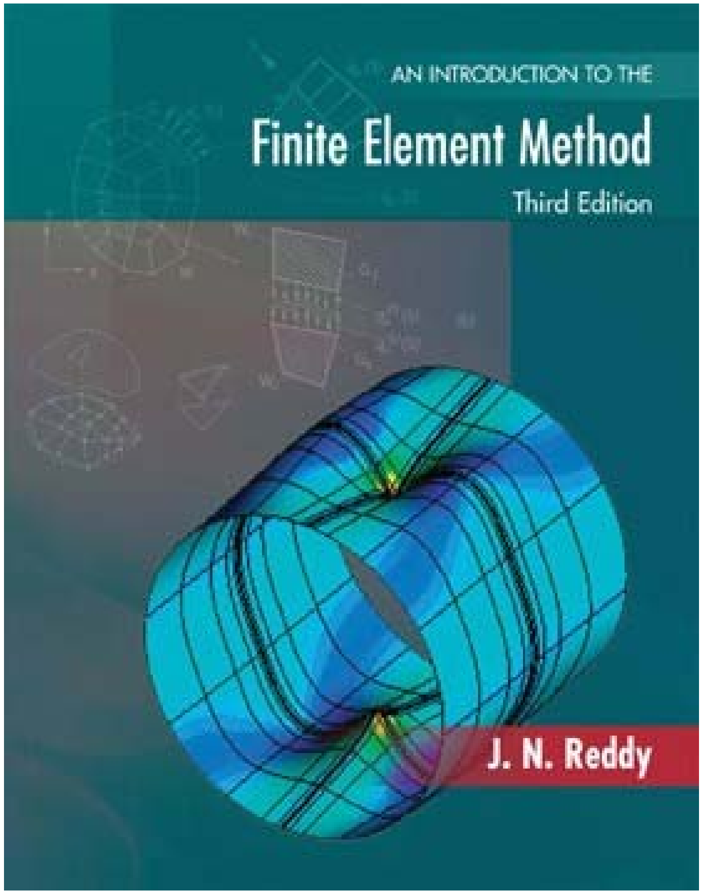
The Finite Element Method

Plane (2D) Truss and Frame Elements

Read: **Sections 4.6 and 5.4**



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- Plane truss element
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- Plane frame element
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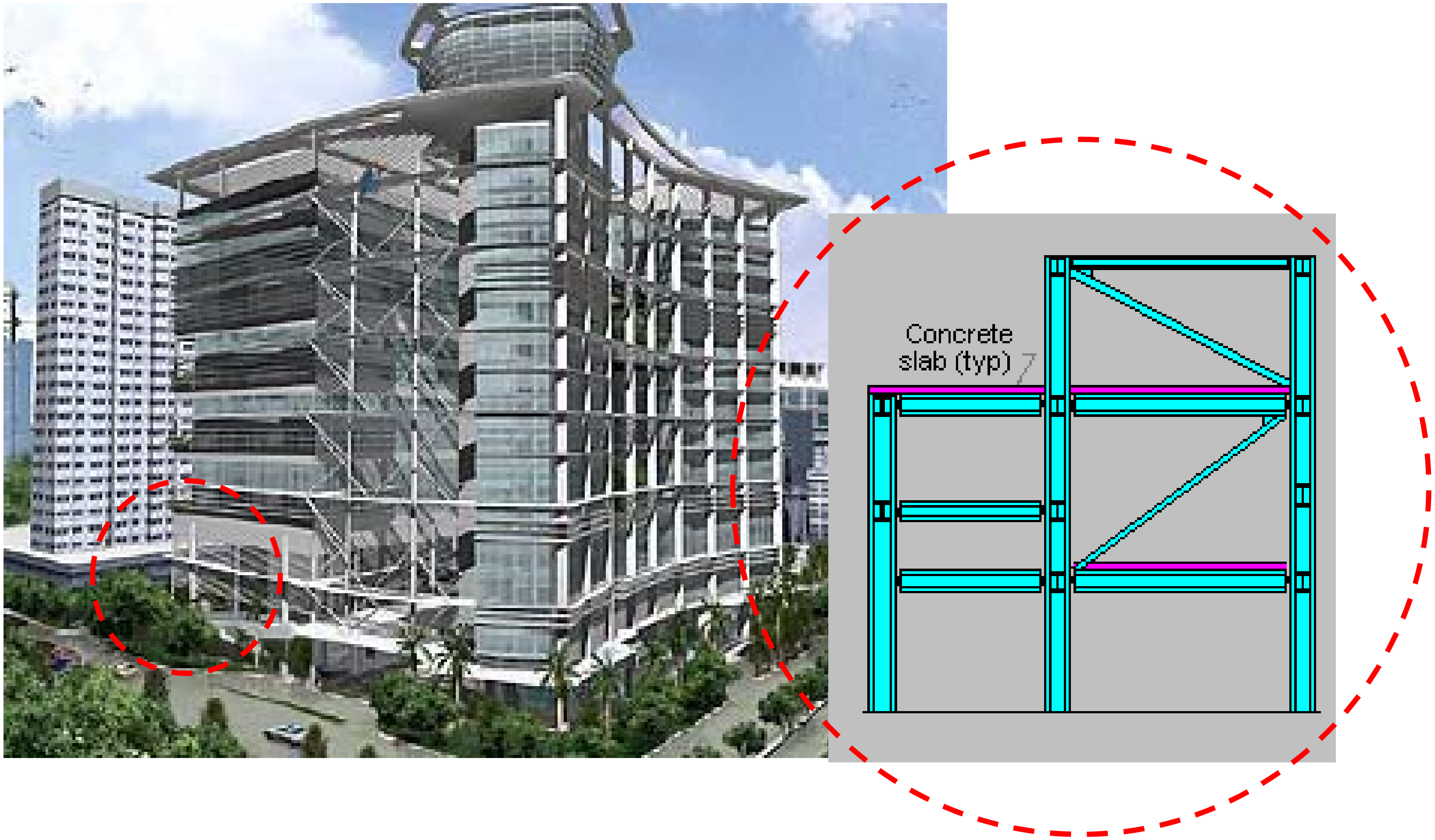
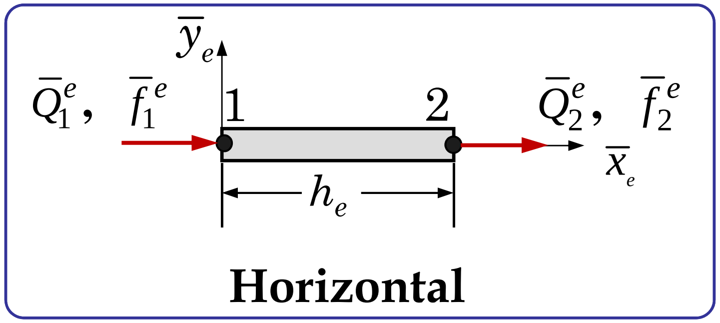
FINITE ELEMENT ANALYSIS OF PLANE TRUSSES AND FRAMES



