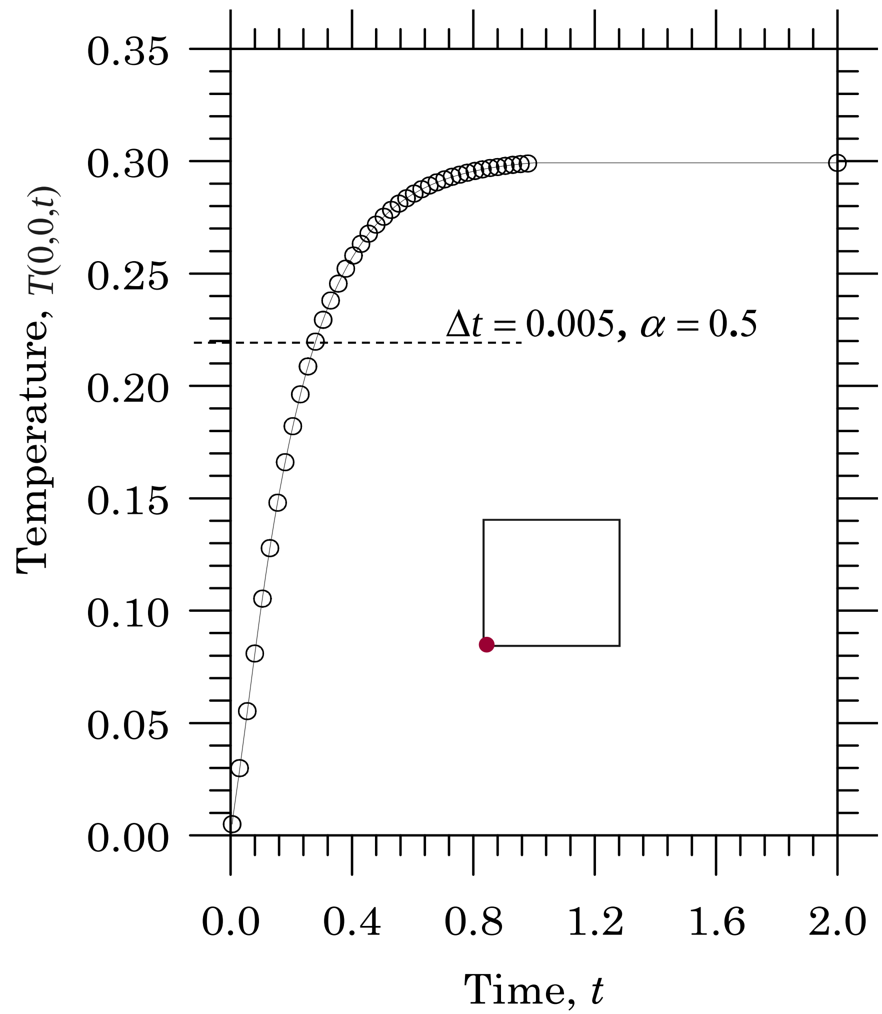


Stability Characteristics

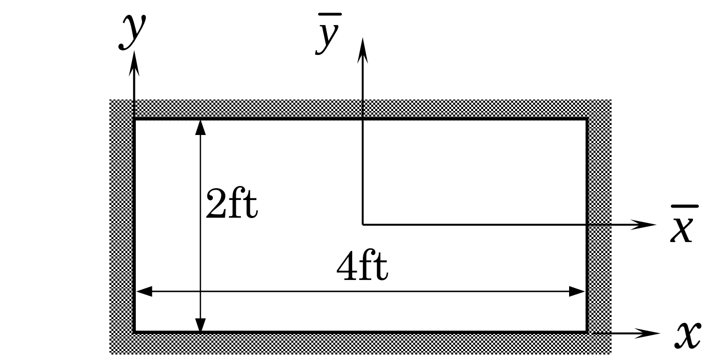


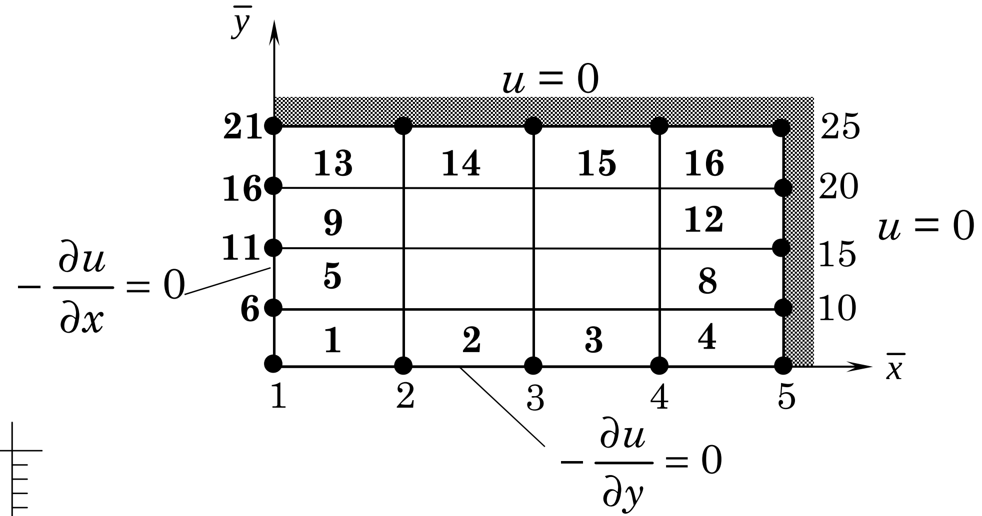
A NUMERICAL EXAMPLE (Parabolic)

