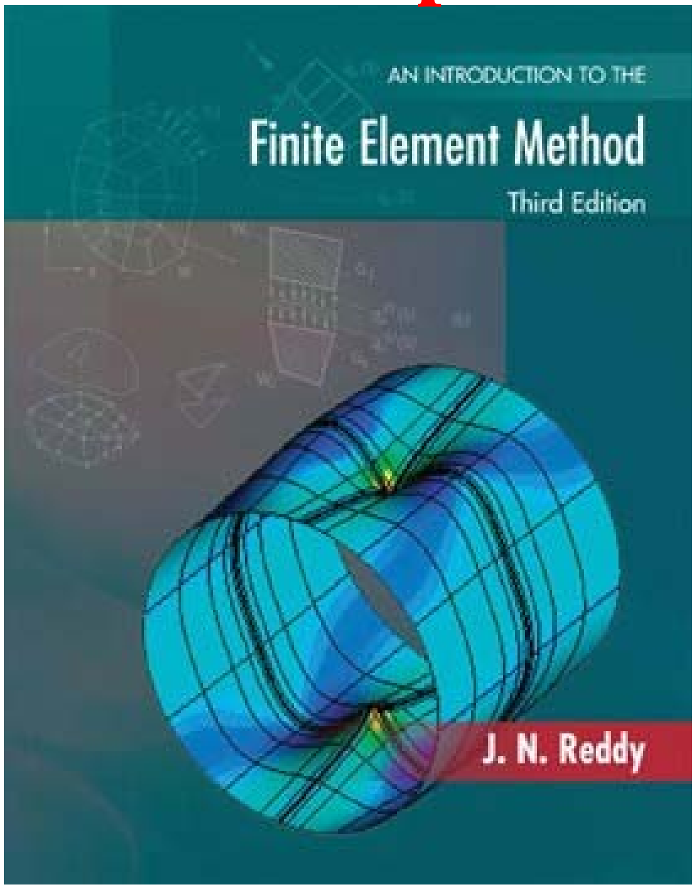
The Finite Element Method

Euler-Bernoulli and Timoshenko Beams

Read: Chapter 5



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Timoshenko beam theory

- Governing Equations
- Finite element model
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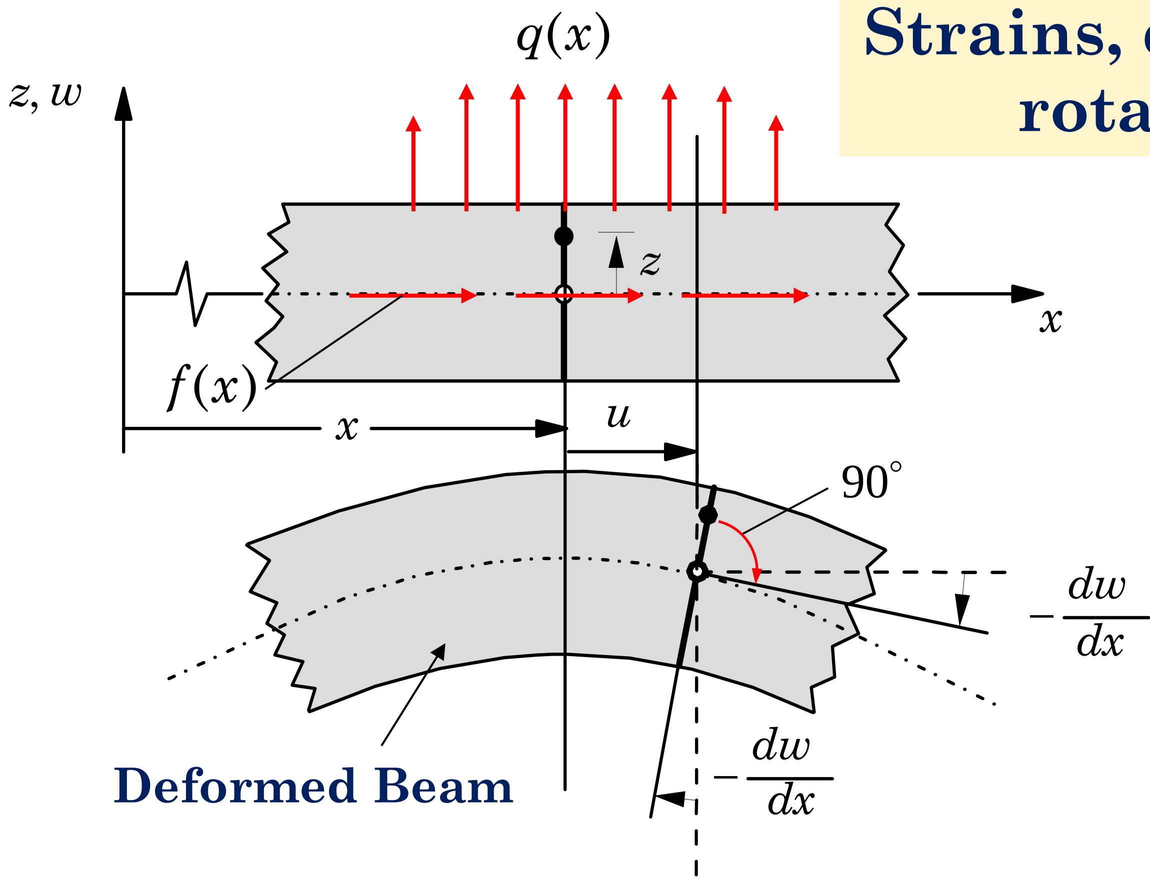
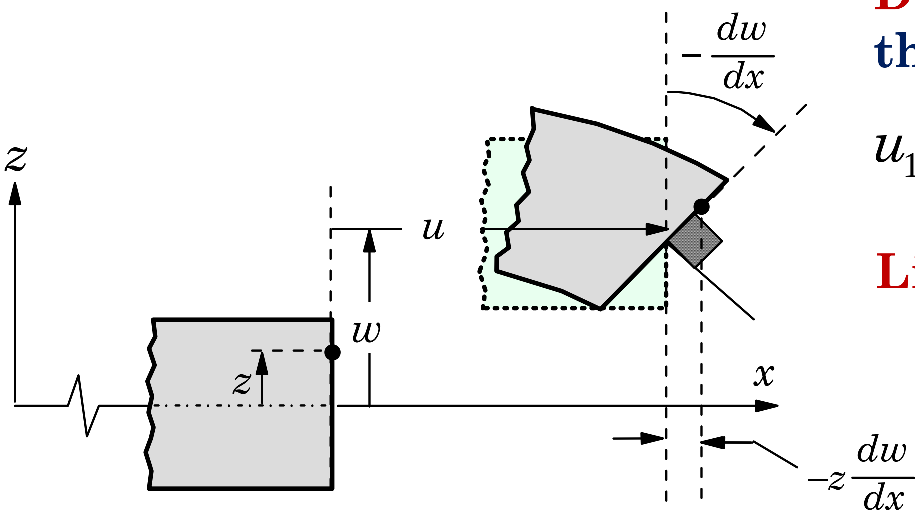
KINEMATICS OF THE LINEARIZED EULER-BERNOULLI BEAM THEORY

Strains, displacements, and rotations are small



Euler-Bernoulli Beam Theory (EBT) is based on the assumptions of ***(1) straightness, (2) inextensibility***, **and** ***(3) normality***

Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



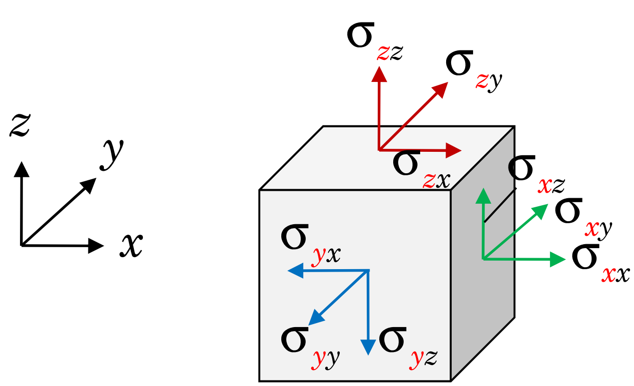
Displacement field (constructed using the hypothesis)

$$u_1(x,z) = u - z\frac{dw}{dx}, \qquad u_2 = 0, \quad u_3 = w(x)$$

Linear strains

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} - z\frac{d^2w}{dx^2},$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\frac{dw}{dx} + \frac{dw}{dx} = 0.$$



