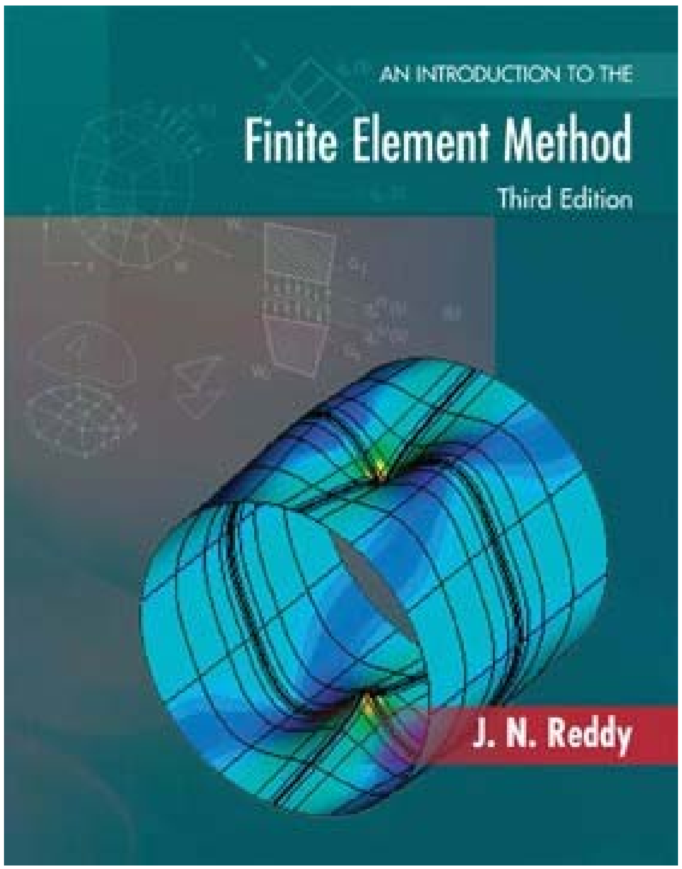
The Finite Element Method

1D Eigenvalue and Time-Dependent Problems

Read: Chapter 6



CONTENTS

- Eigenvalue problems
  - Structural problems
  - Heat transfer-like problems
- Transient problems
  - Semi-discretization
  - Time approximations
  - Mass lumping
  - Stability and accuracy
  - Numerical examples

INTRODUCTION

Equations of motion

Static problem
(set all derivatives with respect to time *t* to zero)

Construct weak form

Construct FE model

Semidiscretization
(construct weak form in spatial coordinates and semi-discrete FEM)

Full discretization
(use time approximations)

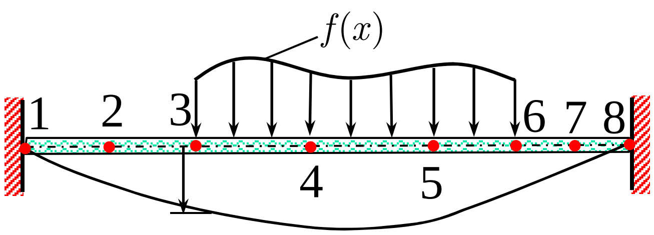
Eigenvalue problem
(replace the solution with *periodic* or *decay* type solution with respect to time *t*)

Construct weak form

Construct FE model

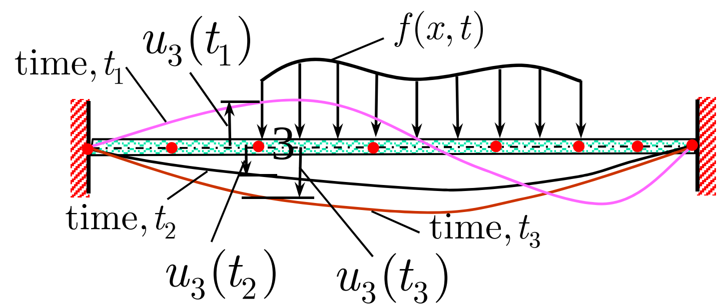
INTRODUCTION (continued)

Equilibrium (static) problem



Equilibrium configuration

Transient problem



Transient configurations

Equation of equilibrium

$$-\frac{d}{dx}\left(T\frac{du}{dx}\right) = f(x)$$

$$u(0)=0, \quad u(L)=0$$

Equation of motion

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) = f(x,t)$$

$$\text{B.C.}: \; u(0,t)=0, \qquad u(L,t)=0$$

$$\text{I.C.}: \; u(x,0)=u_0(x), \;\; \dot{u}(x,0)=v_0(x)$$

EIGENVALUE PROBLEMS

Natural vibration of bars

Governing Equation for transient response

$$\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right)-\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)=f(x,t)$$

Natural vibration

For periodic motion, set $u(x,t)=u_0(x)\,\mathrm{e}^{i\omega t}$

$$\mathrm{e}^{i\omega t}\left\{\rho A\,(i\omega)^2 u_0(x)-\frac{d}{dx}\left[EA\frac{du_0}{dx}\right]\right\}=0$$