REVIEW OF THE BAR ELEMENT

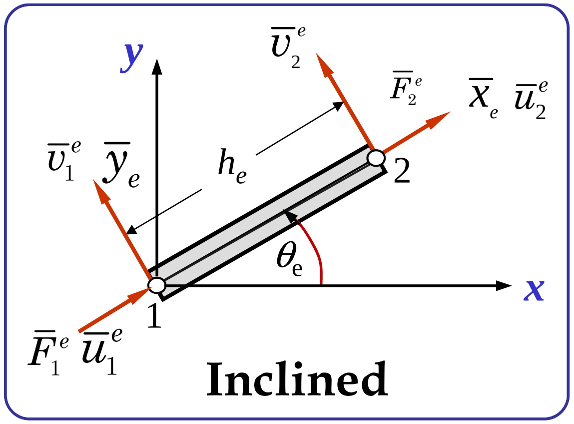
Linear bar element in the element coordinate system



$$\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{F}}^e$$

$$\bar{\mathbf{K}}^e = \frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \bar{\mathbf{F}}^e = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \end{Bmatrix}$$

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_2^e \\ 0 \end{Bmatrix}$$



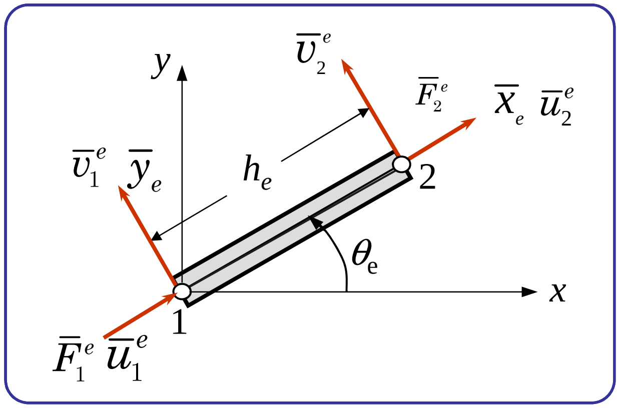
$$\bar{\mathbf{K}}^e \bar{\Delta}^e = \bar{\mathbf{F}}^e$$

Bar Element in Global Coordinates

Bar element in the element coordinates

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_2^e \\ 0 \end{Bmatrix}$$

$$\bar{\mathbf{K}}^e \bar{\Delta}^e = \bar{\mathbf{F}}^e$$

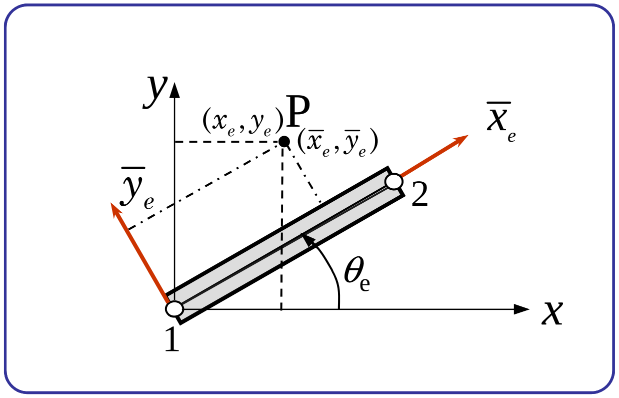


Transformation relations between the two coordinate systems

$$\bar{x}_e = x\,\cos\theta + y\,\sin\theta$$

$$\bar{y}_e = -x\,\sin\theta + y\,\cos\theta$$

$$\begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

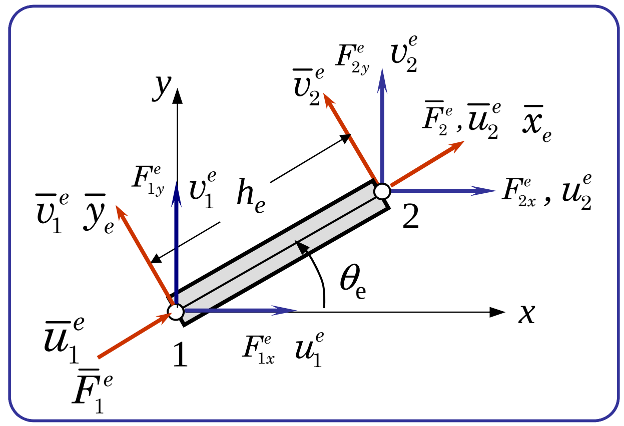


Bar Element in Global Coordinates

Transformation relations between the displacements of the two coordinate systems

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_2^e \\ v_2^e \end{Bmatrix}$$