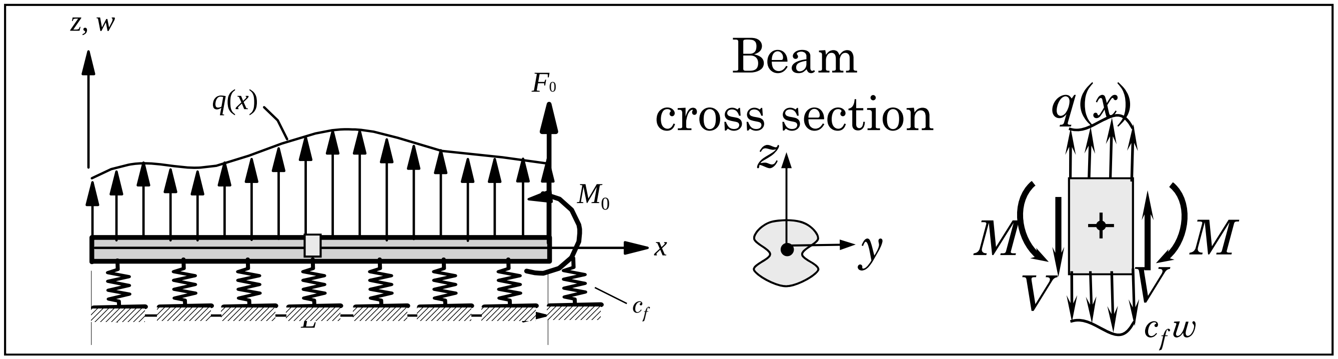
Notation for stress components

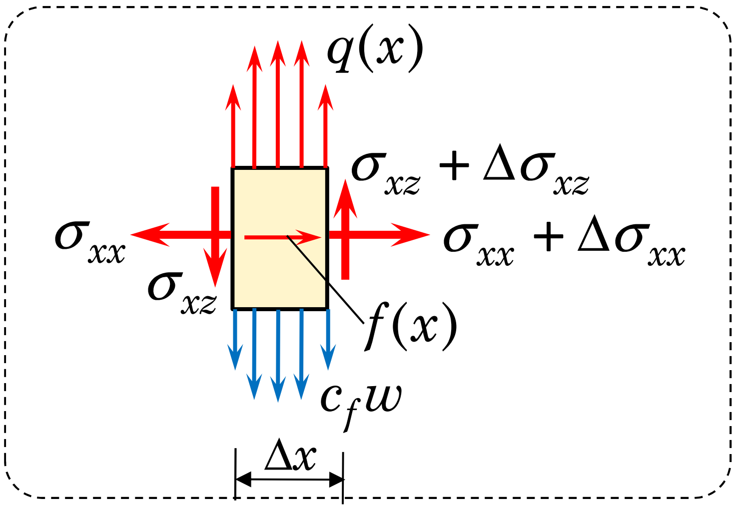
Constitutive relations

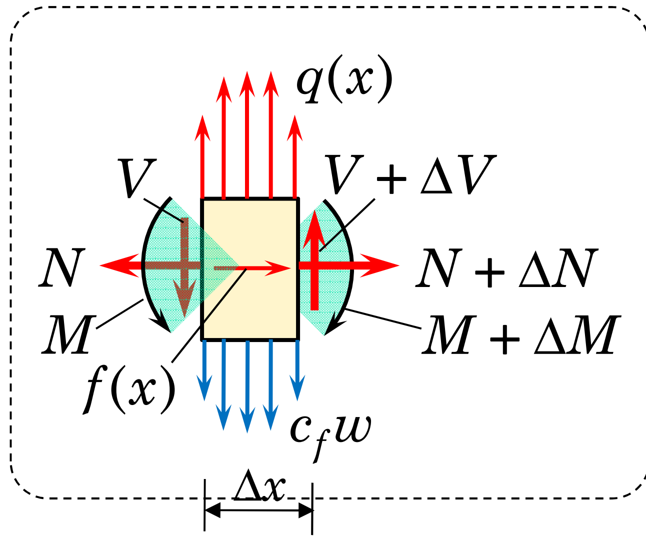
$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx} - Ez\frac{d^2w}{dx^2},$$

$$\sigma_{xz} = G\gamma_{xz} = 0$$

Euler-Bernoulli Beam Theory







Definition of stress resultants

$$N = \int_A \sigma_{xx}\, dA,\;\; M = \int_A \sigma_{xx} \cdot z\, dA,\;\; V = \int_A \sigma_{xz}\, dA.$$

Euler-Bernoulli Beam Theory (Continued)

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + q - c_f w = 0$$

$$\int_A 1 \cdot dA = A, \qquad \int_A z \cdot dA = 0, \qquad \int_A z^2 \cdot dA = I$$

Stress resultants in terms of deflection

$$N = \int_A \sigma_{xx}\; dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) dA = EA\frac{du}{dx}$$

$$M = \int_A \sigma_{xx} \times z\; dA = \int_A \left(E\frac{du}{dx} - Ez\frac{d^2w}{dx^2} \right) z\; dA = -EI\frac{d^2w}{dx^2}$$

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2} \right)$$

Euler-Bernoulli Beam Theory (Continued)

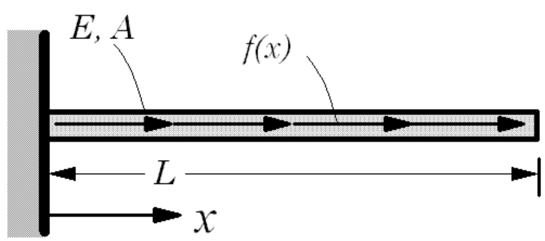
Governing equations in terms of the displacements

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-f=0,\quad 0<x<L$$

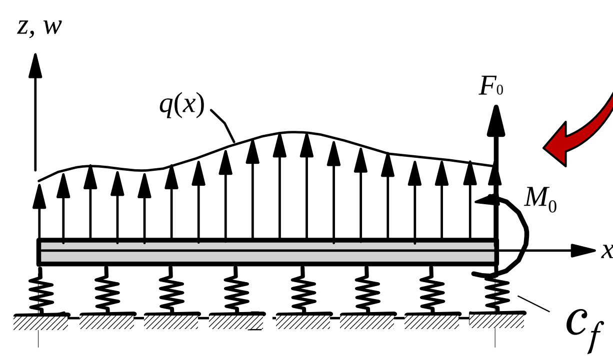
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q=0,\quad 0<x<L$$

Bars
u

Beams
w



Axial deformation of a bar



Bending of a beam

Axial displacement is uncoupled from transverse displacement

Weak Form of the EB Beam Theory

Governing equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q=0,\quad 0<x<L$$

Weak form $\{v_i\}$ – set of weight functions

$$0=\int_{x_a}^{x_b} v_i\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w_h}{dx^2}\right)+c_f w_h-q\right]dx$$

$$=\int_{x_a}^{x_b}\left[-\frac{dv_i}{dx}\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)+c_f v_i w_h-v_i q\right]dx+\left[v_i\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)\right]_{x_a}^{x_b}$$

Implies that the primary variable is *w* (displacement)

Secondary variable (shear force)

$$0=\int_{x_a}^{x_b}\left[-\frac{dv}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)+c_f vw-vq\right]dx-v(x_a)Q_1-v(x_b)Q_3$$

Weak Form (Continued)

$$0=\int_{x_a}^{x_b}\left[EI\frac{d^2v_i}{dx^2}\frac{d^2w_h}{dx^2}+c_f v_i w_h - v_i q\right]dx - v_i(x_a)Q_1 - v_i(x_b)Q_3$$

$$+\left[\left(-\frac{dv_i}{dx}\right)\cdot\left(EI\frac{d^2w_h}{dx^2}\right)\right]_{x_a}^{x_b}$$

Primary Variable, θ Slope/rotation

Secondary variable (Bending Moment)

$$0=\int_{x_a}^{x_b}\left[EI\frac{d^2v_i}{dx^2}\frac{d^2w_h}{dx^2}+c_f v_i w_h - v_i q\right]dx - v_i(x_a)Q_1 - v_i(x_b)Q_3$$

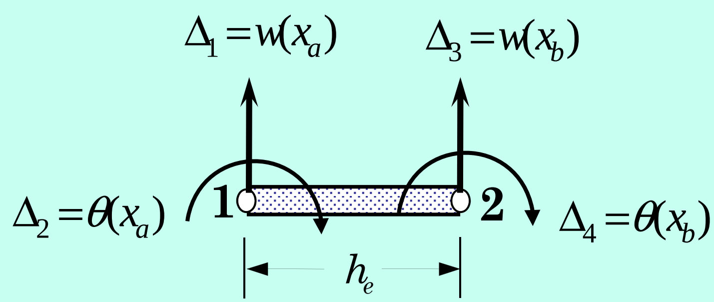
$$-\left(-\frac{dv_i}{dx}\right)_{x_a}\cdot Q_2-\left(-\frac{dv_i}{dx}\right)_{x_b}\cdot Q_4$$

$$Q_1=\left[\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)\right]_{x_a}=-V_h(x_a),\quad Q_3=\left[-\frac{d}{dx}\left(EI\frac{d^2w_h}{dx^2}\right)\right]_{x_b}=V_h(x_b)$$