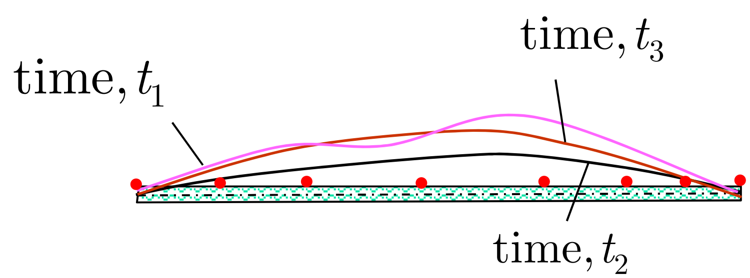
$$-\rho A\,\omega^2 u_0(x)-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right)=0$$

$$\boxed{-\rho A\,\lambda u_0(x)-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right)=0,\quad \lambda=\omega^2}$$

EIGENVALUE PROBLEMS
Heat transfer-like problems

Governing equation for transient response

$$\rho A \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = g(x,t)$$



Transient configurations

Governing equation for eigenvalue analysis: for decay type solution, set

$$g = 0, \;\; T(x,t) = T_0(x)\,\mathrm{e}^{-\lambda t}$$

$$\rho A \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = 0$$

$$\mathrm{e}^{-\lambda t}\left\{\rho A(-\lambda)T_0(x) - \frac{d}{dx}\left(kA\frac{dT_0}{dx}\right)\right\} = 0$$

$$\boxed{-\rho A\,\lambda T_0(x) - \frac{d}{dx}\left(kA\frac{dT_0}{dx}\right) = 0}$$

FE MODELS OF EIGENVALUE PROBLEMS

Weak Form

$$0 = -\int_{x_a}^{x_b} w_i \left\{ \rho A\, \lambda u_h(x) + \frac{d}{dx}\left(EA \frac{du_h}{dx} \right) \right\} dx$$

$$= \int_{x_a}^{x_b} \left(-\rho A\, \lambda w_i u_h(x) + EA \frac{dw_i}{dx}\frac{du_h}{dx} \right) dx - w_i(x_a)\, Q_1 - w_i(x_b)\, Q_2$$

$$Q_1 = -\left(EA \frac{du_h}{dx} \right)_{x_a}, \quad Q_2 = \left(EA \frac{du_h}{dx} \right)_{x_b}$$

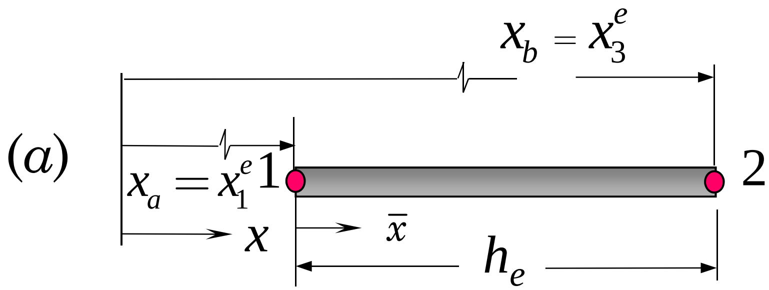
FE Model

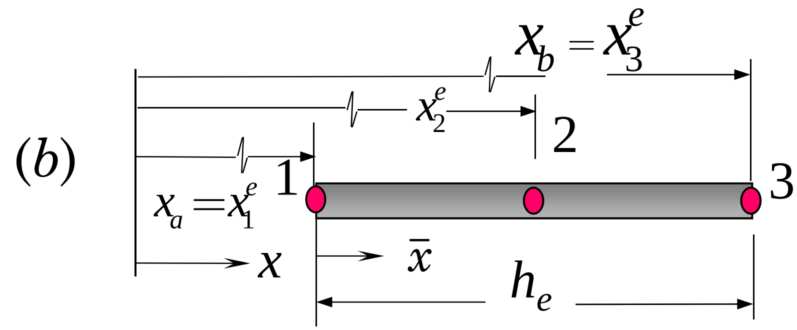
$$u_0(x) \approx u_h(x) = \sum_{j=1}^{n} u_j \psi_j(x)$$

$$\boxed{\mathbf{K}^e \mathbf{u}^e - \lambda \mathbf{M}^e \mathbf{u}^e = \mathbf{Q}^e}$$ Eigenvalue problem

$$K_{ij}^e = \int_{x_a}^{x_b} EA \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\, dx, \quad M_{ij}^e = \int_{x_a}^{x_b} \rho A\, \psi_i \psi_j\, dx$$

Explicit form of the matrices involved





Linear element

$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M^e] = \frac{c_0^e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

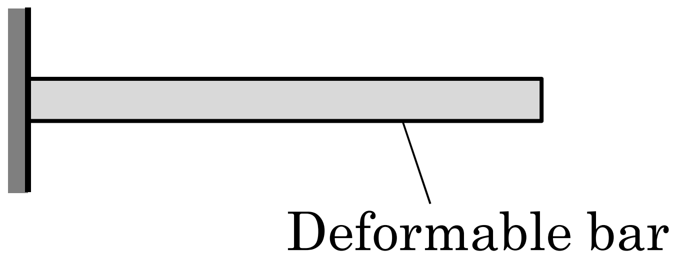
Quadratic element

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$[M^e] = \frac{c_0^e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

EXAMPLES: Axial vibrations of a bar
(fundamental frequency)

