The Finite Element Method

Plane (2D) Truss and Frame Elements

Read: Sections 4.6 and 5.4

CONTENTS

• Review of bar finite element in the local coordinates
• Plane truss element
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• Plane frame element
• Numerical examples

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FINITE ELEMENT ANALYSIS OF PLANE TRUSSES AND FRAMES
**REVIEW OF THE BAR ELEMENT**

Linear bar element in the element coordinate system

\[
\mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e
\]

\[
\mathbf{K}^e = \frac{E_e A_e}{h_e} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}, \quad \mathbf{F}^e = \frac{f_e h_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \end{bmatrix}
\]

\[
\frac{E_e A_e}{h_e} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\bar{u}_1^e \\
\bar{v}_1^e \\
\bar{u}_2^e \\
\bar{v}_2^e
\end{bmatrix} = \begin{bmatrix}
\bar{F}_1^e \\
0 \\
\bar{F}_2^e \\
0
\end{bmatrix}
\]

\[
\mathbf{K}^e \Delta^e = \mathbf{F}^e
\]
Bar Element in Global Coordinates

Bar element in the element coordinates

\[
\frac{E_e A_e}{h_e} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\bar{u}_1^e \\
\bar{v}_1^e \\
\bar{u}_2^e \\
\bar{v}_2^e
\end{bmatrix} = \begin{bmatrix}
\bar{F}_1^e \\
0 \\
\bar{F}_2^e \\
0
\end{bmatrix}
\]

\[
\bar{K}^e \Delta^e = \bar{F}^e
\]

Transformation relations between the two coordinate systems

\[
\begin{align*}
\bar{x}_e &= x \cos \theta + y \sin \theta \\
\bar{y}_e &= -x \sin \theta + y \cos \theta
\end{align*}
\]

\[
\begin{bmatrix}
\bar{x}_e \\
\bar{y}_e
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]
Bar Element in Global Coordinates

Transformation relations between the displacements of the two coordinate systems

\[
\begin{bmatrix}
\bar{u}_1^e \\
\bar{v}_1^e \\
\bar{u}_2^e \\
\bar{v}_2^e
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
u_1^e \\
v_1^e \\
u_2^e \\
v_2^e
\end{bmatrix},
\]

\[
\begin{bmatrix}
\bar{u}_1^c \\
\bar{v}_1^c \\
\bar{u}_2^c \\
\bar{v}_2^c
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_c & \sin \theta_c & 0 & 0 \\
-\sin \theta_c & \cos \theta_c & 0 & 0 \\
0 & 0 & \cos \theta_c & \sin \theta_c \\
0 & 0 & -\sin \theta_c & \cos \theta_c
\end{bmatrix}
\begin{bmatrix}
u_1^c \\
v_1^c \\
u_2^c \\
v_2^c
\end{bmatrix},
\]

\[
\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\}
\]

\[
\{\bar{F}^e\} = [T^e]\{F^e\}
\]
Bar Element in Global Coordinates: Truss

\[
\begin{align*}
\bar{K}^e (\Delta^e) &= \bar{F}^e \\
[T^e]^{-1} &= [T^e]^T \\
[T^e]^T [\bar{K}^e] [T^e] \{\Delta^e\} &= \{F^e\} \quad \text{or} \quad [K^e] \{\Delta^e\} = \{F^e\} \\
[K^e] &= [T^e]^T [\bar{K}^e] [T^e], \quad \{F^e\} = [T^e]^T \{\bar{F}^e\}
\end{align*}
\]

\[
[K^e] = \frac{EA}{h} \begin{bmatrix}
\cos^2 \theta & \frac{1}{2} \sin 2\theta & -\cos^2 \theta & -\frac{1}{2} \sin 2\theta \\
\frac{1}{2} \sin 2\theta & \sin^2 \theta & -\frac{1}{2} \sin 2\theta & -\sin^2 \theta \\
-\cos^2 \theta & -\frac{1}{2} \sin 2\theta & \cos^2 \theta & \frac{1}{2} \sin 2\theta \\
-\frac{1}{2} \sin 2\theta & -\sin^2 \theta & \frac{1}{2} \sin 2\theta & \sin^2 \theta
\end{bmatrix}
\]