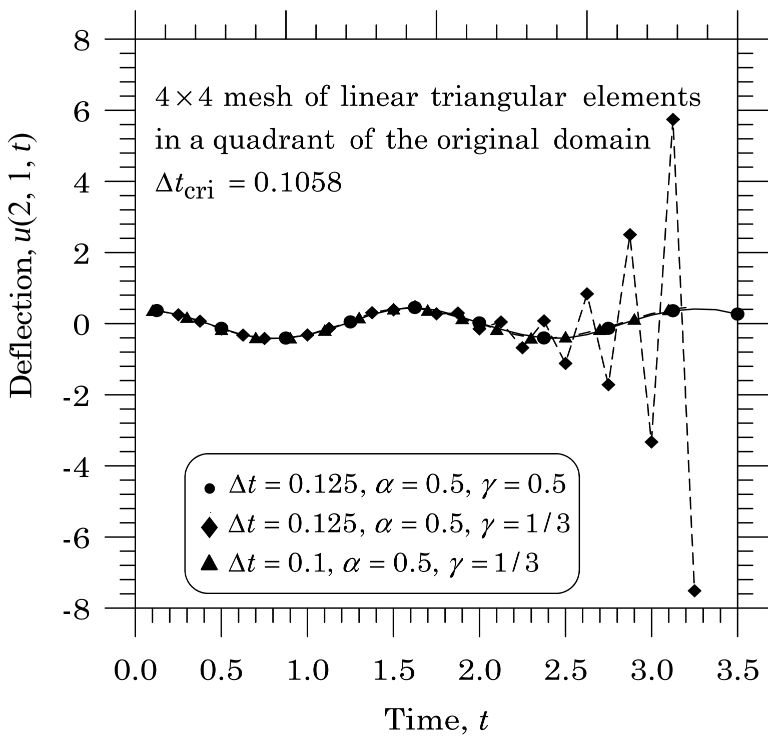


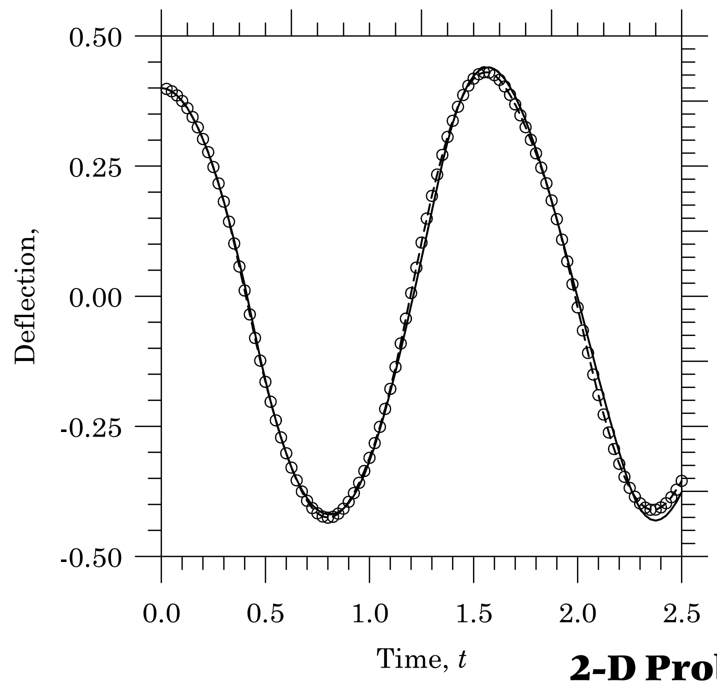


TRANSIENT RESPONSE OF A MEMBRANE









SUMMARY

1. **Starting point is a model differential equation in *u***
2. **Construct an integral statement – weak form,**

   which has three steps. Integration by parts that (a) relaxes ("weakens" differentiability on the variable *u*, and (b) brings in secondary variable into the integral form.
3. **Substitute suitable approximation for *u* and obtain the finite element model (i.e., a set of algebraic relations between nodal values of *u* and *Q)*.**
4. **Assemble equations, impose boundary conditions, and solve the assembled equations.**
5. **Post-computation of variables follows same procedure as in 1-D FEM.**
6. **Several numerical examples are presented.**
7. **Reviewed time-dependent problems and examples**

WEAK FORM DEVELOPMENT

Step 1

$$0 = \int_{\Omega^e} w_i \left[-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x} + a_{12}\frac{\partial u_h}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u_h}{\partial x} + a_{22}\frac{\partial u_h}{\partial y}\right) + a_{00}u - f \right] dxdy$$

$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x} + a_{12}\frac{\partial u_h}{\partial y}\right) + \frac{\partial w_i}{\partial y}\left(a_{21}\frac{\partial u_h}{\partial x} + a_{22}\frac{\partial u_h}{\partial y}\right) + a_{00}w_i u - w_i f \right] dxdy$$

$$- \oint_{\Gamma^e} w_i \left[\left(a_{11}\frac{\partial u_h}{\partial x} + a_{12}\frac{\partial u_h}{\partial y}\right) n_x + \left(a_{21}\frac{\partial u_h}{\partial x} + a_{22}\frac{\partial u_h}{\partial y}\right) n_y \right] ds$$

Step 2

$$= \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x} + a_{12}\frac{\partial u_h}{\partial y}\right) + \frac{\partial w_i}{\partial y}\left(a_{21}\frac{\partial u_h}{\partial x} + a_{22}\frac{\partial u_h}{\partial y}\right) + a_{00}w_i u - w_i f \right] dxdy$$

$$- \oint_{\Gamma^e} w_i \, q_n \, ds$$

Step 3

$$q_n = \left(a_{11}\frac{\partial u_h}{\partial x} + a_{12}\frac{\partial u_h}{\partial y}\right) n_x + \left(a_{21}\frac{\partial u_h}{\partial x} + a_{22}\frac{\partial u_h}{\partial y}\right) n_y ,$$ flux normal to the boundary