Problem 1

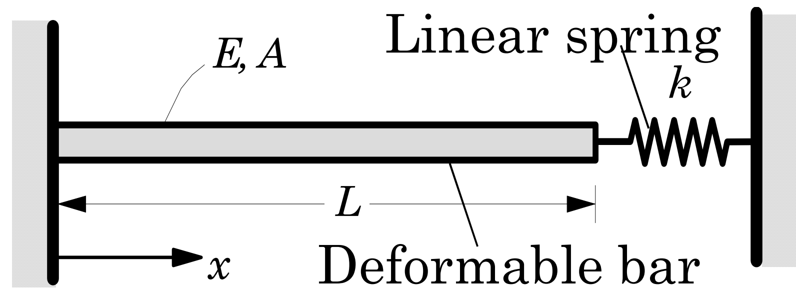


$$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\cancelto{0}{u_1}\\ u_2\end{Bmatrix}-\omega^2\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\cancelto{0}{u_1}\\ u_2\end{Bmatrix}$$

$$=\begin{Bmatrix}Q_1\\ \cancelto{0}{Q_2}\end{Bmatrix}$$

$$\left(\frac{EA}{L}-\omega^2\frac{2\rho AL}{6}\right)u_2=0 \text{ or } \omega=\sqrt{\frac{3E}{\rho L^2}}$$

Problem 2



$$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\cancelto{0}{u_1}\\ u_2\end{Bmatrix}-\omega^2\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\cancelto{0}{u_1}\\ u_2\end{Bmatrix}$$

$$=\begin{Bmatrix}Q_1\\ \cancel{Q_2}\end{Bmatrix}-ku_2$$

$$\left(\frac{EA}{L}+k-\omega^2\frac{\rho AL}{3}\right)u_2=0$$

$$\text{or } \omega=\sqrt{\frac{3(E+kL/\mathrm{A})}{\rho L^2}}$$

TRANSIENT ANALYSIS (steps involved)

Model Equation

$$c_1 \frac{\partial u}{\partial t} + c_2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(b \frac{\partial^2 u}{\partial x^2}\right) + c_0 u = f(x,t)$$

Approximate solution

$$u(x,t) \approx u_h(x,t) = \sum_{j=1}^{n} \Delta_j(t)\,\varphi_j(x)$$

1. Spatial approximation (semidiscretization)

$$\mathbf{C}\dot{\Delta} + \mathbf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F}$$

2. Time approximation (full discretization)

$$\hat{\mathbf{K}}\,\Delta_{s+1} = \mathbf{F}_{s,s+1}$$

SPATIAL APPROXIMATION

Model Equation

$$c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u=f(x,t)$$

Weak Form for semi-discretization

$$0=\int_{x_a}^{x_b} w_i(x)\left[c_1\frac{\partial u_h}{\partial t}+c_2\frac{\partial^2 u_h}{\partial t^2}-\frac{\partial}{\partial x}\left(a\frac{\partial u_h}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial u_h}{\partial x^2}\right)+c_0 u_h-f(x,t)\right]dx$$

$$=\int_{x_a}^{x_b}\left[c_1 w_i\frac{\partial u_h}{\partial t}+c_2 w_i\frac{\partial^2 u_h}{\partial t^2}+\frac{dw_i}{dx}\left(a\frac{\partial u_h}{\partial x}\right)+\frac{d^2 w_i}{dx^2}\left(b\frac{\partial^2 u_h}{\partial x^2}\right)+c_0 w_i u_h-w_i f(x,t)\right]dx$$

$$-w_i(x_a)Q_1-w_i(x_b)Q_3-\left(-\frac{dw_i}{dx}\right)_{x_a}Q_2-\left(-\frac{dw_i}{dx}\right)_{x_b}Q_4$$

SPATIAL DISCRETIZATION
Finite Element Model

Approximation

$$u(x,t) \approx u_h(x,t) = \sum_{j=1}^{n} \Delta_j(t)\varphi_j(x)$$

Finite element model

$$\mathbf{C}\dot{\Delta} + \mathbf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F}$$

$$C_{ij}^e = \int_{x_a}^{x_b} c_1\varphi_i\varphi_j\, dx, \quad M_{ij}^e = \int_{x_a}^{x_b} c_2\varphi_i\varphi_j\, dx$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(b\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2} + a\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + c_0\varphi_i\varphi_j \right) dx$$

$$F_i^e = \int_{x_a}^{x_b} f\varphi_i\, dx + \varphi_i(x_a)Q_1 + \varphi_i(x_b)Q_3 + \left(-\frac{d\varphi_i}{dx}\right)_{x_a} Q_2 + \left(-\frac{d\varphi_i}{dx}\right)_{x_b} Q_4$$

TIME APPROXIMATIONS

PARABOLIC EQUATION (heat transfer, fluid mechanics, and like problems

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

$$C_{ij}^e = \int_{x_a}^{x_b} c_1 \psi_i \psi_j \, dx, \quad K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c_0 \psi_i \psi_j \right) dx$$

$$F_i^e = \int_{x_a}^{x_b} f \, \psi_i \, dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_2$$

HYPERBOLIC EQUATION (structural mechanics problems)