\[
\{F^e\} = \begin{bmatrix}
F_1^e \\
F_2^e \\
F_3^e \\
F_4^e
\end{bmatrix} = \begin{bmatrix}
\bar{P}_1^e \cos \theta_e \\
\bar{P}_1^e \sin \theta_e \\
\bar{P}_2^e \cos \theta_e \\
\bar{P}_2^e \sin \theta_e
\end{bmatrix} + \begin{bmatrix}
\bar{f}_1^e \cos \theta_e \\
\bar{f}_1^e \sin \theta_e \\
\bar{f}_2^e \cos \theta_e \\
\bar{f}_2^e \sin \theta_e
\end{bmatrix}
\]
EXAMPLE 1

Given truss

Finite element discretization

---

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 3</td>
<td>$A, h_1 = L$</td>
<td>$E \theta_1 = 90^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>1 3</td>
<td>$A, h_2 = \sqrt{2}L$</td>
<td>$E \theta_2 = 45^\circ$</td>
</tr>
</tbody>
</table>
EXAMPLE 1 (continued)

The element stiffness matrices are \[1/(2\sqrt{2}) = 0.3536\]

\[
[K^1] = \frac{EA}{L} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

\[
[K^2] = \frac{EA}{L} \begin{bmatrix}
0.3536 & 0.3536 & -0.3536 & -0.3536 \\
0.3536 & 0.3536 & -0.3536 & -0.3536 \\
-0.3536 & -0.3536 & 0.3536 & 0.3536 \\
-0.3536 & -0.3536 & 0.3536 & 0.3536
\end{bmatrix}
\]
EXAMPLE 1  (continued)

Connectivity array for 2 DoF per node truss

\[
[B] = \begin{pmatrix}
1 & 2 & 3 & 4 \\
(1) & 3 & 4 & 5 & 6 \\
(2) & 1 & 2 & 5 & 6 \\
\end{pmatrix}
\]

Assembled stiffness coefficients

\[
K_{11} = K_{11}^{(2)}, \quad K_{12} = K_{12}^{(2)}, \quad K_{13} = 0, \\
K_{14} = 0, \quad K_{15} = K_{13}^{(2)}, \quad K_{16} = K_{14}^{(2)}, \\
K_{22} = K_{22}^{(2)}, \quad K_{23} = 0, \quad K_{24} = 0, \\
K_{25} = K_{23}^{(2)}, \quad K_{26} = K_{24}^{(2)}, \\
K_{33} = K_{11}^{(1)}, \quad K_{34} = K_{12}^{(1)}, \quad K_{35} = K_{13}^{(1)}, \\
K_{36} = K_{14}^{(1)}, \quad K_{44} = K_{22}^{(1)}, \\
K_{45} = K_{23}^{(1)}, \quad K_{46} = K_{24}^{(1)}, \quad K_{55} = K_{33}^{(1)} + K_{33}^{(2)}, \\
K_{56} = K_{34}^{(1)} + K_{34}^{(2)}, \quad K_{66} = K_{44}^{(1)} + K_{44}^{(2)}. \\
\]

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### Assembled system of equations of the truss

\[
\begin{bmatrix}
0.3536 & 0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
E A \\
L \\
symm.
\end{array}
\end{bmatrix}
\begin{bmatrix}
u_1^1 \\
v_2^2 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
0.3536 & 0.3536 & 1.3536
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
U_1 \\
V_2 \\
\vdots
\end{array}
\end{bmatrix}
\begin{bmatrix}
F_{x1} \\
F_{y1} \\
F_{x2} \\
F_{y2} \\
\vdots
\end{bmatrix}
\]