$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix} \longrightarrow \{\bar{\Delta}^e\} = [T^e]\{\Delta^e\}$$

$$\{\bar{F}^e\} = [T^e]\{F^e\}$$

Bar Element in Global Coordinates: Truss

$$\bar{\mathbf{K}}^e \bar{\Delta}^e = \bar{\mathbf{F}}^e$$

$$[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\}, \quad [T^e]^{-1} = [T^e]^{\mathrm{T}}$$

$$[T^e]^{\mathrm{T}}[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \text{ or } \boxed{[K^e]\{\Delta^e\} = \{F^e\}}$$

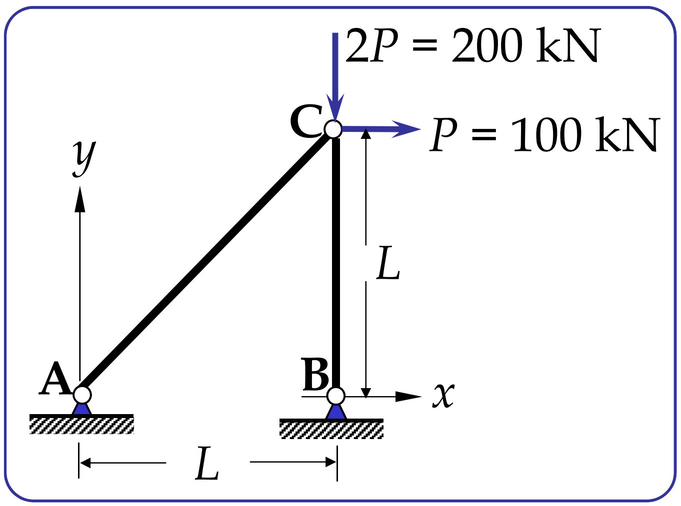
$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \quad \{F^e\} = [T^e]^{\mathrm{T}}\{\bar{F}^e\}$$

$$[K^e] = \frac{EA}{h}\begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta & -\cos^2\theta & -\frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta & -\frac{1}{2}\sin 2\theta & -\sin^2\theta \\ -\cos^2\theta & -\frac{1}{2}\sin 2\theta & \cos^2\theta & \frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & -\sin^2\theta & \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

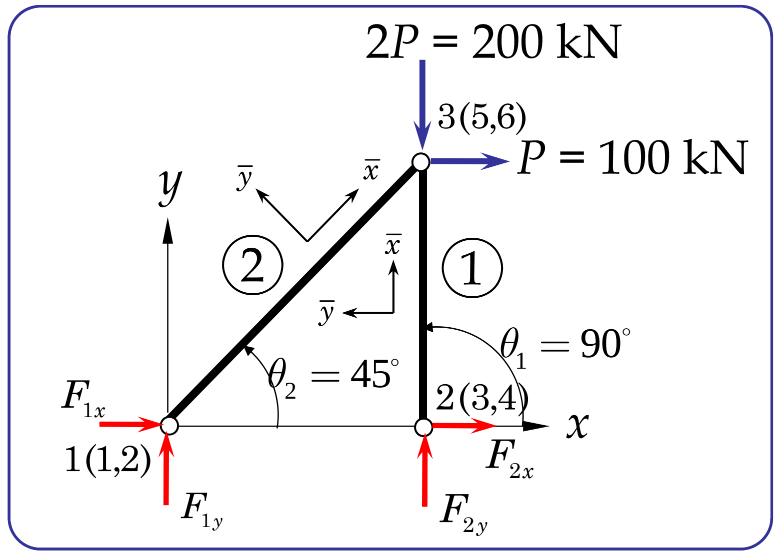
$$\{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{Bmatrix} = \begin{Bmatrix} \bar{P}_1^e \cos\theta_e \\ \bar{P}_1^e \sin\theta_e \\ \bar{P}_2^e \cos\theta_e \\ \bar{P}_2^e \sin\theta_e \end{Bmatrix} + \begin{Bmatrix} \bar{f}_1^e \cos\theta_e \\ \bar{f}_1^e \sin\theta_e \\ \bar{f}_2^e \cos\theta_e \\ \bar{f}_2^e \sin\theta_e \end{Bmatrix}$$