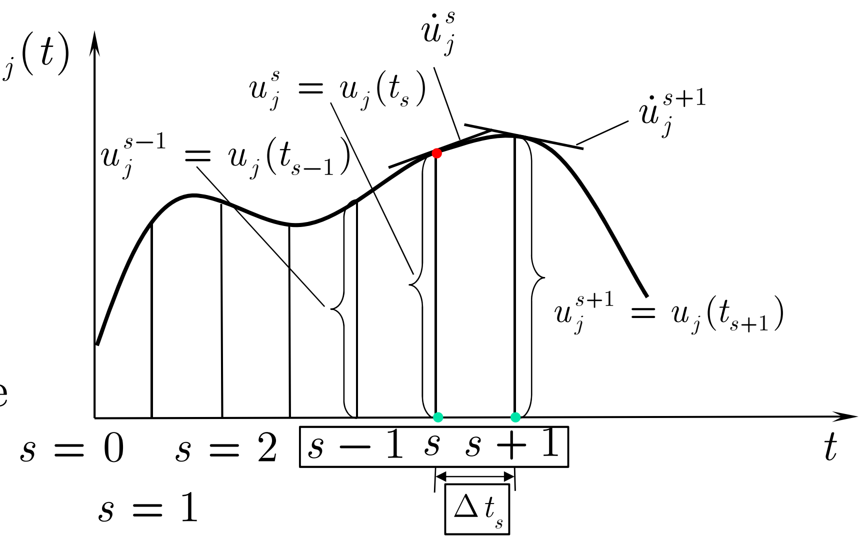
$$\mathbf{C}\dot{\Delta} + \mathbf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F}$$

TIME APPROXIMATIONS OF PARABOLIC EQUATIONS

Approximation of the first derivative

$$\dot{u}_j^s \approx \frac{u_j^{s+1} - u_j^s}{\Delta t_{s+1}}, \text{ forward difference}$$

$$\dot{u}_j^{s+1} \approx \frac{u_j^{s+1} - u_j^s}{\Delta t_{s+1}}, \text{ backward difference}$$



Alfa (α)-family of approximation

$$\alpha \dot{u}_j^{s+1} + (1-\alpha)\dot{u}_j^s \approx \frac{u_j^{s+1} - u_j^s}{\Delta t_{s+1}}, \qquad 0 \le \alpha \le 1$$

$$u_j^{s+1} = u_j^s + \Delta t_{s+1}\left[\alpha \dot{u}_j^{s+1} + (1-\alpha)\dot{u}_j^s\right]$$

TIME APPROXIMATIONS (Parabolic)

Alfa-family of approximation (*Parabolic equation*)

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F},\ \ 0 < t < T \ \Rightarrow \mathbf{C}\dot{\mathbf{u}}^{s} + \mathbf{K}_{s}\mathbf{u}^{s} = \mathbf{F}_{s},\ \mathbf{C}\dot{\mathbf{u}}^{s+1} + \mathbf{K}_{s+1}\mathbf{u}^{s+1} = \mathbf{F}_{s+1}$$

$$\mathbf{u}^{s+1} = \mathbf{u}^{s} + \Delta t_{s+1}\left[\alpha\dot{\mathbf{u}}^{s+1} + (1-\alpha)\dot{\mathbf{u}}^{s}\right]$$

$$\mathbf{C}\mathbf{u}^{s+1} = \mathbf{C}\mathbf{u}^{s} + \Delta t_{s+1}\left[\alpha\mathbf{C}\dot{\mathbf{u}}^{s+1} + (1-\alpha)\mathbf{C}\dot{\mathbf{u}}^{s}\right]$$

$$\mathbf{C}\dot{\mathbf{u}}^{s+1} = \mathbf{F}^{s+1} - \mathbf{K}_{s+1}\mathbf{u}^{s+1} \qquad \mathbf{C}\dot{\mathbf{u}}^{s} = \mathbf{F}^{s} - \mathbf{K}_{s}\mathbf{u}^{s}$$

$$\left(\mathbf{C} + \alpha\Delta t_{s+1}\mathbf{K}_{s+1}\right)\mathbf{u}^{s+1} = \left(\mathbf{C} - (1-\alpha)\Delta t_{s}\mathbf{K}_{s}\right)\mathbf{u}^{s} + \Delta t_{s+1}\left[\alpha\mathbf{F}^{s+1} + (1-\alpha)\mathbf{F}^{s}\right]$$

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}^{s+1} = \hat{\mathbf{F}}^{s+1}$$

where

$$\hat{\mathbf{K}}_{s+1} = \alpha\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C},$$

$$\hat{\mathbf{F}}^{s+1} = \left[(1-\alpha)\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C}\right]\mathbf{u}^{s} + \Delta t_{s+1}\left[\alpha\mathbf{F}^{s+1} + (1-\alpha)\mathbf{F}^{s}\right]$$

TIME APPROXIMATIONS (Hyperbolic)

Semidiscrete FE model

$$\mathbf{C}^e\dot{\mathbf{u}}^e + \mathbf{M}^e\ddot{\mathbf{u}}^e + \mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e$$