FINITE ELEMENT APPROXIMATION

Weak form

$$0=\int_{\Omega^e}\left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}+a_{12}\frac{\partial u_h}{\partial y}\right)+\frac{\partial w_i}{\partial y}\left(a_{21}\frac{\partial u_h}{\partial x}+a_{22}\frac{\partial u_h}{\partial y}\right)+a_{00}w_i u - w_i f\right]-\oint_{\Gamma^e} w_i\, q_n\, ds$$

Approximation

$$u(x,y)\approx u_h(x,y)=\sum_{j=1}^{n}u_j\psi_j(x,y)$$

Finite element model [the ith equation is obtained by replacing the weight function w by ψ_i $(i=1,2,\cdots,n)$]

$$0=\int_{\Omega^e}\left[\frac{\partial \psi_i}{\partial x}\left(a_{11}\frac{\partial u_h}{\partial x}+a_{12}\frac{\partial u_h}{\partial y}\right)+\frac{\partial \psi_i}{\partial y}\left(a_{21}\frac{\partial u_h}{\partial x}+a_{22}\frac{\partial u_h}{\partial y}\right)+a_{00}\psi_i u - \psi_i f\right]-\oint_{\Gamma^e} \psi_i\, q_n\, ds$$

FINITE ELEMENT MODEL DEVELOPMENT (continued)

$$0 = \sum_{j=1}^{n} u_j \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + a_{21} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy - \int_{\Omega^e} \psi_i f \, dxdy - \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

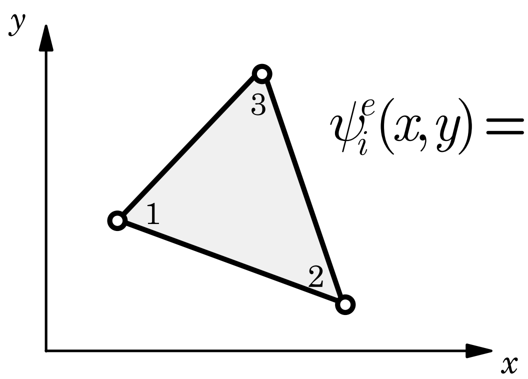
$$= \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e = \sum_{j=1}^{n} K_{ij}^e u_j^e - F_i^e \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e$$

$$K_{ij}^e = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + a_{21} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy,$$

$$F_i^e = \int_{\Omega^e} \psi_i f \, dxdy + \oint_{\Gamma^e} \psi_i \, q_n \, ds$$

NUMERICAL EVALUATION OF COEFFICIENT MATRICES

Linear Triangular Element



$$\psi_i^e(x,y) = \frac{1}{2\Delta^e}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

Δ^e = Area of the triangle

$$K_{ij}^e = \int_{\Delta^e}\left(a_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + a_{12}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y} + a_{21}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x} + a_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy$$

$$= a_{11}^e\int_{\Delta^e}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}dxdy + a_{12}^e\int_{\Delta^e}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y}dxdy + a_{21}^e\int_{\Delta^e}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x}dxdy + a_{22}^e\int_{\Delta^e}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}dxdy$$

$$= \frac{1}{4\Delta^e}\left(a_{11}^e\beta_i\beta_j + a_{12}^e\beta_i\gamma_j + a_{21}^e\gamma_i\beta_j + a_{22}^e\gamma_i\gamma_j\right)$$

$$f_i^e = f_e\int_{\Omega^e}\psi_i\, dxdy = \frac{f_e\Delta^e}{3}$$

Potential (Inviscid) Flows

1. Stream Function ψ Formulation

$$-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0,\quad v_x=+\frac{\partial\psi}{\partial y},\qquad v_y=-\frac{\partial\psi}{\partial x}$$