The displacement continuity conditions are

\[
\begin{align*}
u_1^1 &= u_1^3 = U_1, & v_1^1 &= v_1^3 = V_1 \\
u_2^1 &= u_2^2 = U_2, & v_2^1 &= v_2^2 = V_2 \\
u_2^2 &= u_2^3 = U_3, & v_2^2 &= v_2^3 = V_3
\end{align*}
\]
where the global forces and displacements are

\[ F_1^1 + F_1^3 = F_x^1, \quad F_2^1 + F_2^3 = F_y^1 \]
\[ F_3^1 + F_2^2 = F_x^2, \quad F_4^1 + F_2^2 = F_y^2 \]
\[ F_3^2 + F_3^3 = F_x^3, \quad F_4^2 + F_4^3 = F_y^3 \]

\[
\{ \Delta \} = \begin{bmatrix}
U_1 \\
V_1 \\
U_2 \\
V_2 \\
U_3 \\
V_3
\end{bmatrix}, \quad \{ F \} = \begin{bmatrix}
F_1^1 + F_1^3 \\
F_2^1 + F_2^3 \\
F_3^1 + F_2^2 \\
F_4^1 + F_2^2 \\
F_3^2 + F_3^3 \\
F_4^2 + F_4^3
\end{bmatrix} = \begin{bmatrix}
F_x^1 \\
F_y^1 \\
F_x^2 \\
F_y^2 \\
F_x^3 \\
F_y^3
\end{bmatrix}
\]

Boundary conditions

\[ U_1 = V_1 = U_2 = V_2 = 0, \quad F_x^3 = P, \quad F_y^3 = -2P \]
Solution

\[
\frac{EA}{L} \begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix} \begin{bmatrix} U_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} P \\ -2P \end{bmatrix}
\]

\[
\begin{bmatrix} F_1^x \\ F_1^y \\ F_2^x \\ F_2^y \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} -0.3536 & -0.3536 \\ -0.3536 & -0.3536 \\ 0.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix} \begin{bmatrix} U_3 \\ V_3 \end{bmatrix}
\]

\[U_3 = (3 + 2\sqrt{2}) \frac{PL}{EA} = 5.828 \frac{PL}{EA}, \quad V_3 = -3\frac{PL}{EA} (\text{m})\]

\[F_1^{(1)} = -F_2^{(1)} = 3P; \quad F_1^{(2)} = -F_2^{(2)} = -\sqrt{2}P \quad (\text{N})\]
Post-computation of member displacements and stresses

\[
\sigma^e = -\frac{\bar{P}_1^e}{A_e} = \frac{\bar{P}_2^e}{A_e}
\]

\[
\begin{align*}
\left\{ \bar{P}_1^e \right\} &= \frac{A_e E_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left\{ \bar{u}_1^e \right\} \\
\left\{ \bar{P}_2^e \right\} &= \frac{A_e E_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left\{ \bar{u}_2^e \right\}
\end{align*}
\]

\[
\begin{bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{bmatrix} = \begin{bmatrix} \cos \theta_e & \sin \theta_e & 0 & 0 \\ -\sin \theta_e & \cos \theta_e & 0 & 0 \\ 0 & 0 & \cos \theta_e & \sin \theta_e \\ 0 & 0 & -\sin \theta_e & \cos \theta_e \end{bmatrix} \begin{bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{bmatrix}
\]

We have

\[
u_1^1 = v_1^1 = u_1^2 = v_1^2 = 0; \quad u_2^1 = u_2^2 = U_3 = (3 + 2\sqrt{2}) \frac{PL}{AE}, \quad v_2^1 = v_2^2 = V_3 = -\frac{3PL}{AE}
\]
EXAMPLE 1 (continued)

Post-computation of member displacements and stresses

\[ \bar{u}_2^1 = U_3 \cos \theta_1 + V_3 \sin \theta_1 = V_3 = -\frac{3PL}{A} \]

\[ \bar{u}_2^2 = U_3 \cos \theta_2 + V_3 \sin \theta_2 = \frac{1}{\sqrt{2}}(U_3 + V_3) \]

\[ \bar{P}_1^1 = -\bar{P}_2^1 = 3P, \quad \bar{P}_1^2 = -\bar{P}_2^2 = -\sqrt{2}P \]

\[ \sigma^{(1)} = -\frac{3P}{A}, \quad \sigma^{(2)} = \frac{\sqrt{2}P}{A} \]
Displ. degrees of freedom in the element coordinates

\[ \bar{u}_1^e \quad \bar{u}_2^e \]

\[ \bar{w}_1^e \quad \bar{w}_2^e \]

\[ \bar{S}_1^e \quad \bar{S}_2^e \]