Newmark scheme (*hyperbolic equations*)

$$\mathbf{u}^{s+1} = \mathbf{u}^s + \Delta t\,\dot{\mathbf{u}}^s + \tfrac{1}{2}(\Delta t)^2\,\ddot{\mathbf{u}}^{s,\gamma}$$

$$\dot{\mathbf{u}}^{s+1} = \dot{\mathbf{u}}^s + \Delta t\,\ddot{\mathbf{u}}^{s,\alpha},\quad \ddot{\mathbf{u}}^{s,\theta} \equiv (1-\theta)\ddot{\mathbf{u}}^s + \theta\,\ddot{\mathbf{u}}^{s+1}$$

Fully discretized model

$$\hat{\mathbf{K}}_{s+1}\mathbf{u}^{s+1} = \hat{\mathbf{F}}^{s+1},\qquad \hat{\mathbf{K}}_{s+1} = \mathbf{K}_{s+1} + a_3\mathbf{M}_{s+1} + a_5\mathbf{C}_{s+1}$$

$$\hat{\mathbf{F}}^{s+1} = \mathbf{F}^{s+1} + \mathbf{M}_{s+1}\left(a_3\mathbf{u}^s + a_4\dot{\mathbf{u}}^s + a_5\ddot{\mathbf{u}}^s\right) + \mathbf{C}_{s+1}\left(a_5\mathbf{u}^s + a_6\dot{\mathbf{u}}^s + a_7\ddot{\mathbf{u}}^s\right)$$

EXPLICIT AND IMPLICIT FORMULATIONS

General form of the time-marching scheme

$$\hat{\mathbf{K}}\mathbf{u}^{s+1} = \mathbf{B}\mathbf{u}^{s} + \mathbf{F}^{s,s+1}$$

The scheme is called *explicit* if the coefficient matrix $\hat{\mathbf{K}}$ is diagonal (and hence, no inversion of equations is necessary); otherwise, the scheme is said to be implicit.

$$\hat{\mathbf{K}}^{\mathbf{s+1}}\mathbf{u}^{s+1} = \mathbf{F}^{s+1}$$

$$\hat{\mathbf{K}}^{s+1} = \alpha\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C},\ \hat{\mathbf{F}}^{s+1} = \left[(1-\alpha)\Delta t_{s+1}\mathbf{K}_{s+1} + \mathbf{C}\right]\mathbf{u}^{s} + \Delta t_{s+1}\left[\alpha\mathbf{F}^{s+1} + (1-\alpha)\mathbf{F}^{s}\right]$$

The alfa-family scheme is *explicit* if and only if

$$(1)\ \alpha = 0 \quad \text{and} \quad (2)\ \mathbf{C}\ \text{ is diagonal.}$$

ROW-SUM MASS LUMPING

For the Lagrange linear and quadratic elements we have

$$[M^e]_C = \frac{\rho A_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [M^e]_C = \frac{\rho A_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$[M^e]_L = \frac{\rho A_e h_e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [M^e]_L = \frac{\rho A_e h_e}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

STABILITY OF APPROXIMATIONS

$$\hat{\mathbf{K}}^{s+1}\mathbf{u}^{s+1} = \bar{\mathbf{K}}\mathbf{u}^{s} + \mathbf{F}^{s,s+1}$$

$$\mathbf{u}^{s+1} = \mathbf{A}\mathbf{u}^{s} + \mathbf{B}^{s,s+1}$$

The scheme is called *stable* if the repeated solution of the above equation does not result in unbounded solution $\mathbf{u}_{s+1}$. The necessary and sufficient condition for the above scheme to be stable is that the maximum eigenvalue of the coefficient matrix **A** is less than unity:

$$\lambda_{\max}^{A} \leq 1$$

STABILITY OF APPROXIMATIONS (continued)

Alfa-family of approximation scheme

$\alpha \geq \frac{1}{2}$, the scheme is *stable*

$\alpha < \frac{1}{2}$, the scheme is *conditionally stable*

$$\left(-\lambda \mathbf{C} + \mathbf{K}\right)\mathbf{u} = \mathbf{0}$$

$$\boxed{\text{Stability condition: } \Delta t \leq (\Delta t)_{\text{crit}} = \frac{2}{(1-2\alpha)\lambda_{\max}}}$$

$\alpha = 0.0$, Forward difference (Euler) scheme (conditionally stable)

$\alpha = 0.5$, Crank-Nicolson's scheme (stable)

$\alpha = \dfrac{2}{3}$, Galerkin's scheme (stable)

$\alpha = 1.0$, Backward difference scheme (stable)

STABILITY OF APPROXIMATIONS (continued)

Newmark's scheme for Structural Dynamics

$$\left(-\lambda \mathbf{M} + \mathbf{K}\right)\mathbf{u} = \mathbf{0}$$

$$\text{Stability condition:} \quad \Delta t \leq (\Delta t)_{\text{crit}} = \frac{2}{(\alpha - \gamma)\lambda_{\max}}$$

$\alpha = 0.5,\ \gamma = 2\beta = 0.5,$ Constant-average acceleration scheme (stable)

$\alpha = 0.5,\ \gamma = 2\beta = \frac{1}{3},$ Linear acceleration scheme (conditionally stable)

$\alpha = 1.5,\ \gamma = 2\beta = 1.6,$ Galerkin's scheme (stable)

$\alpha = 1.5,\ \gamma = 2\beta = 2.0,$ Backward difference scheme (stable)

$(\Delta t)_{\text{crit}}$ gets smaller as the mesh is refined.