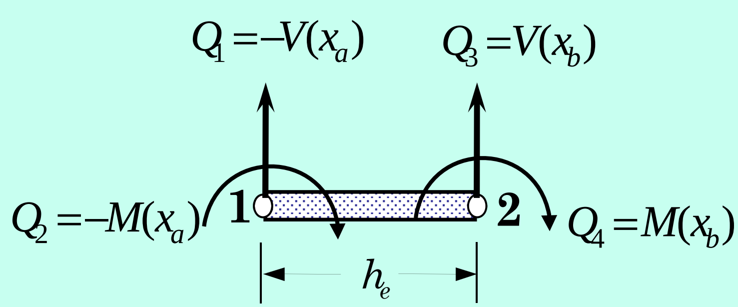
$$Q_2=\left(EI\frac{d^2w_h}{dx^2}\right)_{x_a}=-M_h(x_a),\qquad Q_4=\left(-EI\frac{d^2w_h}{dx^2}\right)_{x_b}=M_h(x_b)$$

Beam Element Degrees of Freedom

Generalized displacements



Generalized forces



FINITE ELEMENT APPROXIMATION: Some Remarks

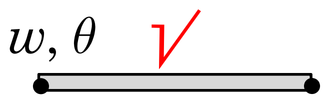
- Continuity requirement based on the weak form, which requires that the second derivative of w exists and square-integrable.
- Continuity based on the primary variables, which requires carrying w and its first derivative as the nodal variables, requires cubic approximation w.
- Post-computation of secondary variables (bending moment and shear force) requires the third derivative of w to exist.

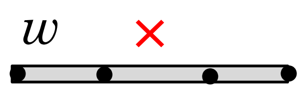
FINITE ELEMENT APPROXIMATION

Primary variables (serve as the nodal variables that must be continuous across elements)

$$w,\ \theta = -\frac{dw}{dx}$$

$$w(x) \approx c_0 + c_1 x + c_2 x^2 + c_3 x^3$$





$$w(x_a) \approx c_0 + c_1 x_a + c_2 x_a^2 + c_3 x_a^3 \equiv \Delta_1$$

$$w(x_b) \approx c_0 + c_1 x_b + c_2 x_b^2 + c_3 x_b^3 \equiv \Delta_3$$

$$\theta(x_a) \approx -c_1 - 2c_2 x_a - 3c_3 x_a^2 \equiv \Delta_2$$

$$\theta(x_b) \approx -c_1 - 2c_2 x_b - 3c_3 x_b^2 \equiv \Delta_4$$

Hermite cubic polynomials

$$\phi_1^e = 1 - 3\left(\frac{x - x_a}{h_e}\right)^2 + 2\left(\frac{x - x_a}{h_e}\right)^3$$

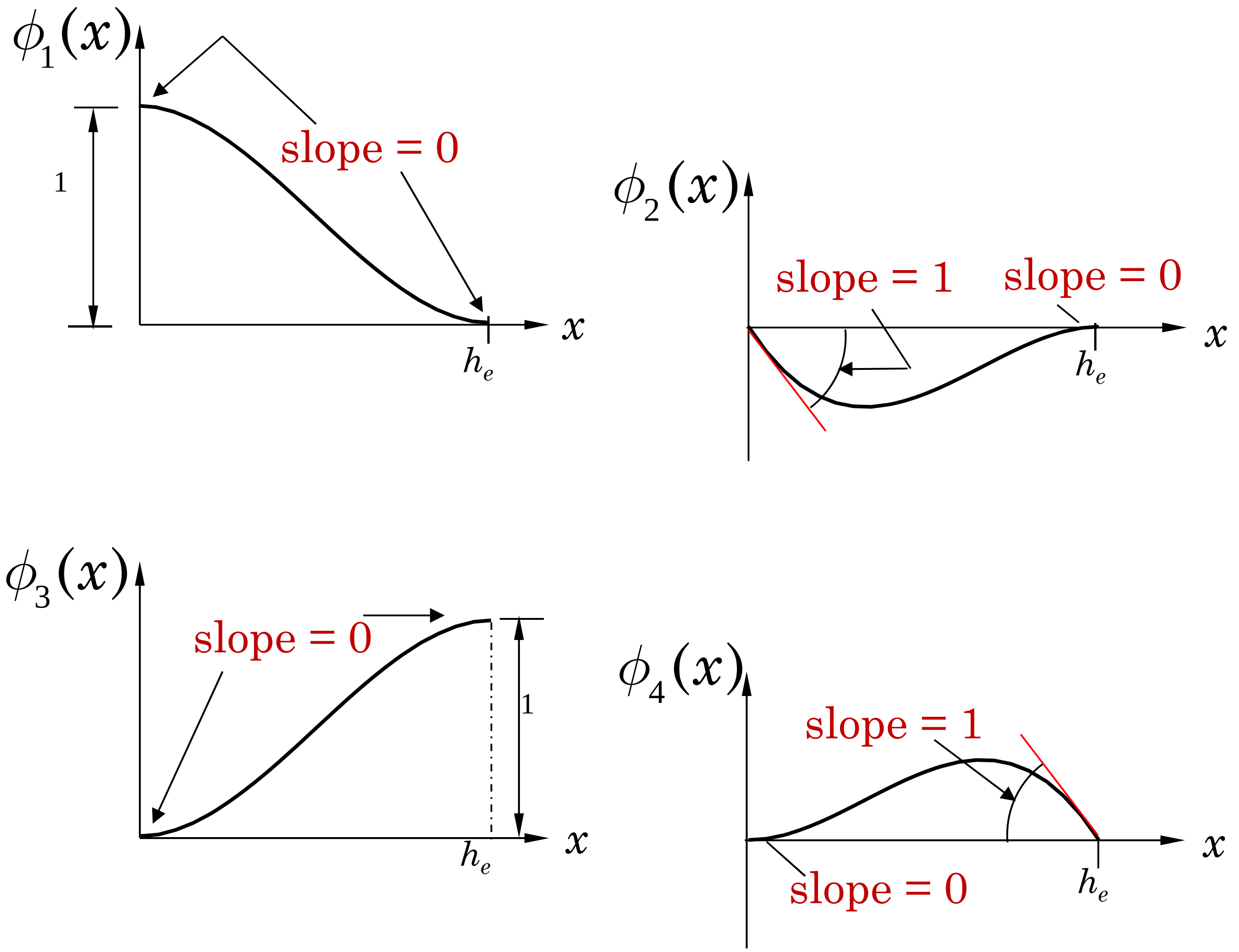
$$\phi_2^e = -(x - x_a)\left(1 - \frac{x - x_a}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_a}{h_e}\right)^2 - 2\left(\frac{x - x_a}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_a)\left[\left(\frac{x - x_a}{h_e}\right)^2 - \frac{x - x_a}{h_e}\right]$$

$$w(x) \approx c_0 + c_1 x + c_2 x^2 + c_3 x^3 = \sum_{j=1}^{4} \Delta_j \phi_j(x)$$

HERMITE CUBIC INTERPOLATION FUNCTIONS $\phi_i(x)$



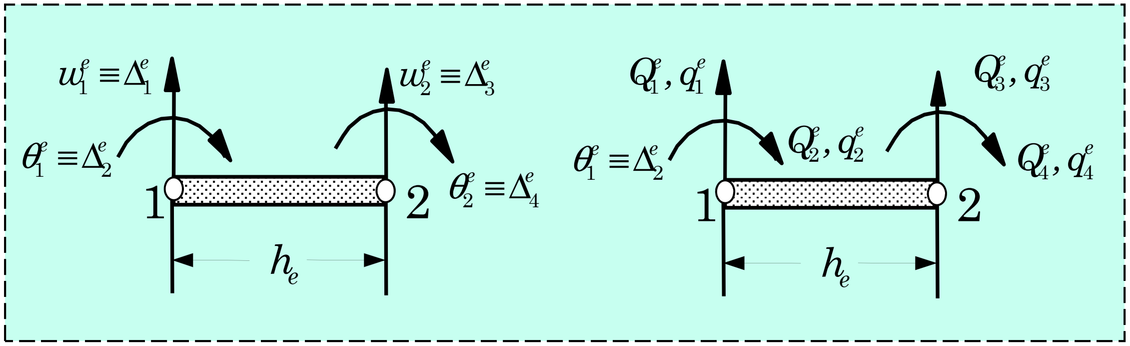
FINITE ELEMENT MODEL

$$w(x) \approx \sum_{j=1}^{4} \Delta_j \phi_j(x)$$

$$\sum_{j=1}^{4} K_{ij}^e \Delta_j^e - F_i^e = 0 \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\}$$