EXAMPLE 1

Given truss



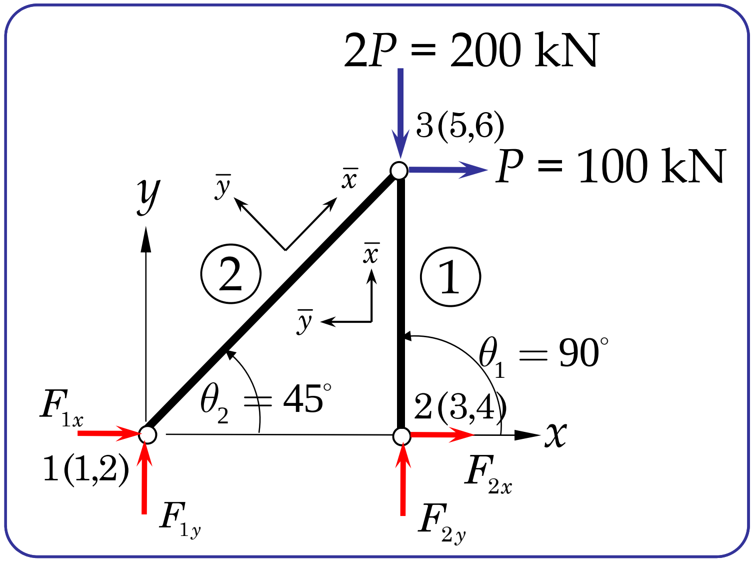
Finite element discretization



Element number	Global nodes	Geom. prop.	Mater. prop.	Orient.
1	2 3	$A,\ h_1 = L$	E	$\theta_1 = 90^\circ$
2	1 3	$A,\ h_2 = \sqrt{2}L$	E	$\theta_2 = 45^\circ$

EXAMPLE 1 (continued)

The element stiffness matrices are $[1/(2\sqrt{2}) = 0.3536]$



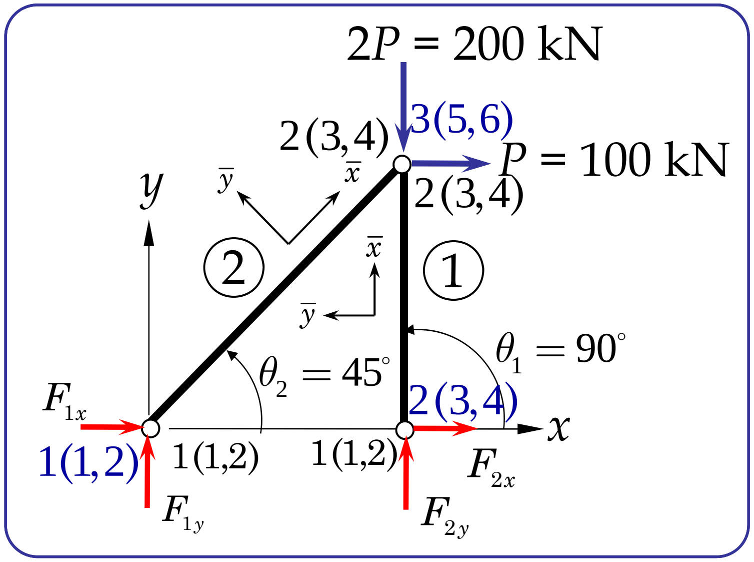
$$[K^1] = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K^2] = \frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \end{bmatrix}$$

EXAMPLE 1 (continued)

Connectivity array for 2 DoF per node truss

$$
[\mathrm{B}] = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} (1) \\ (2) \end{array} & \begin{bmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 \end{bmatrix} \end{array}
$$



Assembled stiffness coefficients

$$K_{11} = K_{11}^{(2)},\ K_{12} = K_{12}^{(2)},\ K_{13} = 0,$$

$$K_{14} = 0,\ K_{15} = K_{13}^{(2)},\ K_{16} = K_{14}^{(2)},$$

$$K_{22} = K_{22}^{(2)},\ K_{23} = 0,\ K_{24} = 0,$$

$$K_{25} = K_{23}^{(2)},\ K_{26} = K_{24}^{(2)},$$

$$K_{33} = K_{11}^{(1)},\ K_{34} = K_{12}^{(1)},\ K_{34} = K_{12}^{(1)},$$

$$K_{35} = K_{13}^{(1)},\ K_{36} = K_{14}^{(1)}, K_{34} = K_{12}^{(1)},$$

$$K_{35} = K_{13}^{(1)},\ K_{36} = K_{14}^{(1)},\ K_{44} = K_{22}^{(1)},$$

$$K_{45} = K_{23}^{(1)},\ K_{46} = K_{24}^{(1)}, K_{55} = K_{33}^{(1)} + K_{33}^{(2)},$$

$$K_{56} = K_{34}^{(1)} + K_{34}^{(2)},\ K_{66} = K_{44}^{(1)} + K_{44}^{(2)}.$$

EXAMPLE 1 (continued)

Assembled system of equations of the truss

$$
\frac{EA}{L}\left[\begin{array}{cccc|cc}
0.3536 & 0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
 & 0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
 & & 0.0 & 0.0 & 0.0 & 0.0 \\
 & & & 1.0 & 0.0 & -1.0 \\
\text{symm.} & --- & --- & --- & ---- & ---- \\
 & & & & 0.3536 & 0.3536 \\
 & & & & & 1.3536
\end{array}\right]
\left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \\ \hdashline U_3 \\ V_3 \end{array}\right\}
=
\left\{\begin{array}{c} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ \hdashline F_{x3} \\ F_{y3} \end{array}\right\}
$$