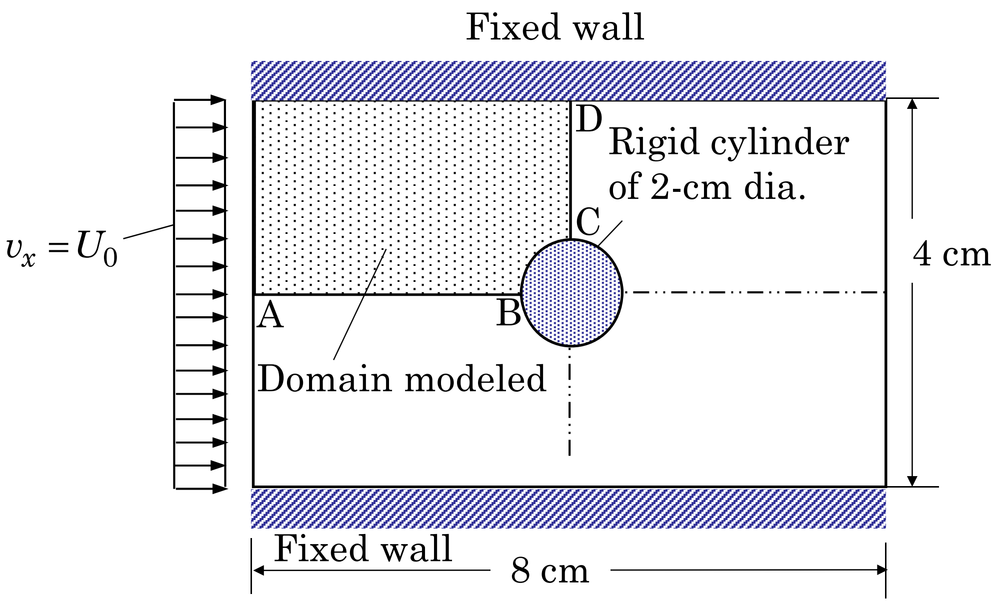
(v_x, v_y) – **Velocity components**

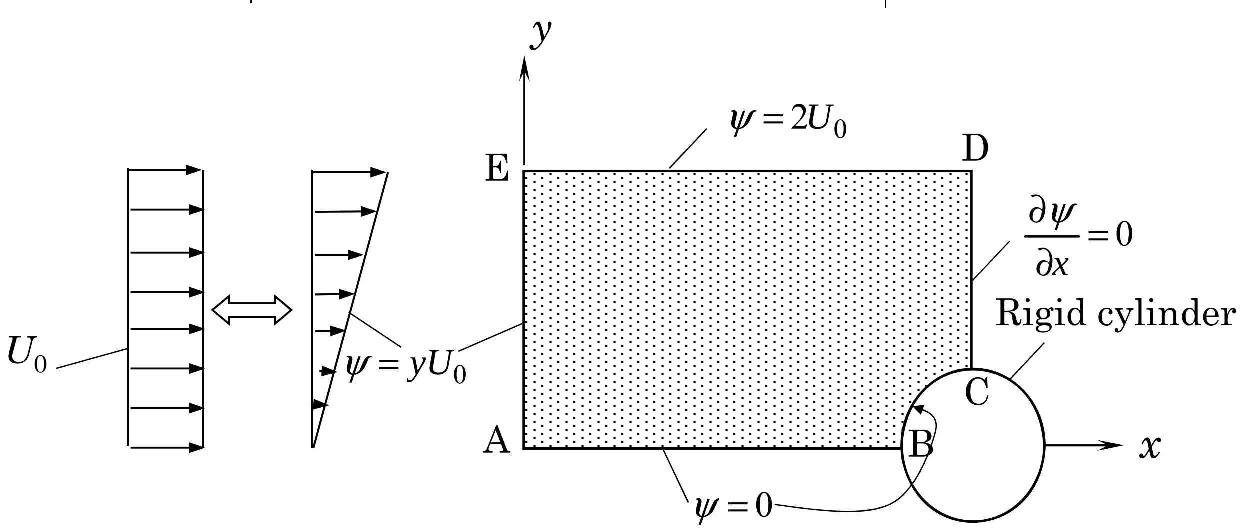
2. Velocity Potential ϕ Formulation

$$-\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0,\quad v_x=-\frac{\partial\phi}{\partial x},\qquad v_y=-\frac{\partial\phi}{\partial y}$$