NUMERICAL EXAMPLES

Heat transfer in a rod

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0$$

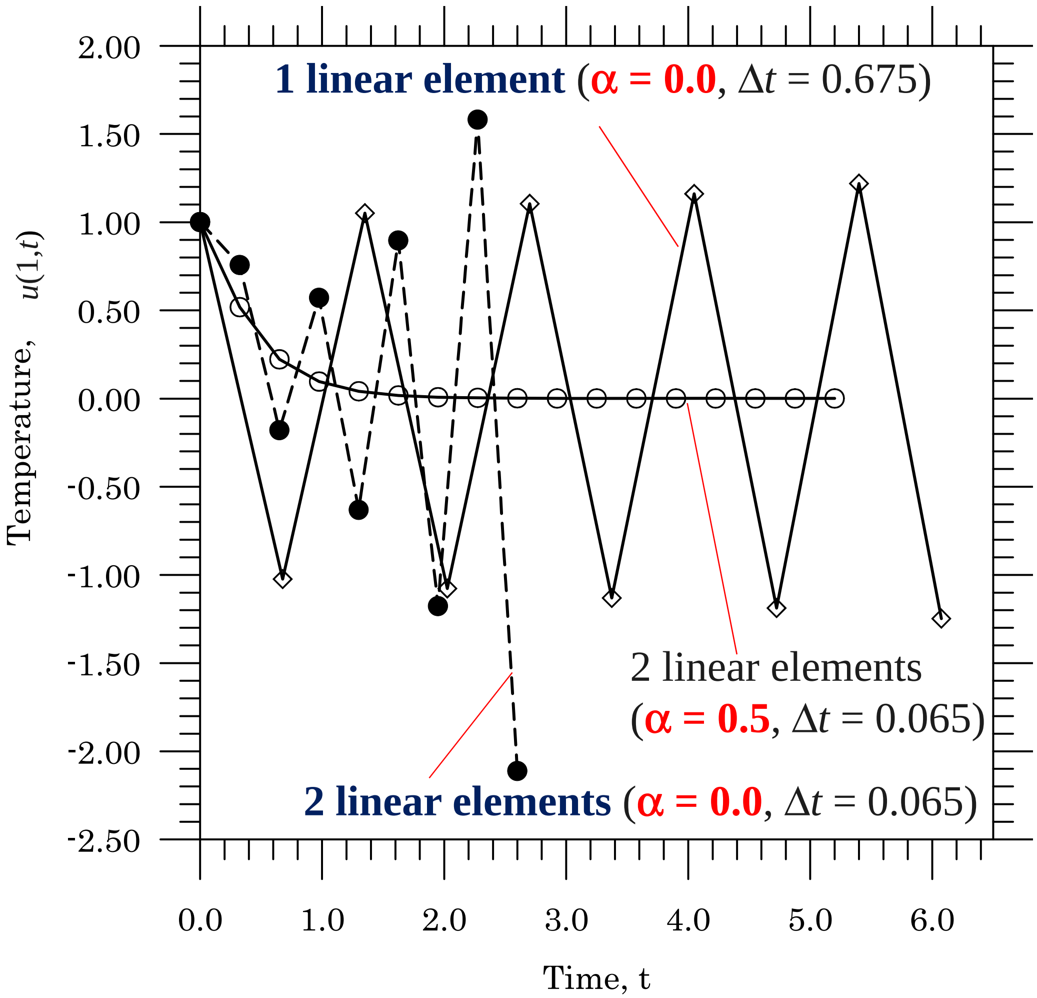
$$u(x,0) = 1$$

one-element model:

$$\Delta t_{crit} = 2/3 = 0.66667$$

two-element model:

$$\Delta t_{crit} = 2/31.689 = 0.063$$



NUMERICAL EXAMPLES (continued)