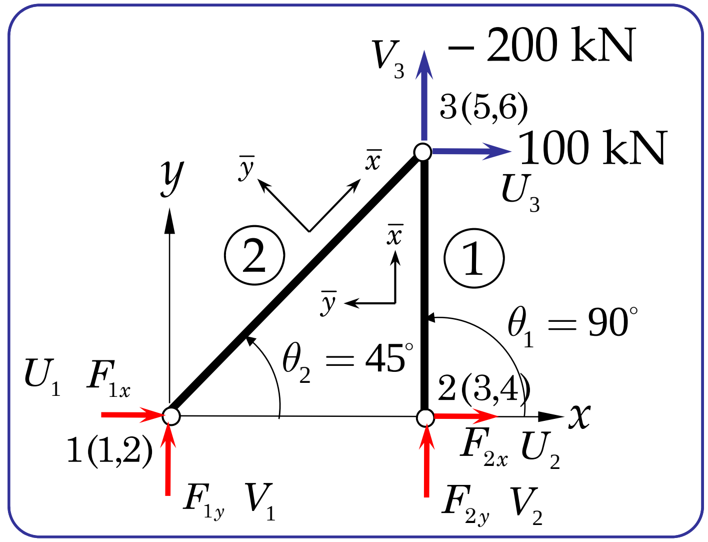
The displacement continuity conditions are

$$
u_1^1 = u_1^3 = U_1, \quad v_1^1 = v_1^3 = V_1
$$

$$
u_2^1 = u_1^2 = U_2, \quad v_2^1 = v_1^2 = V_2
$$

$$
u_2^2 = u_2^3 = U_3, \quad v_2^2 = v_2^3 = V_3
$$

EXAMPLE 1 (continued)



where the global forces and displacements are

$$F_1^1 + F_1^3 = F_x^1, \quad F_2^1 + F_2^3 = F_y^1$$
$$F_3^1 + F_1^2 = F_x^2, \quad F_4^1 + F_2^2 = F_y^2$$
$$F_3^2 + F_3^3 = F_x^3, \quad F_4^2 + F_4^3 = F_y^3$$

$$\{\Delta\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_2^3 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 + F_3^3 \\ F_4^2 + F_4^3 \end{Bmatrix} = \begin{Bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \\ F_x^3 \\ F_y^3 \end{Bmatrix}$$

Boundary conditions

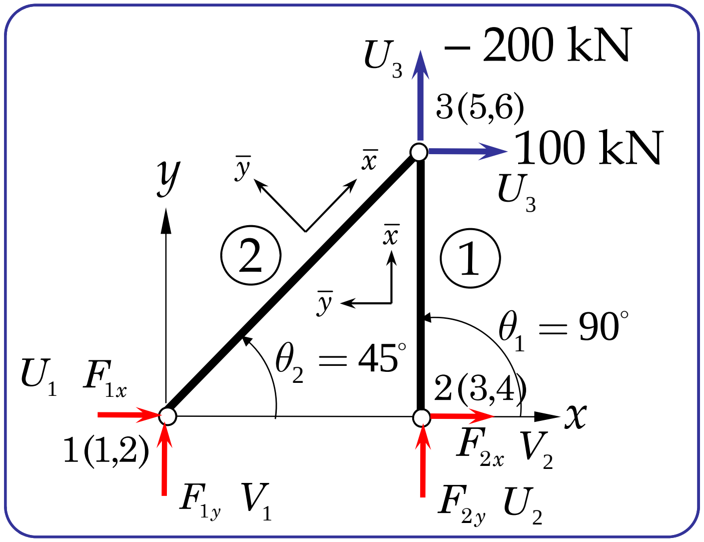
$$U_1 = V_1 = U_2 = V_2 = 0, \quad F_x^3 = P, \quad F_y^3 = -2P$$

EXAMPLE 1 (continued)

Solution

$$\frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$$

$$\begin{Bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} -0.3536 & -0.3536 \\ -0.3536 & -0.3536 \\ 0.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix}$$



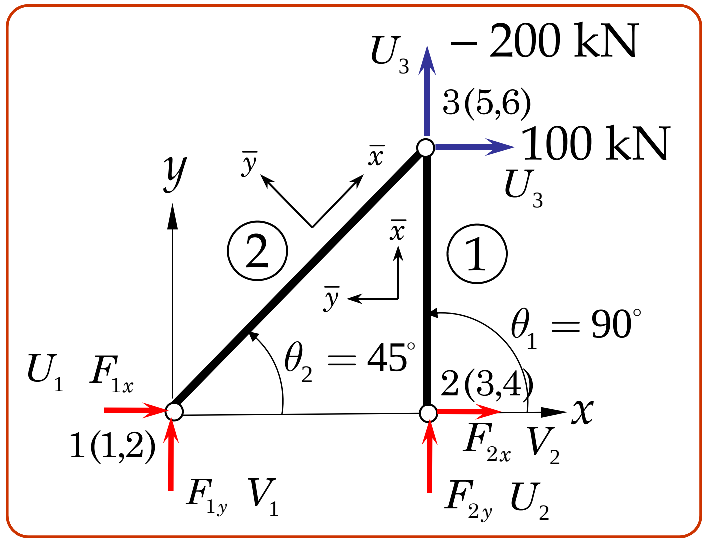
$(P = 10^5\ \text{N},\ EA = 10^8\ \text{N})$

$$U_3 = (3+2\sqrt{2})\frac{PL}{EA} = 5.828\,\frac{PL}{EA},\quad V_3 = -\frac{3PL}{EA}\ (\text{m})$$

$$F_1^{(1)} = -F_2^{(1)} = 3P;\ \ F_1^{(2)} = -F_2^{(2)} = -\sqrt{2}P \qquad (\text{N})$$

EXAMPLE 1 (continued)

Post-computation of member displacements and stresses



$$\sigma^e = -\frac{\bar{P}_1^e}{A_e} = \frac{\bar{P}_2^e}{A_e}$$

$$\begin{Bmatrix} \bar{P}_1^e \\ \bar{P}_2^e \end{Bmatrix} = \frac{A_e E_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix}$$

We have

$$u_1^1 = v_1^1 = u_1^2 = v_1^2 = 0;\ u_2^1 = u_2^2 = U_3 = (3 + 2\sqrt{2})\frac{PL}{AE},\ v_2^1 = v_2^2 = V_3 = -\frac{3PL}{AE}$$