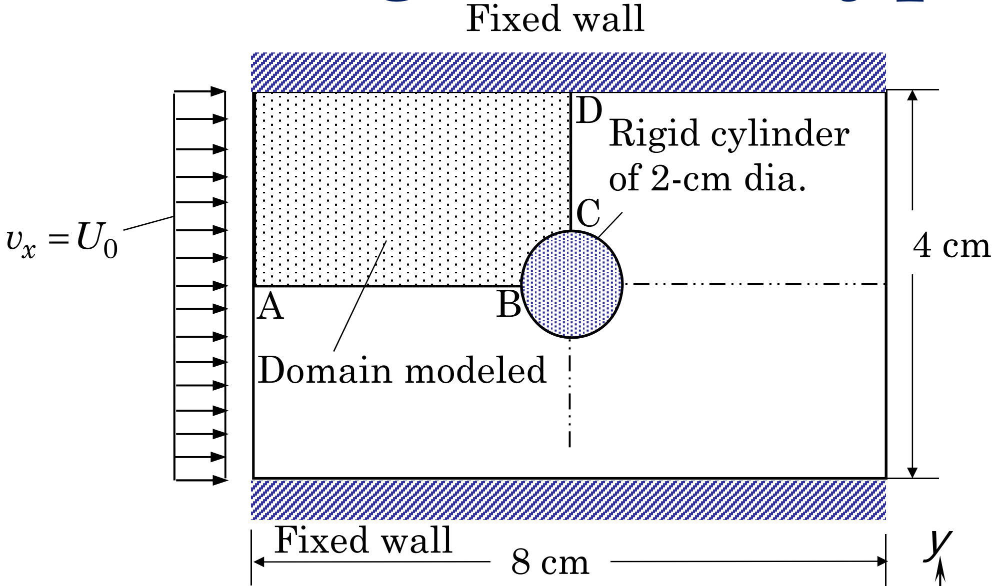
(v_x, v_y) – **Velocity components**

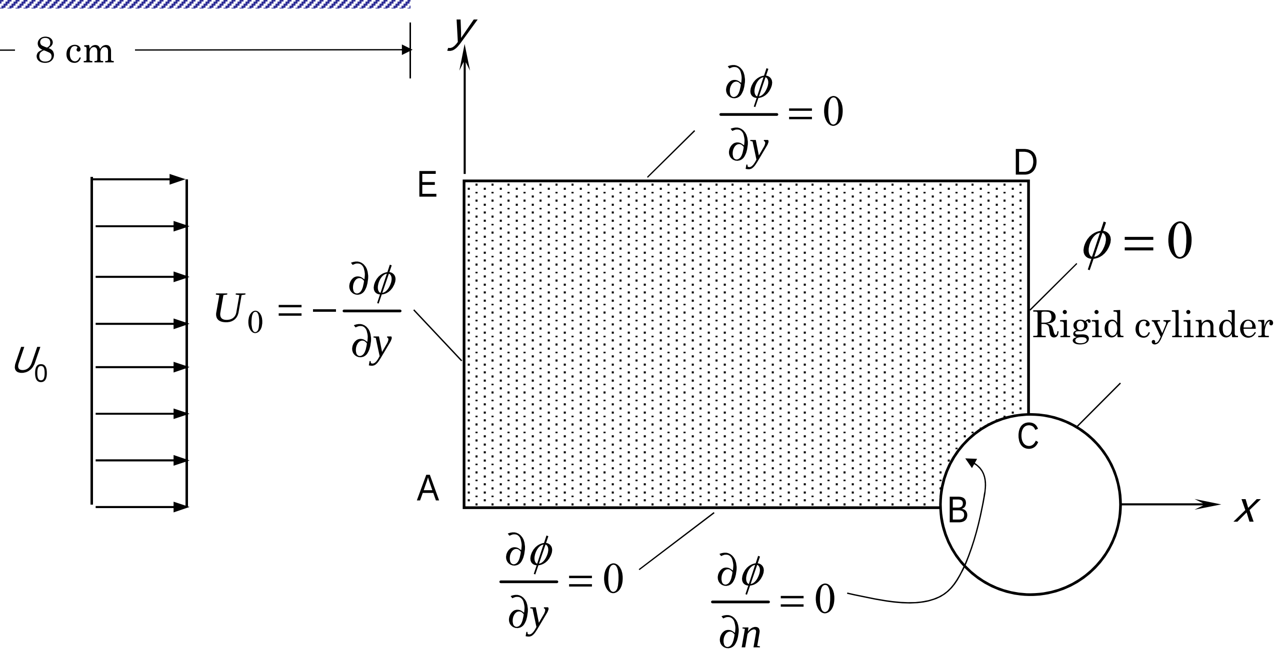
Flow of an inviscid fluid around a cylinder using the stream function formulation



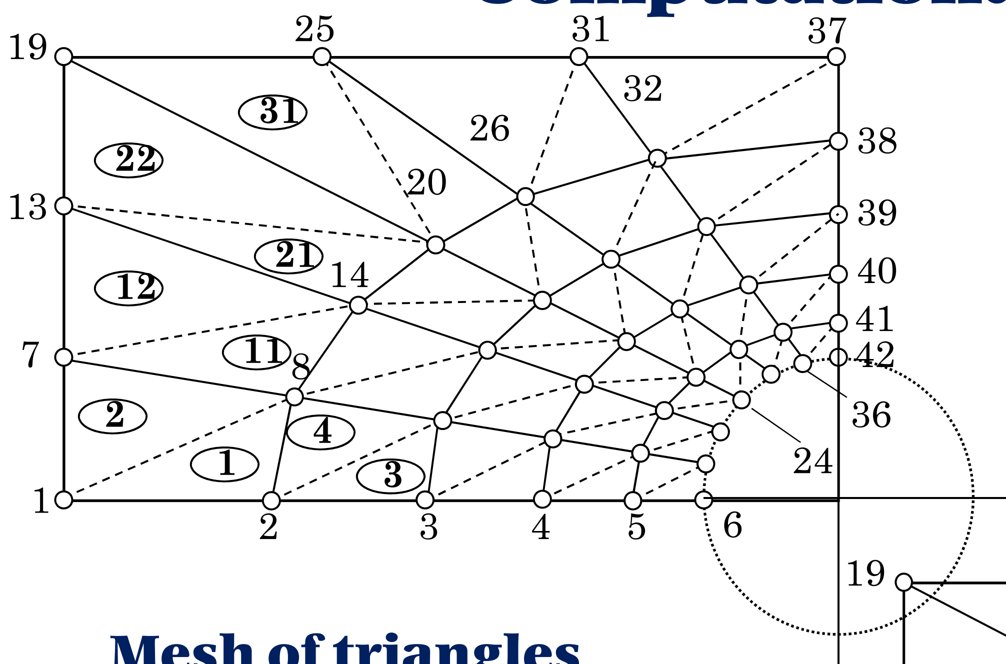


Flow of an inviscid fluid around a cylinder using the velocity potential formulation