Force degrees of freedom in the element coordinates

\[ \bar{F}_2^e = \bar{Q}_2^e + \bar{q}_1^e \]
\[ \bar{F}_5^e = \bar{Q}_5^e + \bar{q}_3^e \]
\[ \bar{F}_3^e = \bar{Q}_1^e + \bar{q}_2^e \]
\[ \bar{F}_1^e = \bar{Q}_1^e + \bar{f}_1^e \]
\[ \bar{F}_6^e = \bar{Q}_4^e + \bar{q}_4^e \]
\[ \bar{F}_4^e = \bar{Q}_4^e + \bar{f}_2^e \]

Displ. degrees of freedom in the local coordinates

Displ. degrees of freedom in the global coordinates
Frame Element in Global Coordinates

\[
\begin{align*}
\begin{bmatrix}
\bar{u}_1^e \\
\bar{w}_1^e \\
\bar{\theta}_1^e
\end{bmatrix} &= 
\begin{bmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1^e \\
w_1^e \\
\theta_1^e
\end{bmatrix}, \quad
\begin{bmatrix}
\bar{u}_2^e \\
\bar{w}_2^e \\
\bar{\theta}_2^e
\end{bmatrix} &= 
\begin{bmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_2^e \\
w_2^e \\
\theta_2^e
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
\bar{u}_1^e \\
\bar{w}_1^e \\
\bar{\theta}_1^e \\
\bar{u}_2^e \\
\bar{w}_2^e \\
\bar{\theta}_2^e
\end{bmatrix} = 
\begin{bmatrix}
\cos \alpha & \sin \alpha & 0 & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \cos \alpha & \sin \alpha \\
0 & 0 & 0 & -\sin \alpha & \cos \alpha \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1^e \\
w_1^e \\
\theta_1^e \\
u_2^e \\
w_2^e \\
\theta_2^e
\end{bmatrix}
\]

\[
\bar{\Delta}^e = T^e \Delta^e
\]

\[
[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\}, \quad [T^e]^{-1} = [T^e]^T
\]

\[
[T^e]^T[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\}
\]

\[
[K^e] = [T^e]^T[\bar{K}^e][T^e], \quad \{F^e\} = [T^e]^T\{\bar{F}^e\}
\]
Frame Element in Global Coordinates
(the Euler-Bernoulli beam frame element)

\[
K^e = \frac{2E_e I_e}{h_e^3} \begin{bmatrix}
\kappa_e \cos^2 \alpha_e + 6 \sin^2 \alpha_e & (\kappa_e - 6) \cos \alpha_e \sin \alpha_e & 3h_e \sin \alpha_e \\
(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & \kappa_e \sin^2 \alpha_e + 6 \cos^2 \alpha_e & -3h_e \cos \alpha_e \\
3h_e \sin \alpha_e & -3h_e \cos \alpha_e & 2h_e^2 \\
-(\kappa_e \cos^2 \alpha_e + 6 \sin^2 \alpha_e) & -(\kappa_e - 6) \sin \alpha_e \cos \alpha_e & -3h_e \sin \alpha_e \\
-(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & -(\kappa_e \sin^2 \alpha_e + 6 \cos^2 \alpha_e) & 3h_e \cos \alpha_e \\
3h_e \sin \alpha_e & -3h_e \cos \alpha_e & h_e^2 \\
-(\kappa_e \cos^2 \alpha_e + 6 \sin^2 \alpha_e) & -(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & 3h_e \sin \alpha_e \\
-(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & -(\kappa_e \sin^2 \alpha_e + 6 \cos^2 \alpha_e) & -3h_e \cos \alpha_e \\
3h_e \sin \alpha_e & 3h_e \cos \alpha_e & h_e^2 \\
\kappa_e \cos^2 \alpha_e + 6 \sin^2 \alpha_e & -(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & 3h_e \sin \alpha_e \\
(\kappa_e - 6) \cos \alpha_e \sin \alpha_e & \kappa_e \sin^2 \alpha_e + 6 \cos^2 \alpha_e & -3h_e \cos \alpha_e \\
-3h_e \sin \alpha_e & 3h_e \cos \alpha_e & 2h_e^2
\end{bmatrix}
\]