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} + c_f \, \phi_i^e \phi_j^e \right) dx \qquad F_i^e = \int_{x_a}^{x_b} \phi_i^e q \, dx + Q_i^e$$



Finite Element Model (Continued)

For element-wise constant values of $E_e I_e$ and q_e (and $c_f = 0$):

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} \quad \{F^e\} = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Postprocessing

$$M(x) = -EI\frac{d^2w}{dx^2} = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e}{dx^2}$$

$$V(x) = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^3\phi_j^e}{dx^3}$$

$$\sigma_x(x,z) = -\frac{M(x)z}{I} = Ez\frac{d^2w}{dx^2} = Ez\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e(x)}{dx^2}$$

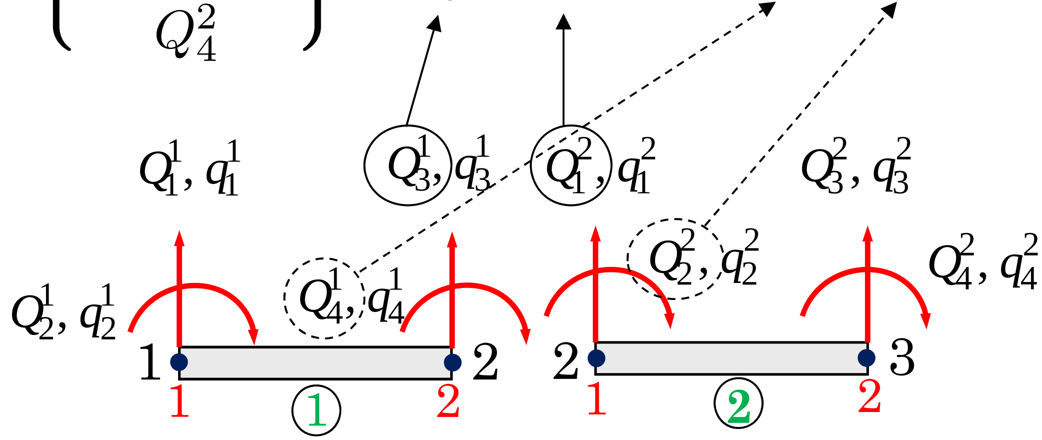
ASSEMBLY OF TWO BEAM ELEMENTS

connected end-to-end

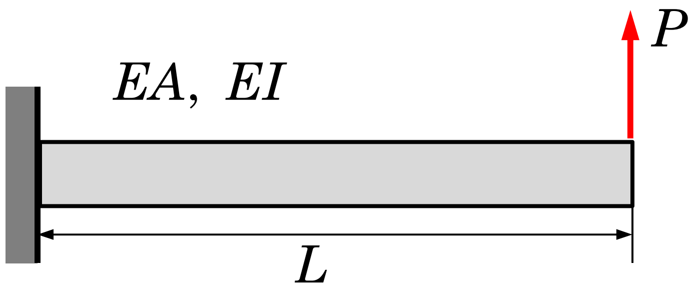
$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \qquad h = L/2$$

$$= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0$$



A SIMPLE EXAMPLE - 1

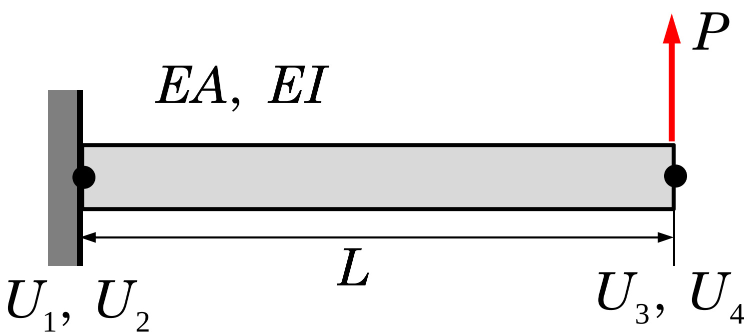


Given problem

Exact solution (according to the Euler-Bernoulli beam theory)

$$w(L) = \frac{PL^3}{3EI}$$

One element discretization



$$[K^e]\{\Delta^e\} = \{q^e\} + \{Q^e\} \qquad h_e = L$$

$$\{F^e\} = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

0

Boundary conditions:

$$U_1 = U_2 = 0, \quad Q_3 = P, \; Q_4 = 0$$

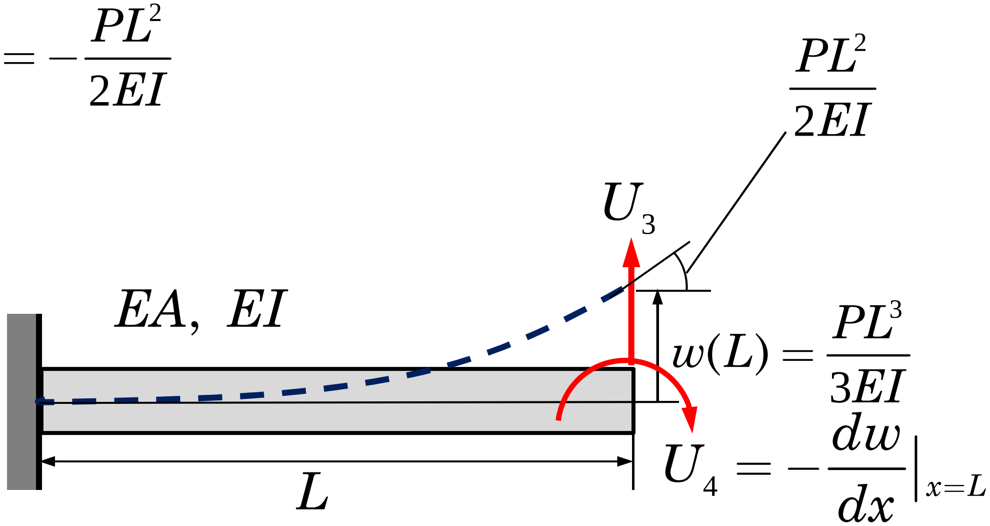
$$[K^e] = \frac{2E_e I_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

A SIMPLE EXAMPLE – 1 (continued)

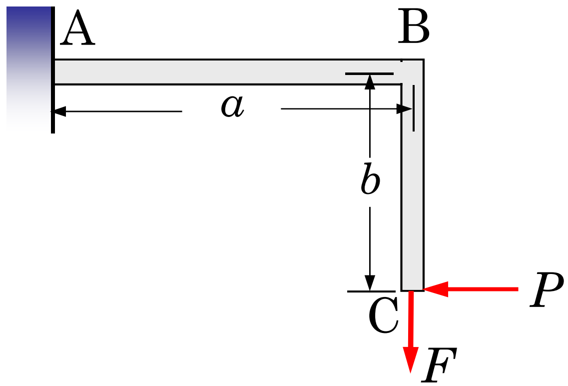
Solution using Cramer's rule

$$\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\begin{vmatrix} P & \frac{6EI}{L^2} \\ 0 & \frac{4EI}{L} \end{vmatrix}}{\begin{vmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{vmatrix}} = \frac{(4PEI / L)}{\left[12(EI)^2 / L^4\right]} = \frac{PL^3}{3EI}$$

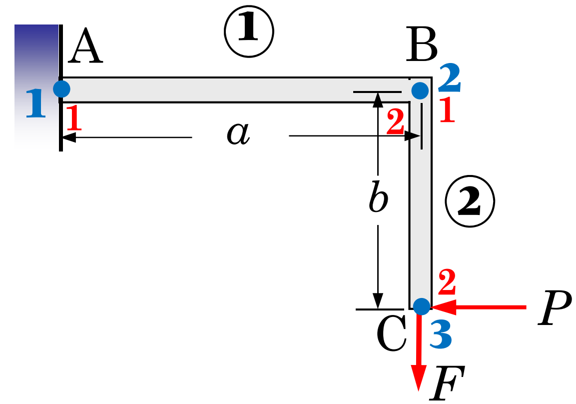
$$U_4 = \frac{\begin{vmatrix} \frac{12EI}{L^3} & P \\ \frac{6EI}{L^2} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{vmatrix}} = \frac{-(6PEI / L^2)}{\left[12(EI)^2 / L^4\right]} = -\frac{PL^2}{2EI}$$



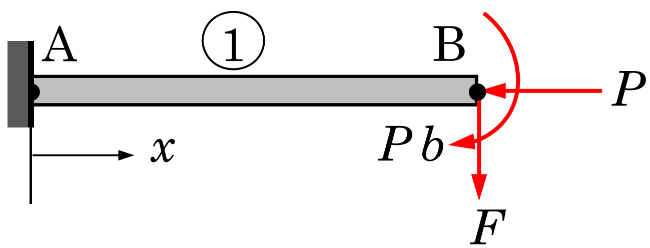
EXAMPLE – 2: A determinate frame structure