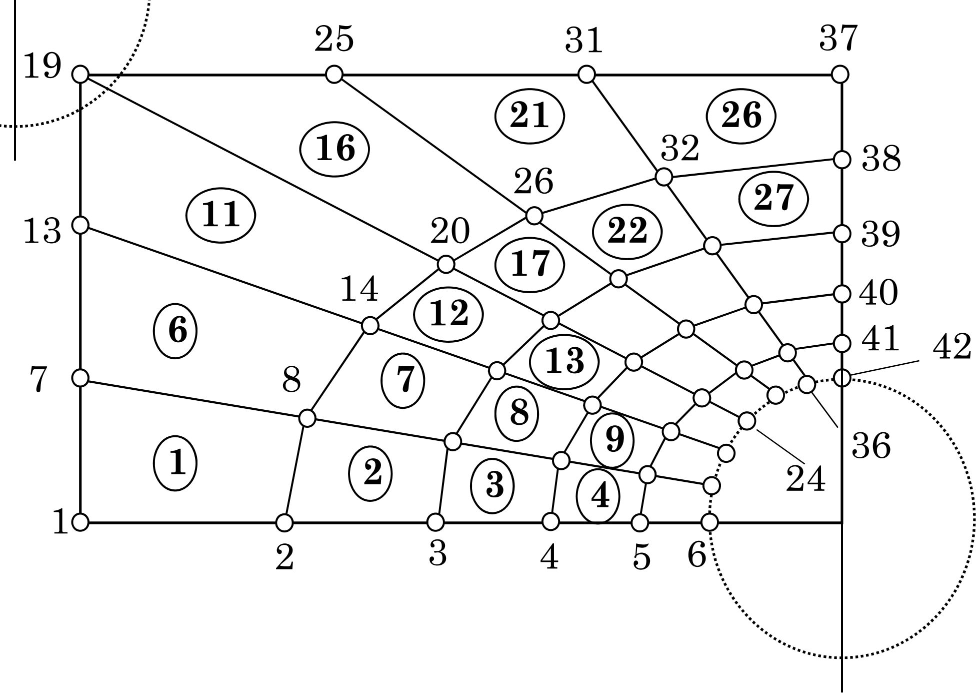


Finite Element Meshes in the Computational Domains



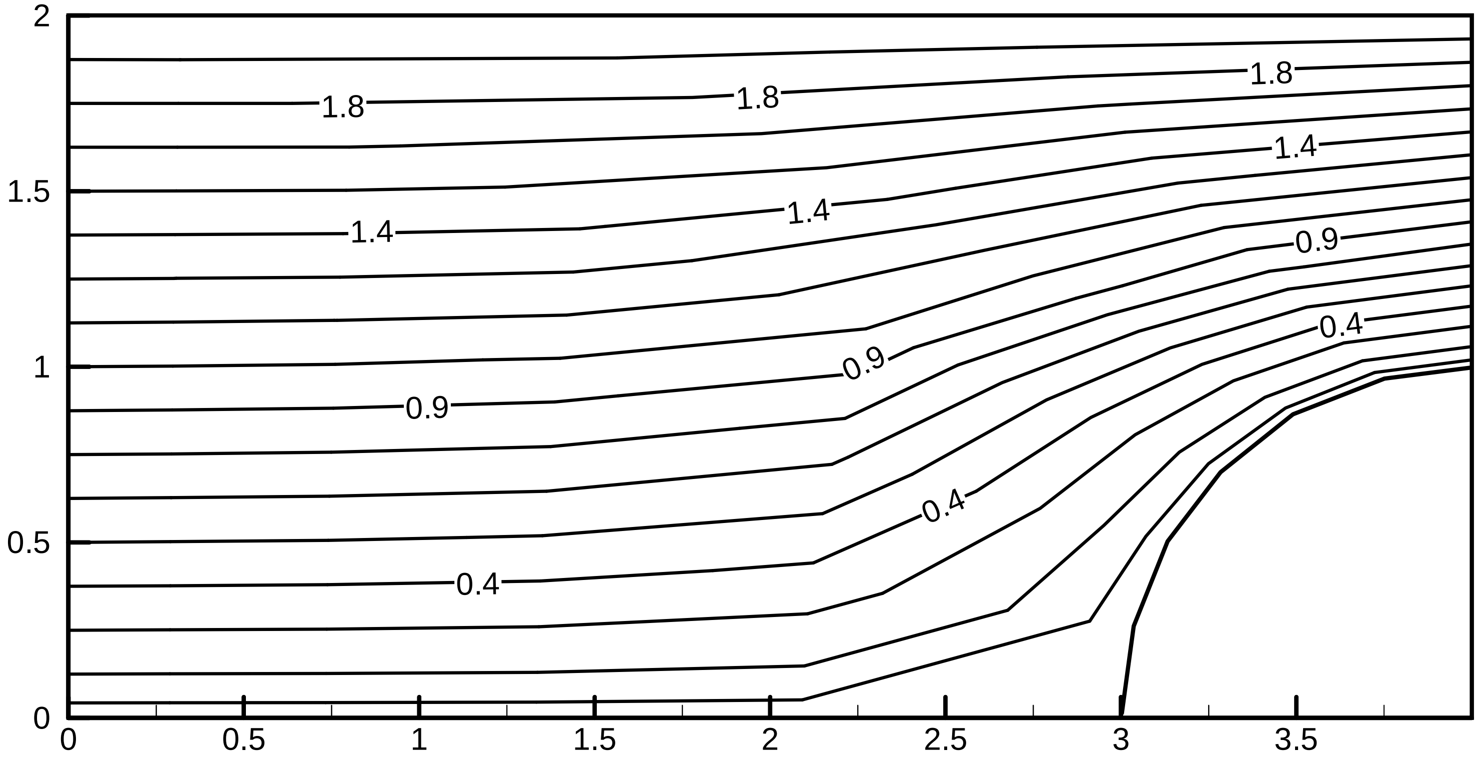
Mesh of triangles

Mesh of quadrilaterals



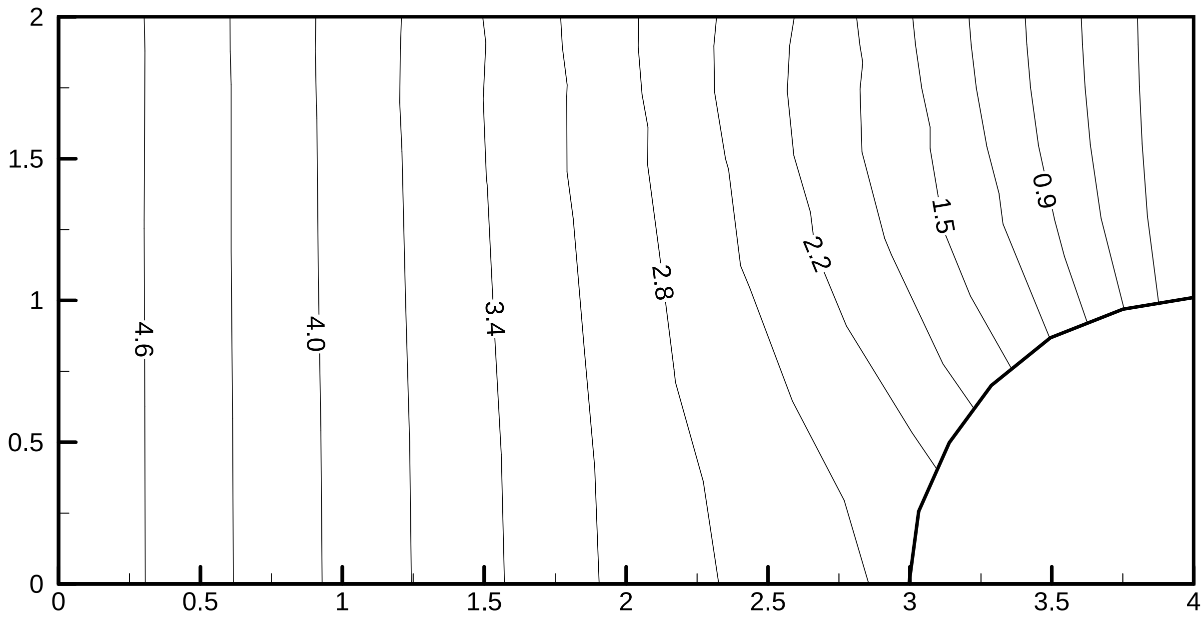
Results obtained using the stream function formulation