\[
F^e = \begin{bmatrix}
F_1^e \cos \alpha_e - F_2^e \sin \alpha_e \\
F_1^e \sin \alpha_e + F_2^e \cos \alpha_e \\
F_3^e \\
F_4^e \cos \alpha_e - F_5^e \sin \alpha_e \\
F_4^e \sin \alpha_e + F_5^e \cos \alpha_e \\
F_6^e
\end{bmatrix}, \quad \kappa_e = \frac{A_e h_e^2}{2I_e}
\]
AN EXAMPLE OF A FRAME STRUCTURE

Given structure

Finite element discretization

Connectivity array for 3 DoF per node

\[
[B] = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 & 8 & 9 \\
\end{bmatrix}
\]

\[\alpha_1 = 0^0, \quad \alpha_2 = -90^0 \text{ or } 270^0\]
Global stiffness coefficients in terms of element stiffness coefficients

\[ K_{11} = K_{11}^{(1)}, \ K_{12} = K_{12}^{(1)}, \ldots, K_{16} = K_{16}^{(1)} \]

\[ K_{22} = K_{22}^{(1)}, \ K_{23} = K_{23}^{(1)}, \ldots, K_{26} = K_{26}^{(1)} \]

\[ K_{33} = K_{33}^{(1)}, \ K_{34} = K_{34}^{(1)}, \ K_{35} = K_{35}^{(1)}, K_{36} = K_{36}^{(1)} \]

\[ K_{44} = K_{44}^{(1)} + K_{11}^{(2)}, \ K_{45} = K_{45}^{(1)} + K_{12}^{(2)}, \]

\[ K_{46} = K_{46}^{(1)} + K_{13}^{(2)}, \ K_{55} = K_{55}^{(1)} + K_{22}^{(2)}, \]

\[ K_{56} = K_{56}^{(1)} + K_{23}^{(2)}, \ K_{66} = K_{66}^{(1)} + K_{33}^{(2)}, \]

\[ K_{47} = K_{14}^{(2)}, \ K_{48} = K_{15}^{(2)}, \ K_{49} = K_{16}^{(2)}, \]

\[ K_{57} = K_{24}^{(2)}, \ K_{58} = K_{25}^{(2)}, \ K_{59} = K_{26}^{(2)}, \]

\[ K_{67} = K_{24}^{(2)}, \ K_{68} = K_{35}^{(2)}, \ K_{69} = K_{26}^{(2)}, \]

\[ K_{77} = K_{44}^{(2)}, \ K_{78} = K_{45}^{(2)}, \ K_{79} = K_{46}^{(2)}, \]

\[ K_{88} = K_{55}^{(2)}, \ K_{89} = K_{56}^{(2)}, \ K_{99} = K_{66}^{(2)}. \]
EXAMPLE 2 (from the textbook)

The $y$-axis is into the plane of the paper, and $\alpha$ is measured counterclockwise from $x$-axis to $\bar{x}$-axis

$A = 10 \text{ in.}^2$, $I = 10 \text{ in.}^4$, $E = 30 \times 10^6 \text{ psi}$

The rest of the calculations can be found on pp. 278-281 of the Book.
SUMMARY

In this lecture we have covered the following topics:

• Plane truss element in the local and global coordinates
• Transformation of element equations from element coordinates to global coordinates
• Examples, illustrating assembly, application of boundary conditions, and calculation of element forces and stresses
• Plane frame element in the local and global coordinates
• Transformation of element equations from element coordinates to global coordinates