EXAMPLE 1 (continued)

Post-computation of member displacements and stresses

$$\bar{u}_2^1 = U_3 \cos\theta_1 + V_3 \sin\theta_1 = V_3 = -\frac{3PL}{A}$$

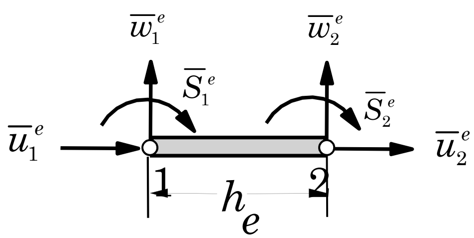
$$\bar{u}_2^2 = U_3 \cos\theta_2 + V_3 \sin\theta_2 = \frac{1}{\sqrt{2}}\left(U_3 + V_3\right)$$

$$\bar{P}_1^1 = -\bar{P}_2^1 = 3P,\ \bar{P}_1^2 = -\bar{P}_2^2 = -\sqrt{2}P$$

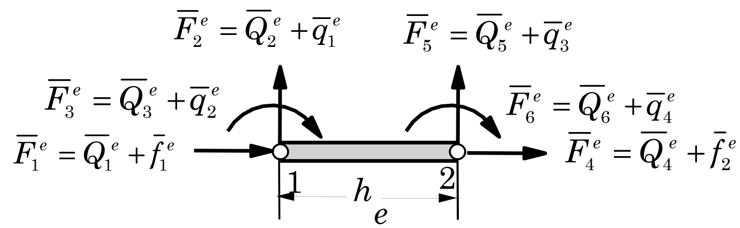
$$\sigma^{(1)} = -\frac{3P}{A},\ \ \sigma^{(2)} = \frac{\sqrt{2}P}{A}$$

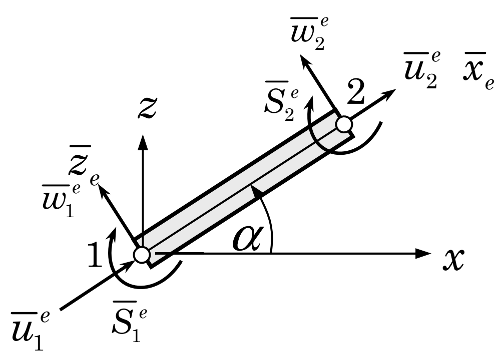
PLANE FRAME STRUCTURES

Displ. degrees of freedom in the element coordinates

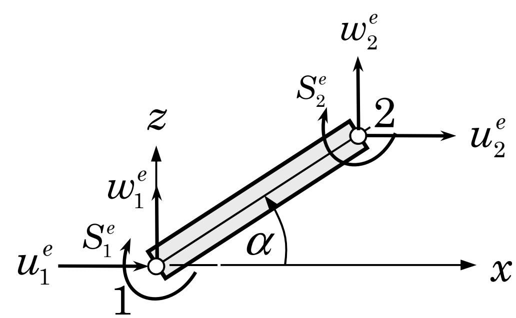


Force degrees of freedom in the element coordinates





Displ. degrees of freedom in the local coordinates



Displ. degrees of freedom in the global coordinates

Frame Element in Global Coordinates

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{w}_1^e \\ \bar{\theta}_1^e \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ w_1^e \\ \theta_1^e \end{Bmatrix}, \quad \begin{Bmatrix} \bar{u}_2^e \\ \bar{w}_2^e \\ \bar{\theta}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2^e \\ w_2^e \\ \theta_2^e \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{w}_1^e \\ \bar{\theta}_1^e \\ \bar{u}_2^e \\ \bar{w}_2^e \\ \bar{\theta}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & & & \\ -\sin\alpha & \cos\alpha & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \\ & & & \cos\alpha & \sin\alpha & 0 \\ & \mathbf{0} & & -\sin\alpha & \cos\alpha & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ w_1^e \\ \theta_1^e \\ u_2^e \\ w_2^e \\ \theta_2^e \end{Bmatrix} \qquad \bar{\Delta}^e = \mathbf{T}^e \Delta^e$$

$$[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\}, \quad [T^e]^{-1} = [T^e]^{\mathrm{T}}$$

$$[T^e]^{\mathrm{T}}[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \text{ or } [K^e]\{\Delta^e\} = \{F^e\}$$

$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \quad \{F^e\} = [T^e]^{\mathrm{T}}\{\bar{F}^e\}$$

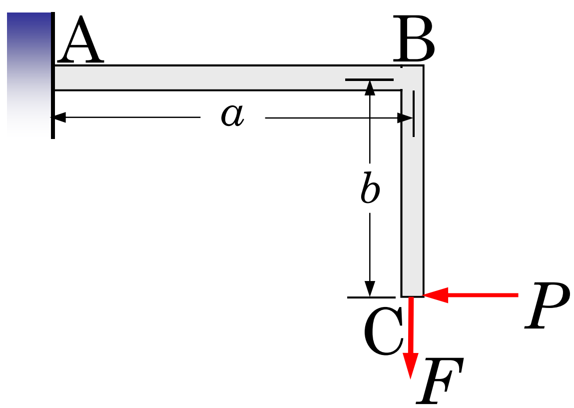
Frame Element in Global Coordinates

(the Euler-Bernoulli beam frame element)