Iso stream lines

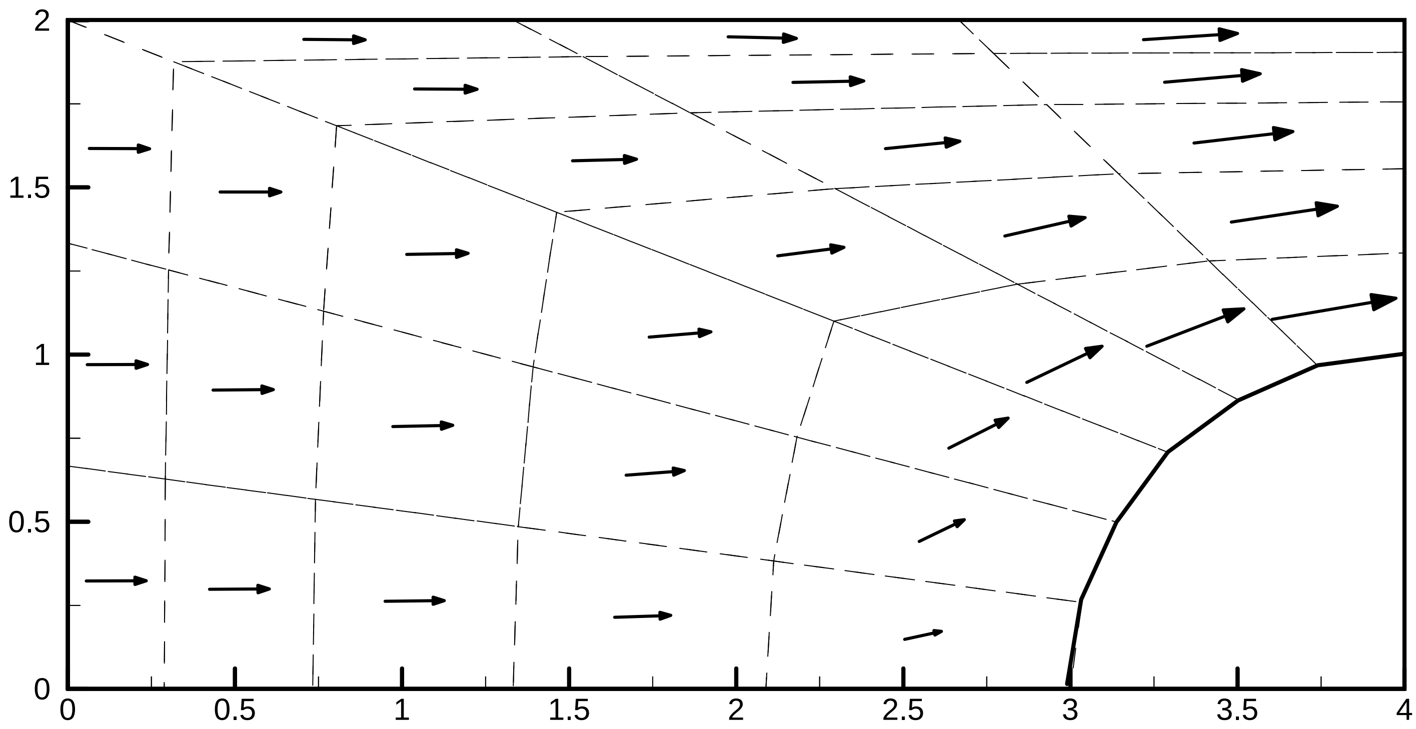


Results obtained using the Velocity potential formulation

Iso potential lines

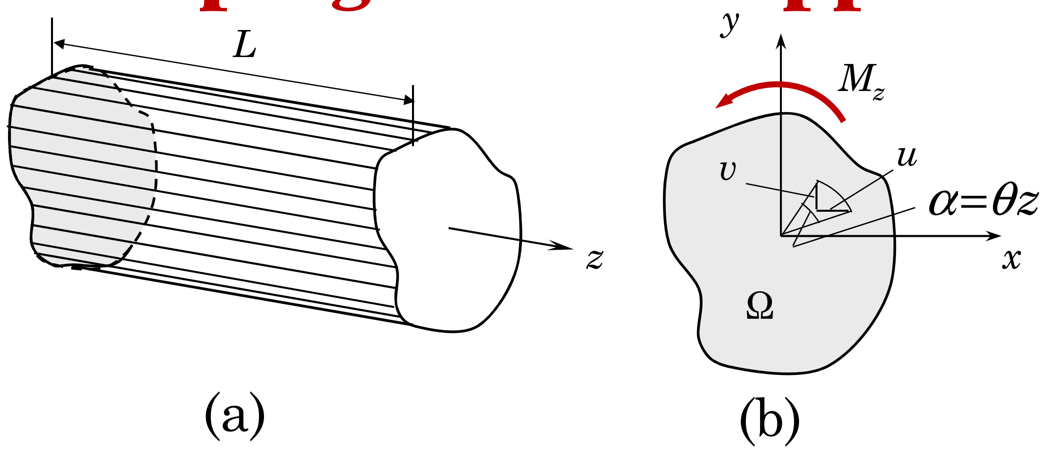


Velocity vectors for the flow around a cylinder



TORSION OF CYLINDRICAL MEMBERS

Warping function approach



$\phi(x, y)$ **– warping function**

$$u = -\theta z y, \quad v = -\theta z x, \quad w = \theta \phi(x, y)$$

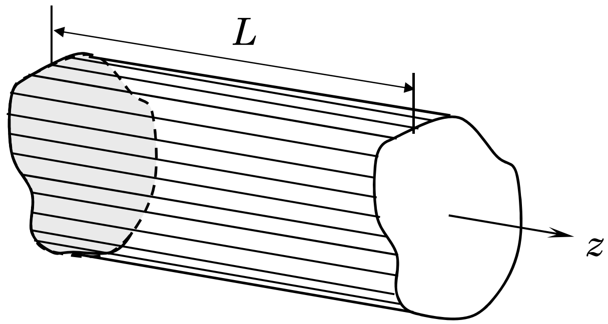
$$\sigma_{xz}(x, y) = G\theta\left(\frac{\partial \phi}{\partial x} - y\right), \quad \sigma_{yz}(x, y) = G\theta\left(\frac{\partial \phi}{\partial y} + x\right)$$

$$-G\theta\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \text{ in } \boldsymbol{\Omega}$$

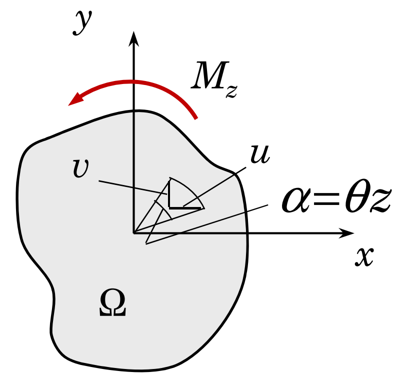
$$\left(\frac{\partial \phi}{\partial x} - y\right)n_x + \left(\frac{\partial \phi}{\partial y} - x\right)n_y = 0 \text{ on } \boldsymbol{\Gamma}$$

TORSION OF CYLINDRICAL MEMBERS