Bending of a clamped beam

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \qquad 0 < x < 1$$

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0,$$

$$w(1,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0$$

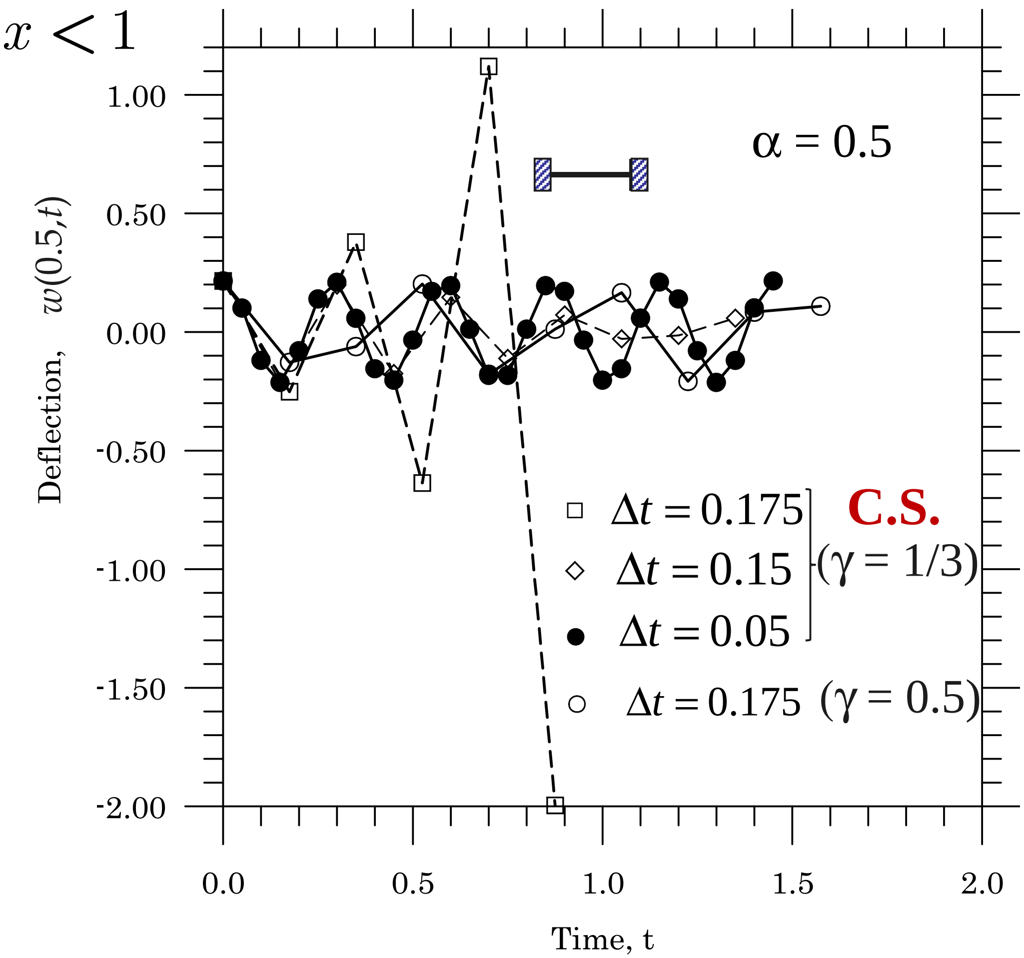
$$w(x,0) = \sin \pi x - \pi x(1-x)$$

one-element model:

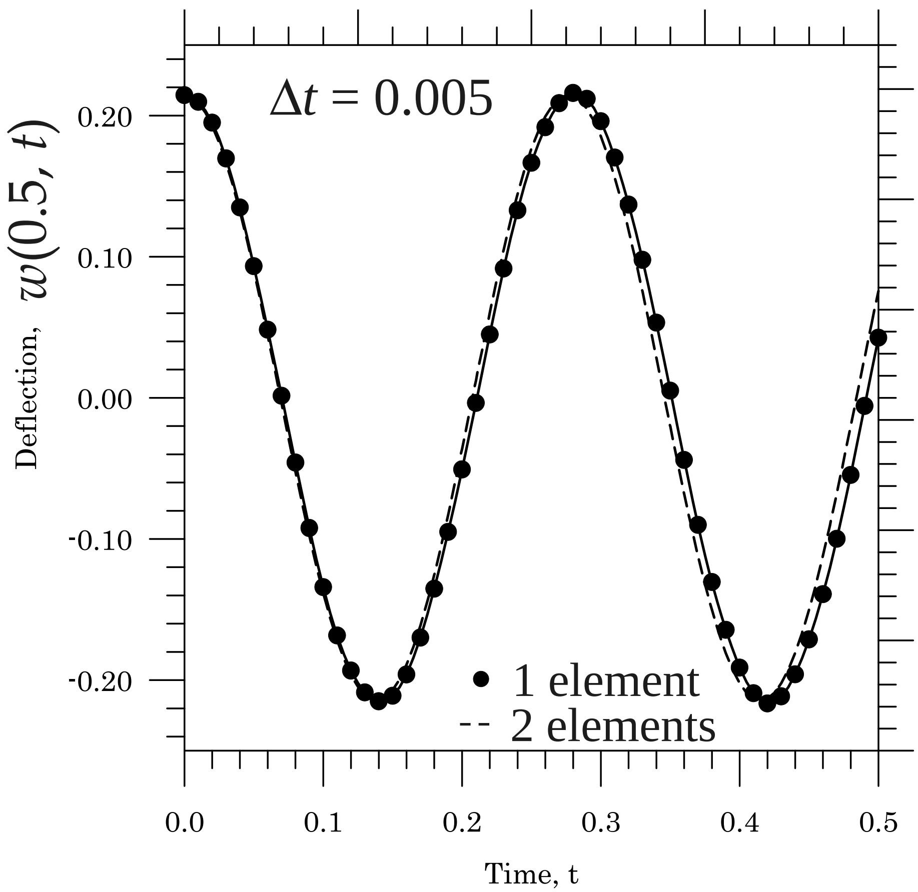
$$\Delta t_{crit} = 12/516.93$$
$$= 0.023214$$

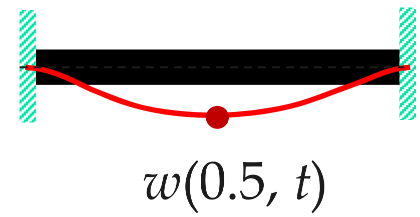
two-element model:

$$\Delta t_{crit} = 0.00897$$



NUMERICAL EXAMPLES (continued)