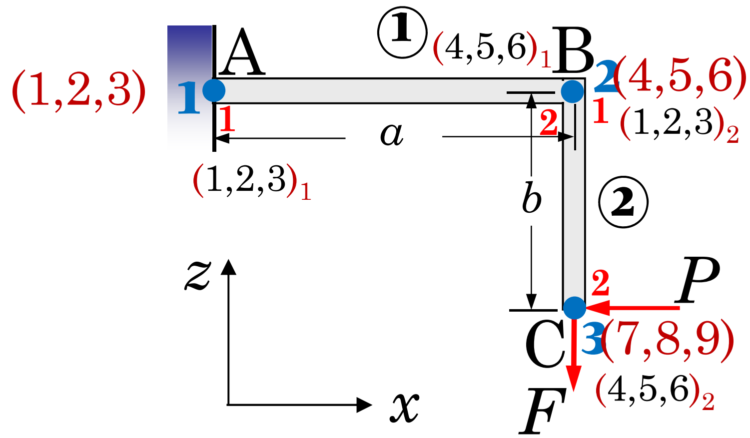
$$
\mathbf{K}^e = \frac{2E_eI_e}{h_e^3}\begin{bmatrix}
\kappa_e\cos^2\alpha_e + 6\sin^2\alpha_e & (\kappa_e-6)\cos\alpha_e\sin\alpha_e & 3h_e\sin\alpha_e & -(\kappa_e\cos^2\alpha_e + 6\sin^2\alpha_e) & -(\kappa_e-6)\cos\alpha_e\sin\alpha_e & 3h_e\sin\alpha_e \\
(\kappa_e-6)\cos\alpha_e\sin\alpha_e & \kappa_e\sin^2\alpha_e + 6\cos^2\alpha_e & -3h_e\cos\alpha_e & -(\kappa_e-6)\cos\alpha_e\sin\alpha_e & -(\kappa_e\sin^2\alpha_e + 6\cos^2\alpha_e) & -3h_e\cos\alpha_e \\
3h_e\sin\alpha_e & -3h_e\cos\alpha_e & 2h_e^2 & -3h_e\sin\alpha_e & 3h_e\cos\alpha_e & h_e^2 \\
-(\kappa_e\cos^2\alpha_e + 6\sin^2\alpha_e) & -(\kappa_e-6)\sin\alpha_e\cos\alpha_e & -3h_e\sin\alpha_e & \kappa_e\cos^2\alpha_e + 6\sin^2\alpha_e & (\kappa_e-6)\cos\alpha_e\sin\alpha_e & -3h_e\sin\alpha_e \\
-(\kappa_e-6)\cos\alpha_e\sin\alpha_e & -(\kappa_e\sin^2\alpha_e + 6\cos^2\alpha_e) & 3h_e\cos\alpha_e & (\kappa_e-6)\cos\alpha_e\sin\alpha_e & \kappa_e\sin^2\alpha_e + 6\cos^2\alpha_e & 3h_e\cos\alpha_e \\
3h_e\sin\alpha_e & -3h_e\cos\alpha_e & h_e^2 & -3h_e\sin\alpha_e & 3h_e\cos\alpha_e & 2h_e^2
\end{bmatrix}
$$

$$
\mathbf{F}^e = \begin{Bmatrix}
F_1^e\cos\alpha_e - F_2^e\sin\alpha_e \\
F_1^e\sin\alpha_e + F_2^e\cos\alpha_e \\
F_3^e \\
F_4^e\cos\alpha_e - F_5^e\sin\alpha_e \\
F_4^e\sin\alpha_e + F_5^e\cos\alpha_e \\
F_6^e
\end{Bmatrix}, \qquad \kappa_e = \frac{A_eh_e^2}{2I_e}
$$

AN EXAMPLE OF A FRAME STRUCTURE



Given structure



Finite element discretization

Connectivity array for 3 DoF per node

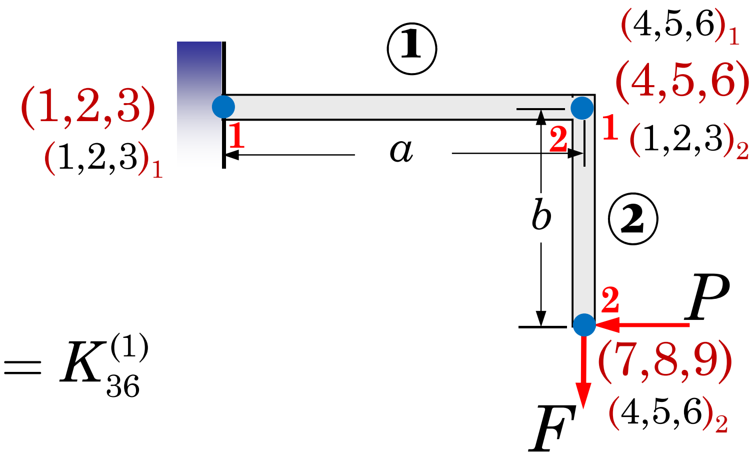
$$
[B] = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} (1) \\ (2) \end{array} & \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix} \end{array}
$$

$$
\alpha_1 = 0^0, \;\; \alpha_2 = -90^0 \text{ or } 270^0
$$

JN Reddy

ASSEMBLY OF ELEMENT MATRICES

Global stiffness coefficients in terms of element stiffness coefficients



$$K_{11} = K_{11}^{(1)},\ K_{12} = K_{12}^{(1)},\ \ldots,\ K_{16} = K_{16}^{(1)}$$

$$K_{22} = K_{22}^{(1)},\ K_{23} = K_{23}^{(1)},\ \ldots,\ K_{26} = K_{26}^{(1)}$$