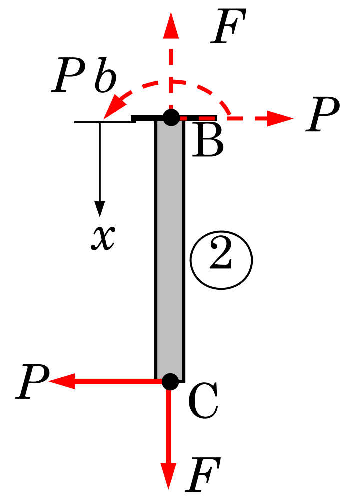


Given structure

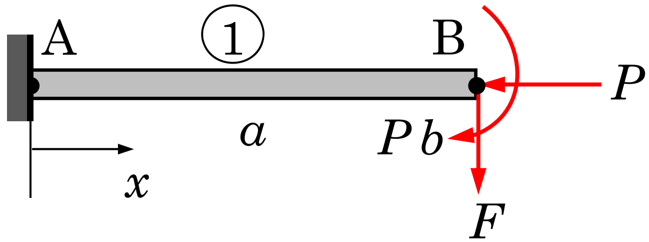


Finite element discretization

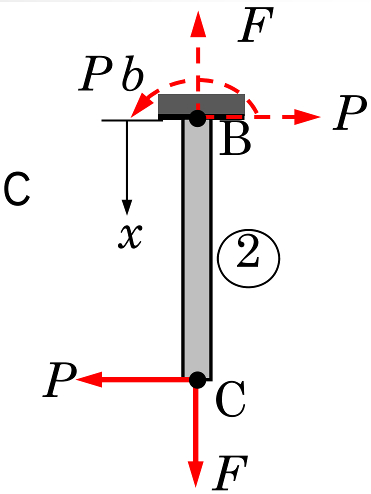




EXAMPLE – 2 (continued)



Displacements at C relative to point B



Bar element, AB

$$\frac{E_1A_1}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_A \\ u_B \end{Bmatrix} = \begin{Bmatrix} Q_A \\ Q_B \end{Bmatrix}$$

$$u_A = 0,\ Q_B = -P \ \Rightarrow\ u_B = -\frac{Pa}{E_1A_1}$$

Bar element, BC

$$u_C = \frac{Fb}{E_2A_2}$$

Beam element, AB

$$\frac{2E_1I_1}{a^3}\begin{bmatrix} 6 & -3a & -6 & -3a \\ -3a & 2a^2 & 3a & a^2 \\ -6 & 3a & 6 & 3a \\ -3a & a^2 & 3a & 2a^2 \end{bmatrix}\begin{Bmatrix} w_A \\ \theta_A \\ w_B \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} Q_1^A \\ Q_2^A \\ Q_1^B \\ Q_2^B \end{Bmatrix}$$

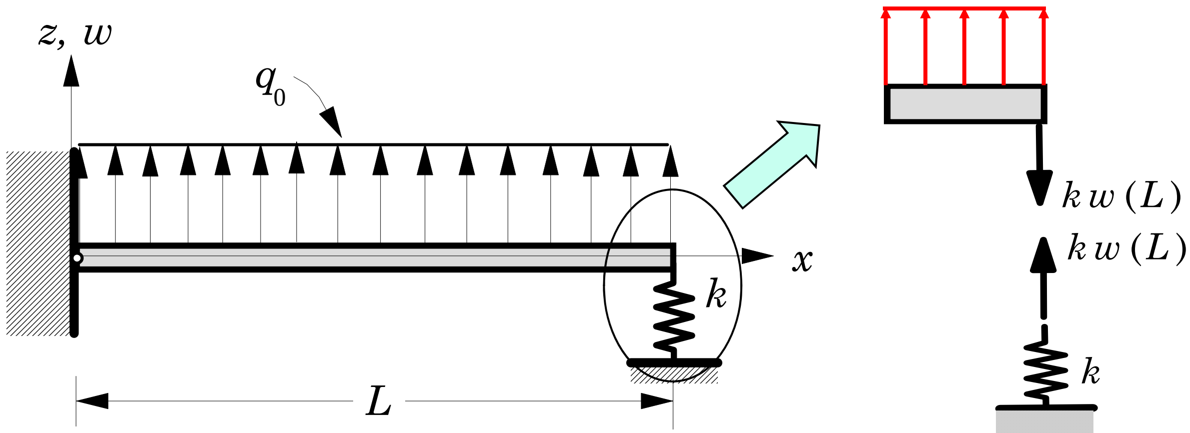
$$w_A = 0,\ \theta_A = 0,\ Q_1^B = -F,\ Q_2^B = Pb$$

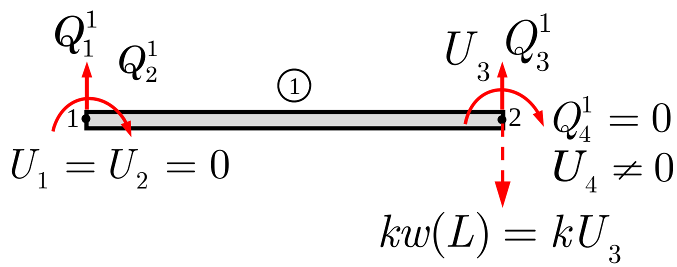
Beam element, BC

$$\frac{2E_2I_2}{b^3}\begin{bmatrix} 6 & -3b & -6 & -3b \\ -3b & 2b^2 & 3b & b^2 \\ -6 & 3b & 6 & 3b \\ -3b & b^2 & 3b & 2b^2 \end{bmatrix}\begin{Bmatrix} w_B \\ \theta_B \\ w_C \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} Q_1^B \\ Q_2^B \\ Q_1^C \\ Q_2^C \end{Bmatrix}$$

$$w_B = 0,\ \theta_B = 0,\ Q_1^C = -P,\ Q_2^C = 0$$

EXAMPLE – 3: Handling of a vertical spring





$\Rightarrow$ $Q_3^1 = -kw(L) = -kU_3$

Alternatively,

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^s \\ u_2^s \end{Bmatrix} = \begin{Bmatrix} Q_1^s \\ Q_2^s \end{Bmatrix}, \qquad u_1^s = 0,\; u_2^s = U_3 \;\Rightarrow Q_2^s = kU_3$$

SOLUTION TO THE SPRING-SUPPORTED BEAM

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} \begin{Bmatrix} U_1 = w_1 \to 0 \\ U_2 = \theta_1 \to 0 \\ U_3 = w_2 \\ U_4 = \theta_2 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \to -kU_3 \\ Q_4 \to 0 \end{Bmatrix}$$

Boundary conditions

$$w_1 = 0,\ \theta_1 = 0,\ Q_3 = -kU_3,\ Q_4 = 0$$

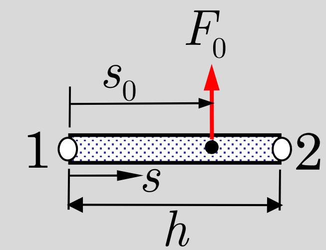
Condensed equations for the unknown generalized nodal displacements

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \end{Bmatrix}$$

HANDLING OF A POINT SOURCES INSIDE AN ELEMENT

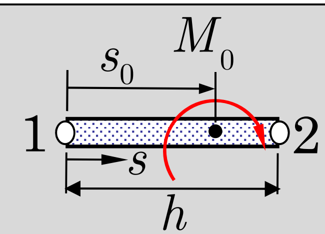
$$q_i = \int_0^h q(s)\phi_i(s)\,ds$$

$$q_i = \int_0^h q(s)\phi_i(s)\,ds = F_0\,\phi_i(s_0),\; i = 1,2,3,4$$



$$q(s) = F_0\delta(s - s_0)$$

$$q_i = \int_0^h q(s)\phi_i(s)\,ds = -M_0 \left.\frac{d\phi_i}{ds}\right|_{s=s_0},\; i = 1,2,3,4$$



$$q(s) = M_0\delta'(s - s_0)$$

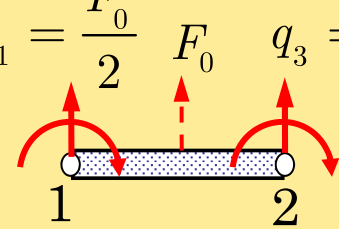
$$\phi_1(s) = 1 - 3\left(\frac{s}{h}\right)^2 + 2\left(\frac{s}{h}\right)^3,\;\; \phi_2(s) = -s\left(1 - \frac{s}{h}\right)^2$$