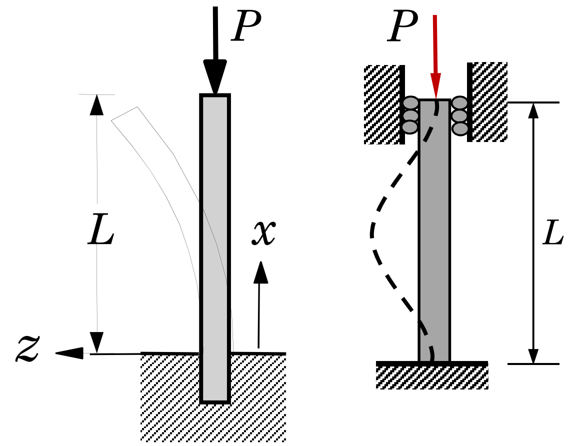




STRUCTURAL STABILITY (Buckling)

Governing Equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+P\frac{d^2w}{dx^2}=0$$



Beam-column

Weak Form over an Element

$$0=\int_{x_a}^{x_b} v\left\{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+P\frac{d}{dx}\left(\frac{dw}{dx}\right)\right\}dx$$

$$=\int_{x_a}^{x_b}\left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}-P\frac{dv}{dx}\frac{dw}{dx}\right)dx$$

$$-v(x_a)Q_1-v(x_b)Q_3-\left(-\frac{dv}{dx}\right)_{x_a}Q_2-\left(-\frac{dv}{dx}\right)_{x_b}Q_4$$

Finite Element Model

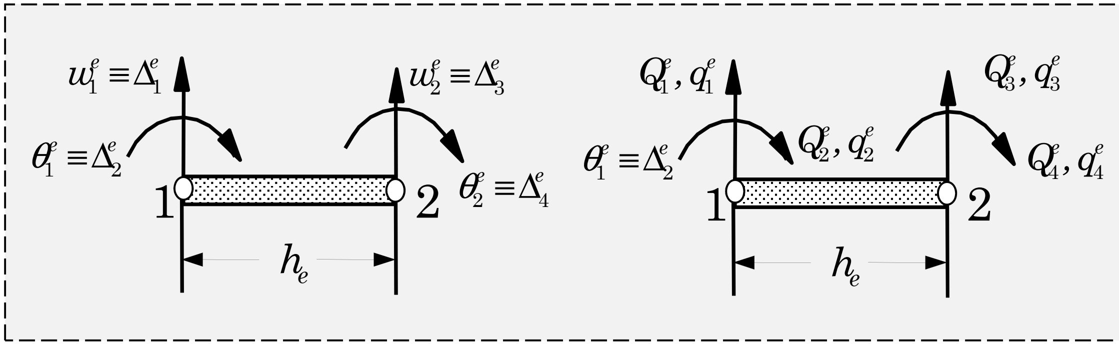
$$\left(\mathbf{K}^e-P\mathbf{G}^e\right)\boldsymbol{\Delta}^e=\mathbf{Q}^e$$

STRUCTURAL STABILITY (continued)

$$\left(\mathbf{K}^e - P\mathbf{G}^e\right)\boldsymbol{\Delta}^e = \mathbf{Q}^e$$

$$K_{ij}^e = \int_{x_a}^{x_b} EI \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}\,dx,$$

$$G_{ij}^e = \int_{x_a}^{x_b} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$$



For element-wise constant values of $E_e I_e$

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

$$\mathbf{G}^e = \frac{1}{30h_e}\begin{bmatrix} 36 & -3h_e & 36 & -3h_e \\ -3h_e & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix}$$

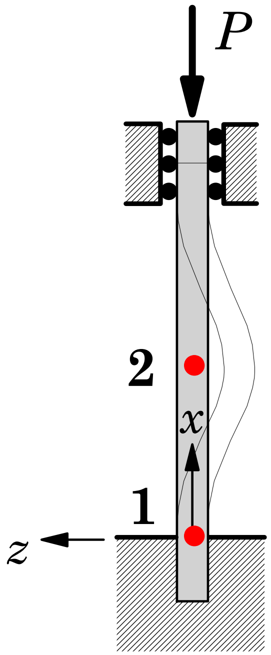
EXAMPLE: Buckling of a clamped-clamped beam

One-element (EBT) in half beam

$$\left(\frac{2EI}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 4h_e^2 \end{bmatrix} - \frac{P}{30h_e}\begin{bmatrix} 36 & -3h_e & 36 & -3h_e \\ -3h_e & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \end{Bmatrix}$$

Boundary conditions

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ U_3 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ 0 \\ Q_4^{(1)} \end{Bmatrix}$$



Solution

$$\left(\frac{12EI}{L^3} - P\frac{36}{30L}\right)U_3 = 0 \Rightarrow P = \frac{12EI}{L^3}\frac{30L}{36} = 10\frac{EI}{L^2}$$

SUMMARY

In this lecture the following topics were covered:

- Transient and eigenvalue problems and their finite element formulations
- Time-approximations of first-order and second-order equations
- Explicit and implicit schemes, mass lumping, as well as stability of the schemes
- Numerical examples to illustrate to stability and accuracy.
- Buckling of beams