Stress function approach



(a)



(b)

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial \phi}{\partial y} - x, \quad \frac{\partial \Psi}{\partial y} = \frac{\partial \phi}{\partial x} - y$$

$$\sigma_{xz}(x,y) = G\theta\left(\frac{\partial \phi}{\partial x} - y\right) = G\theta\frac{\partial \Psi}{\partial y},$$

$\Psi(x, y)$ – **stress function**

$$\sigma_{yz}(x,y) = G\theta\left(\frac{\partial \phi}{\partial y} + x\right) = -G\theta\frac{\partial \Psi}{\partial x},$$

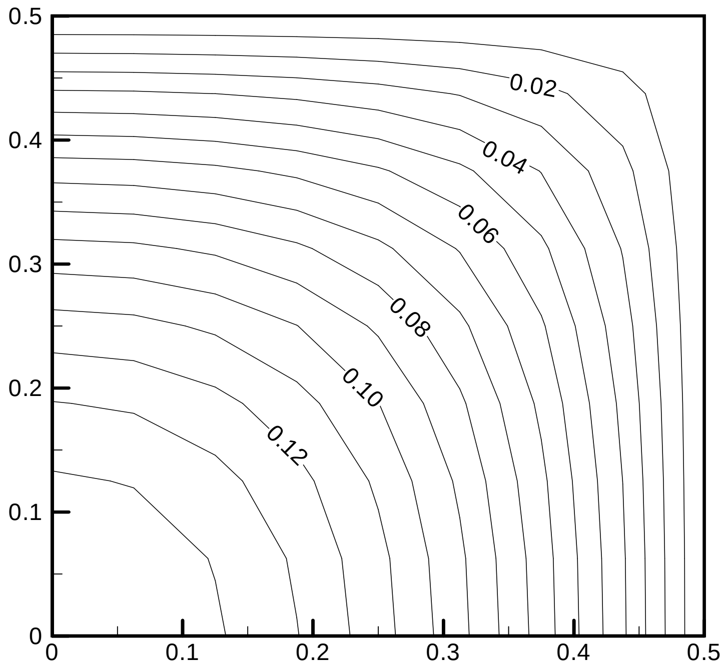
$$-\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = 2 \text{ in } \Omega$$

$$\Psi = \text{constant}, 0 \text{ on } \Gamma$$

Torsion of a Square- Section Shaft

Results obtained with 8 x 8 mesh of linear rectangular elements in a quadrant

Plots of equi-stress Function contours



Shear stress vectors

$\sigma_{xz}\,\hat{\mathbf{e}}_x + \sigma_{yz}\,\hat{\mathbf{e}}_y$