$$\phi_3(s) = 3\left(\frac{s}{h}\right)^2 - 2\left(\frac{s}{h}\right)^3,\;\; \phi_4(s) = -s\left[\left(\frac{s}{h}\right)^2 - \frac{s}{h}\right]$$

for F_0 placed at $s_0 = 0.5h$

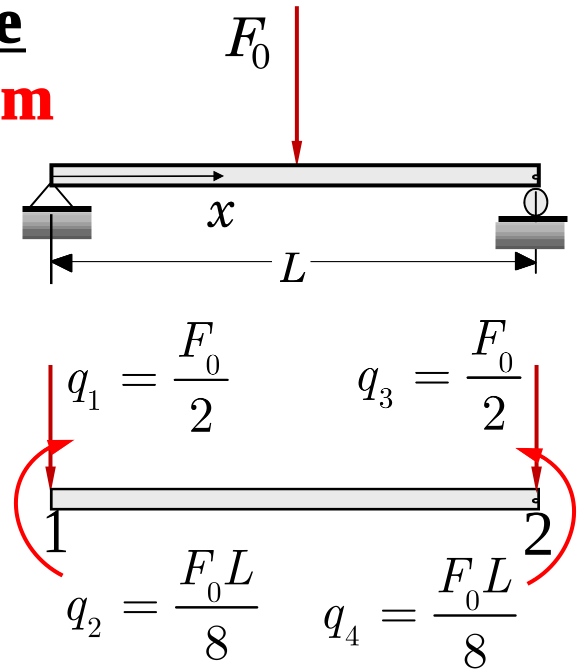
$$q_1 = \frac{F_0}{2} \qquad q_3 = \frac{F_0}{2}$$

$$q_2 = -\frac{F_0 h}{8} \qquad q_4 = \frac{F_0 h}{8}$$



EXAMPLE – 4: A simply-supported beam

(a) Find the center deflection using <u>one</u> Euler-Bernoulli element in full beam



$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} \begin{Bmatrix} U_1 = w_1 \\ U_2 = \theta_1 \\ U_3 = w_2 \\ U_4 = \theta_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

($U_1 \to 0$, $U_3 \to 0$; $q_2 \to \frac{F_0 L}{8}$, $q_4 \to -\frac{F_0 L}{8}$; $Q_2 \to 0$, $Q_4 \to 0$)

<u>Condensed equations</u>

$$\frac{2EI}{L^3}\begin{bmatrix} 2L^2 & L^2 \\ L^2 & 2L^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \frac{F_0 L}{8}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$U_2 = \frac{F_0 L^2}{16EI}, \quad U_4 = -\frac{F_0 L^2}{16EI}$$

$$\begin{aligned} w(x) &= U_1\phi_1(x) + U_2\phi_2(x) + U_3\phi_3(x) + U_4\phi_4(x) \\ &= U_2\phi_2(x) + U_4\phi_4(x) \\ &= \frac{F_0 L^2}{16EI}\left\{\left[-x\left(1 - \frac{x}{L}\right)^2\right] + x\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right]\right\} \end{aligned}$$

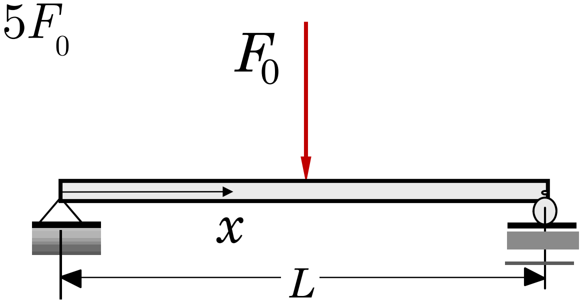
$$w(0.5L) = \frac{F_0 L^2}{16EI}\left(-\frac{L}{8} - \frac{L}{8}\right) = -\frac{F_0 L^3}{64EI}$$

EXAMPLE – 4: A simply-supported beam

(b) Find the center deflection using <u>one</u> Euler-Bernoulli element in half beam

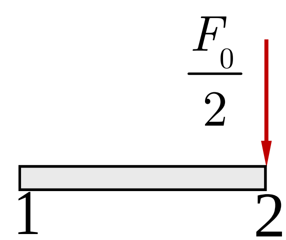
$$\frac{16EI}{L^3}\begin{bmatrix} 6 & -1.5L & -6 & -1.5L \\ -1.5L & 0.5L^2 & 1.5L & 0.25L^2 \\ -6 & 1.5L & 6 & 1.5L \\ -3L & 0.25L^2 & 1.5L & 0.5L^2 \end{bmatrix}\begin{Bmatrix} U_1 = w_1 \\ U_2 = \theta_1 \\ U_3 = w_2 \\ U_4 = \theta_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$U_1 = w_1 \to 0$, $U_4 = \theta_2 \to 0$; $Q_2 \to 0$, $Q_3 \to -0.5F_0$



<u>Condensed equations</u>

$$\frac{16EI}{L^3}\begin{bmatrix} 0.5L^2 & 1.5L \\ 1.5L & 6 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 0.5F_0\begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$



$U_1 = 0,\quad U_4 = 0,$

$Q_2 = 0 \quad Q_3 = -0.5F_0$

$$U_2 = \frac{F_0L^3}{32EI}\frac{4}{3L^2}\frac{1.5L}{1} = \frac{F_0L^2}{16EI},$$

$$U_3 = \frac{F_0L^3}{32EI}\frac{4}{3L^2}\frac{0.5L^2}{1} = \frac{F_0L^3}{48EI}$$

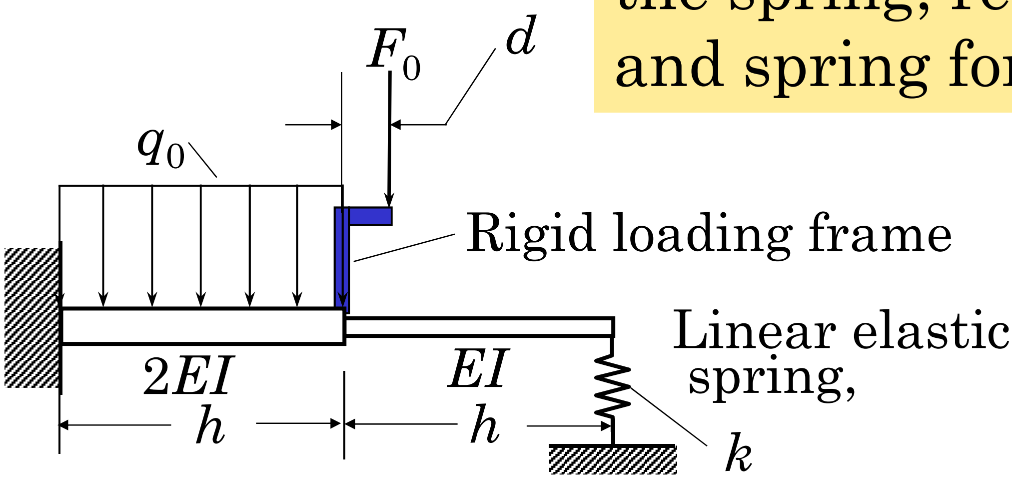
EXERCISE PROBLEM

Problem: Develop weak form and the finite element model of the following equation, where w and P are unknowns:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+P\frac{d^2w}{dx^2}=0,\quad 0<x<L$$