$$K_{33} = K_{33}^{(1)},\ K_{34} = K_{34}^{(1)},\ K_{35} = K_{35}^{(1)},\ K_{36} = K_{36}^{(1)}$$

$$K_{44} = K_{44}^{(1)} + K_{11}^{(2)},\ K_{45} = K_{45}^{(1)} + K_{12}^{(2)},$$

$$K_{46} = K_{46}^{(1)} + K_{13}^{(2)},\ K_{55} = K_{55}^{(1)} + K_{22}^{(2)},$$

$$K_{56} = K_{56}^{(1)} + K_{23}^{(2)},\ K_{66} = K_{66}^{(1)} + K_{33}^{(2)},$$

$$K_{47} = K_{14}^{(2)},\ K_{48} = K_{15}^{(2)},\ K_{49} = K_{16}^{(2)},$$

$$K_{57} = K_{24}^{(2)},\ K_{58} = K_{25}^{(2)},\ K_{59} = K_{26}^{(2)},$$

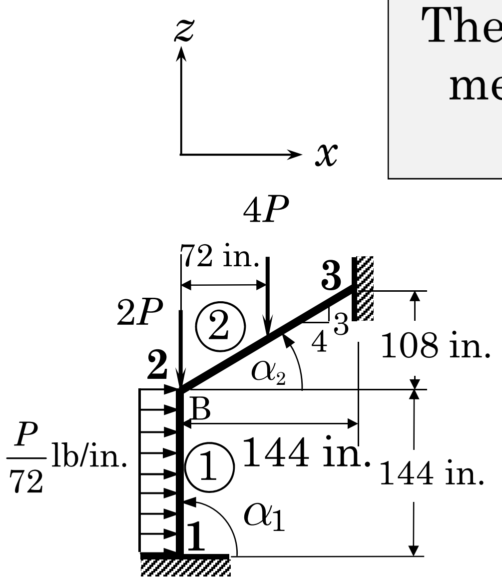
$$K_{67} = K_{24}^{(2)},\ K_{68} = K_{35}^{(2)},\ K_{69} = K_{26}^{(2)},$$

$$K_{77} = K_{44}^{(2)},\ K_{78} = K_{45}^{(2)},\ K_{79} = K_{46}^{(2)},$$

$$K_{88} = K_{55}^{(2)},\ K_{89} = K_{56}^{(2)},\ K_{99} = K_{66}^{(2)}.$$

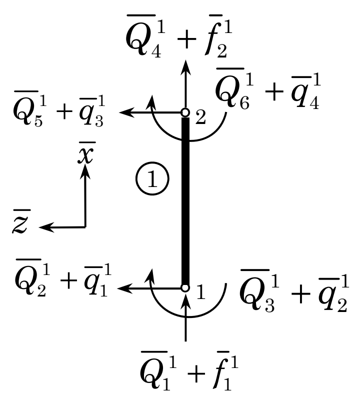
$$[B] = \begin{array}{c} \\ (1) \\ (2) \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix} \end{array}$$

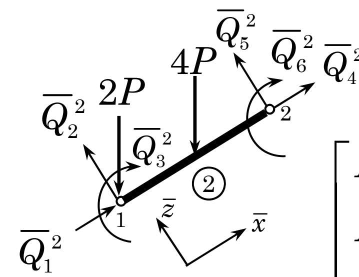
EXAMPLE 2 (from the textbook)



The y-axis is <u>into</u> the plane of the paper, and α is measured counterclockwise from x-axis to $\bar{x}$-axis

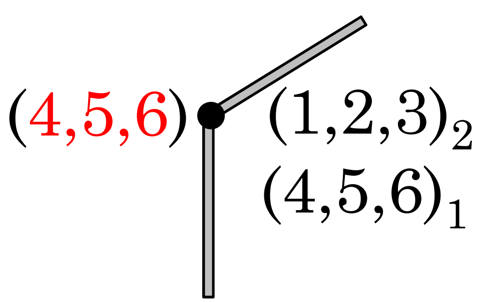
$A = 10$ in.^{2}, $I = 10$ in.^{4}, $E = 30\times10^6$ psi





$$\begin{bmatrix} K_{44} & K_{45} & K_{46} \\ K_{54} & K_{55} & K_{56} \\ K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

$$\alpha_1 = 90^\circ,\ \ \alpha_2 = \tan^{-1}\left(\tfrac{3}{4}\right)$$



$$K_{44} = K_{44}^{(1)} + K_{11}^{(2)},\ K_{45} = K_{45}^{(1)} + K_{12}^{(2)},$$
$$K_{46} = K_{46}^{(1)} + K_{13}^{(2)},\ K_{55} = K_{55}^{(1)} + K_{22}^{(2)},$$
$$K_{56} = K_{56}^{(1)} + K_{23}^{(2)},\ K_{66} = K_{66}^{(1)} + K_{33}^{(2)},$$
$$F_4 = F_4^{(1)} + F_1^{(2)} = P$$
$$F_5 = F_5^{(1)} + F_2^{(2)} = (-2P) + (-2P) = -4\,\mathrm{P}$$
$$F_6 = F_6^{(1)} + F_3^{(2)} = 24P - 72P = -48P$$

The rest of the calculations can be found on pp. 278-281 of the Book.

SUMMARY

In this lecture we have covered the following topics:

- Plane truss element in the local and global coordinates
- Transformation of element equations from element coordinates to global coordinates
- Examples, illustrating assembly, application of boundary conditions, and calculation of element forces and stresses
- Plane frame element in the local and global coordinates
- Transformation of element equations from element coordinates to global coordinates