EXERCISE PROBLEM

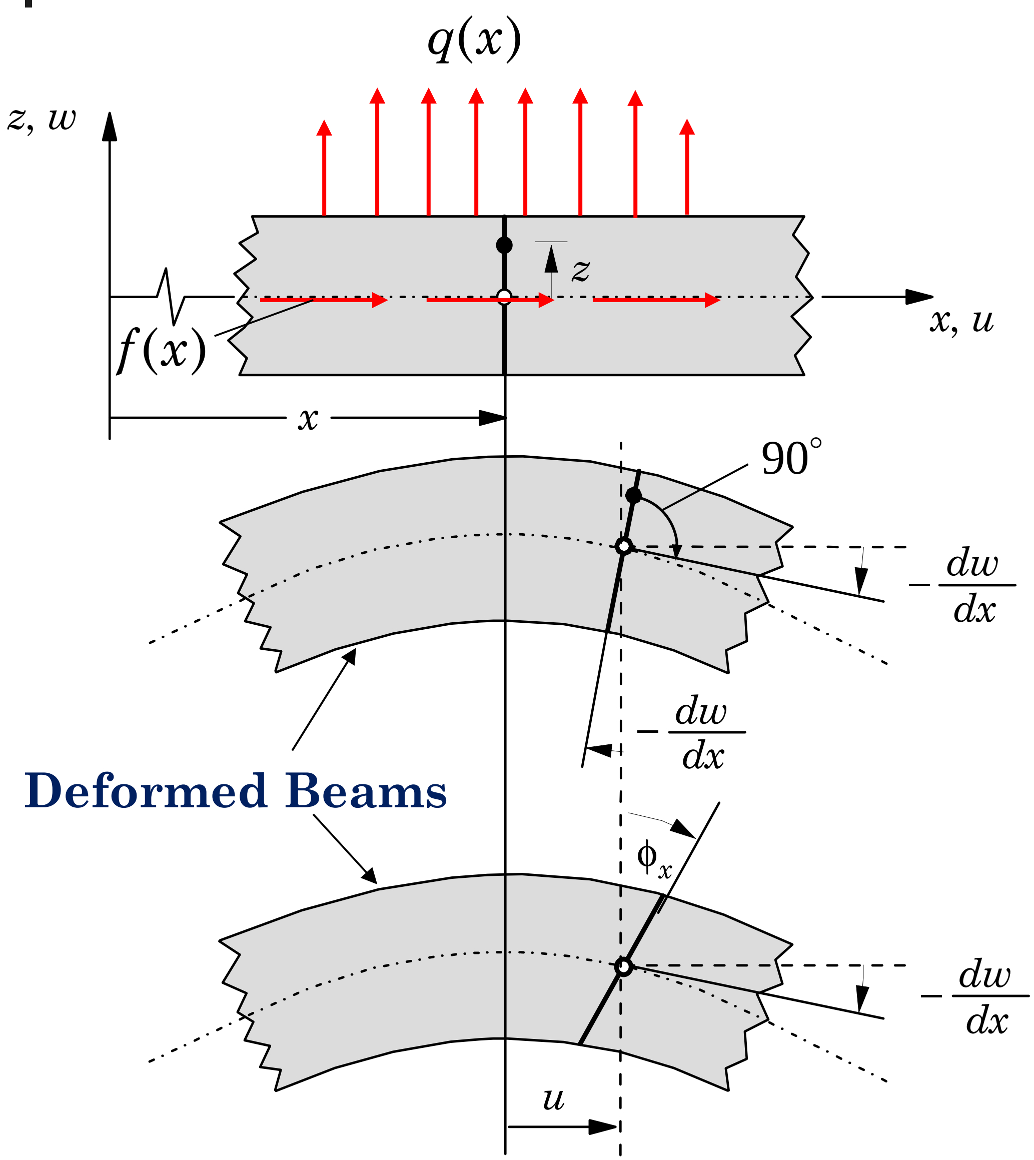
Problem: Use the minimum number of EBT elements to find the compression in the spring, reactions at the fixed support, and spring force.



TIMOSHENKO BEAM THEORY and its Finite Element Model

- Governing Equations
- Finite element model
- Shear locking
- Numerical example

Kinematics of Timoshenko Beam Theory



Undeformed Beam

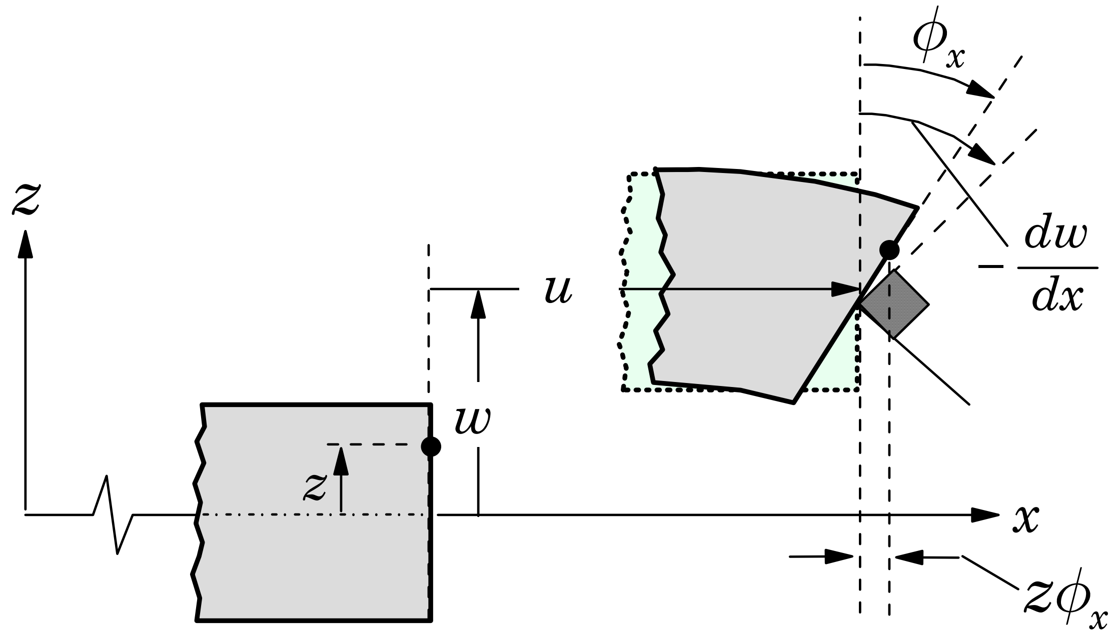
Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)
Straightness and inextensibility

Timoshenko Beam Theory

Kinematic Relations

$$u_1(x,z) = u(x) + z\phi_x(x),$$
$$u_2 = 0, \quad u_3(x,z) = w(x)$$
$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} = \frac{du}{dx} + z\frac{d\phi_x}{dx},$$
$$\gamma_{xz} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \phi_x + \frac{dw}{dx}$$



Constitutive Equations

$$\sigma_{xx} = E\,\varepsilon_{xx} = E\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right)$$
$$\sigma_{xz} = G\,\gamma_{xz} = G\left(\phi_x + \frac{dw}{dx}\right)$$

Timoshenko Beam Theory (Continued)

Equilibrium Equations $$\frac{dN}{dx}+f=0,\quad -\frac{dV}{dx}-q+c_f w=0,\quad -\frac{dM}{dx}+V=0.$$

Beam Constitutive Equations

$$N=\int_A \sigma_{xx}\, dA=\int_A E\left(\frac{du}{dx}+z\frac{d\phi_x}{dx}\right)dA=EA\frac{du}{dx}$$

$$M=\int_A \sigma_{xx} z\, dA=\int_A E\left(\frac{du}{dx}+z\frac{d\phi}{dx}\right)z\,dA=EI\frac{d\phi}{dx}$$

$$V=K_s\int_A \sigma_{xz}\, dA=GK_s\left(\phi+\frac{dw}{dx}\right)\int_A dA=GAK_s\left(\phi+\frac{dw}{dx}\right)$$

Governing Equations in terms of the displacements

$$-\frac{d}{dx}\left[GAK_s\left(\phi+\frac{dw}{dx}\right)\right]+c_f w=q \qquad (1)$$

$$-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s\left(\phi+\frac{dw}{dx}\right)=0 \qquad (2)$$

WEAK FORMS OF TBT

Weak Form of Eq. (1)

$$0 = \int_{x_a}^{x_b} v_1 \left\{ -\frac{d}{dx}\left[GAK_s\left(\phi + \frac{dw}{dx}\right)\right] + c_f w - q \right\} dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{dv_1}{dx}\left[GAK_s\left(\phi + \frac{dw}{dx}\right)\right] + c_f v_1 w - v_1 q \right\} dx - \left[v_1 \cdot GAK_s\left(\phi + \frac{dw}{dx}\right)\right]_{x_a}^{x_b}$$

$$= \int_{x_a}^{x_b} \left\{ \frac{dv_1}{dx}\left[GAK_s\left(\phi + \frac{dw}{dx}\right)\right] + c_f v_1 w - v_1 q \right\} dx$$

$$- v_1(x_a) \cdot \left[-GAK_s\left(\phi + \frac{dw}{dx}\right)\right]_{x_a} - v_1(x_b) \cdot \left[GAK_s\left(\phi + \frac{dw}{dx}\right)\right]_{x_b}$$

$$= \int_{x_a}^{x_b} \left\{ GAK_s \frac{dv_1}{dx}\left(\phi + \frac{dw}{dx}\right) + c_f v_1 w - v_1 q \right\} dx - v_1(x_a) \cdot Q_1 - v_1(x_b) \cdot Q_3$$

Weak Forms of TBT (continued)

Weak Form of Eq. (2)

$$0=\int_{x_a}^{x_b} v_2\left[-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s\left(\phi+\frac{dw}{dx}\right)\right]dx$$

$$=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-\left[v_2\cdot EI\frac{d\phi}{dx}\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-v_2(x_a)\cdot\left(-EI\frac{d\phi}{dx}\right)_{x_a}-v_2(x_b)\cdot\left(EI\frac{d\phi}{dx}\right)_{x_b}$$

$$0=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}\left(EI\frac{d\phi}{dx}\right)+GAK_s v_2\left(\phi+\frac{dw}{dx}\right)\right]dx-v_2(x_a)\cdot Q_2-v_2(x_b)\cdot Q_4$$

Total Potential Energy

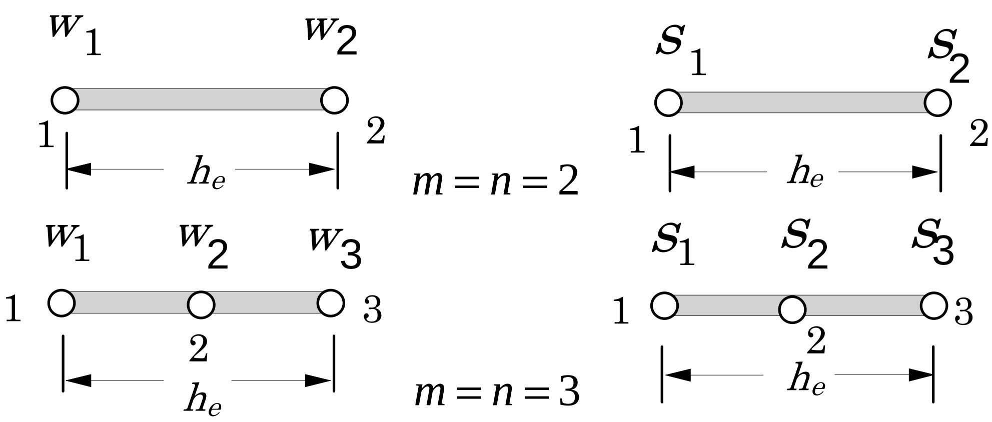
$$\Pi(w,\phi_x)=\int_{x_a}^{x_b}\left[\frac{EI}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{GAK_s}{2}\left(\phi+\frac{dw}{dx}\right)^2+\frac{c_f}{2}w^2\right]dx$$

$$-\int_{x_a}^{x_b} wq\,dx+w(x_a)Q_1+w(x_b)Q_3+\phi(x_a)Q_2+\phi(x_b)Q_4$$

FINITE ELEMENT MODELS OF TIMOSHENKO BEAMS

Finite Element Approximation

$$w \approx \sum_{j=1}^{m} w_j \psi_j(x), \quad \phi \approx \sum_{j=1}^{n} S_j \varphi_j(x)$$



$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c_f \psi_i \psi_j \right) dx, \quad K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i}{dx} \varphi_j \; dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left[EI \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + GAK_s \varphi_i \varphi_j \right] dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} GAK_s \; \varphi_i \frac{d\psi_j}{dx} \, dx$$

$$F_i^1 = \int_{x_a}^{x_b} q\psi_i \; dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_3, \qquad F_i^2 = \varphi_i(x_a) Q_2 + \varphi_i(x_b) Q_4$$

Shear Locking in Timoshenko Beams

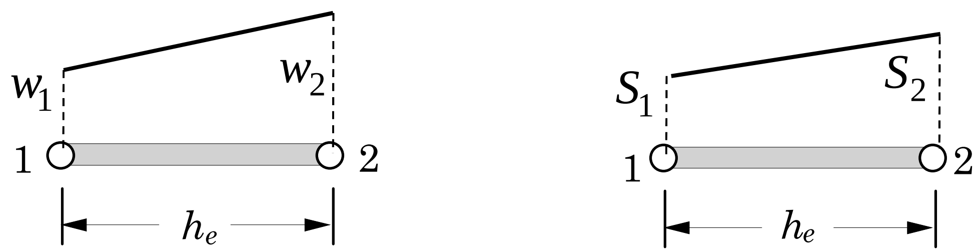
(1) Thick beam experiences shear deformation, $\phi_x \neq -\dfrac{dw}{dx}$

(2) Shear deformation is negligible in thin beams, $\phi_x = -\dfrac{dw}{dx}$

Linear interpolation of both w, ϕ_x:

$$w \approx \sum_{j=1}^{2} w_j^e \psi_j^e(x), \quad \phi_x \approx \sum_{j=1}^{2} S_j^e \psi_j^e(x)$$

$$w(x) \approx w_1\psi_1(x) + w_2\psi_2(x), \quad \phi_x(x) \approx S_1\psi_1(x) + S_2\psi_2(x)$$



Thus, in the **thin beam limit** it is not possible for the element to realize the requirement

$$\phi_x = -\frac{dw}{dx}$$

SHEAR LOCKING - REMEDY

In the thin beam limit, ϕ should become constant so that it matches dw/dx. However, if ϕ is a constant then the bending energy becomes zero. If we can mimic the two states (constant and linear) in the formulation, we can overcome the problem. Numerical integration of the coefficients allows us to evaluate both ϕ and $d\phi/dx$ as constants. The terms highlighted should be evaluated using "reduced integration".

$$K_{ij}^{11} = \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} + c_f \psi_i^{(1)} \psi_j^{(1)} \right) dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^{(1)}}{dx} \psi_j^{(2)} \, dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left[EI \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + GAK_s \psi_i^{(2)} \psi_j^{(2)} \right] dx$$

STIFFNESS MATRICES OF TIMOSHENKO BEAM ELEMENT

(for constant *EI* and *GA*)

Reduced integration linear element (RIE) Linear approximation of both w and ϕ

$$\frac{2E_eI_e}{\mu_0^e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2\xi_e & 3h_e & h_e^2\zeta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\zeta_e & 3h_e & h_e^2\xi_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

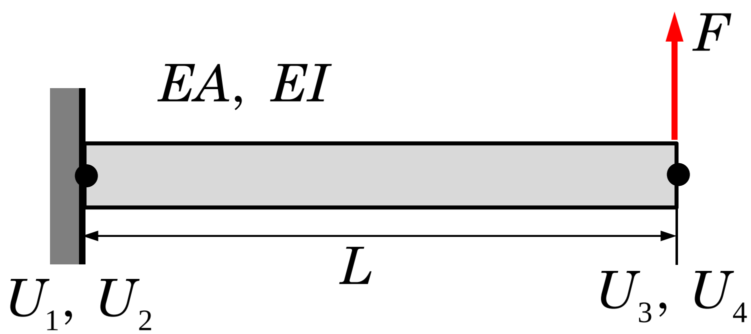
$$\xi_e = 1.5 + 6\Lambda_e,\ \zeta_e = 1.5 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_0^e = 12\Lambda_e$$

Consistent interelement element (CIE) Hermite cubic approximation of w and dependent quadratic approximation of ϕ

$$\frac{2E_eI_e}{\mu_e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2\Sigma_e & 3h_e & h_e^2\Theta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\Theta_e & 3h_e & 2h_e^2\Sigma_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\Sigma_e = 1.0 + 3\Lambda_e,\ \Theta_e = 1.0 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_e = 1 + 12\Lambda_e$$

AN EXAMPLE of TBT



Exact solution (according to the E-B beam theory)

$$w(L) = \frac{FL^3}{3EI}$$

One element discretization using the RIE element

$$\frac{2E_eI_e}{\mu_0^e h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2\xi_e & 3h_e & h_e^2\zeta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\zeta_e & 3h_e & h_e^2\xi_e \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\xi_e = 1.5 + 6\Lambda_e,\ \zeta_e = 1.5 - 6\Lambda_e,\ \Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2},\ \mu_0^e = 12\Lambda_e$$

Boundary conditions:

$$U_1 = U_2 = 0,\quad Q_3 = F,\ Q_4 = 0$$

AN EXAMPLE (TBT) (continued)

$$\frac{2EI}{\mu_0 L^3}\begin{bmatrix} 6 & 3L \\ 3L & \xi L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\mu_0 L^3}{2EI}\frac{FL^2\xi}{\left(6L^2\xi - 9L^2\right)} = \frac{12\Lambda FL^3(1.5+6\Lambda)}{6EI(12\Lambda)}$$

$$\text{When } \Lambda = \frac{EI}{GAK_s L^2} = 0 \Rightarrow U_3 = \frac{1.5FL^3}{6EI} = \frac{FL^3}{4EI} \ (\text{too stiff})$$

$$\text{When } \Lambda \neq 0, \quad \text{then } U_3 = \frac{FL^3(1.5+6\Lambda)}{6EI} = (0.75+3\Lambda)\frac{FL^3}{3EI}$$

$$\Lambda = \frac{EI}{GAK_s L^2} = \frac{2(1+\nu)H^2}{12L^2 K_s} = \frac{(1+\nu)}{6K_s}\left(\frac{H}{L}\right)^2 = \frac{1.3}{5}\left(\frac{H}{L}\right)^2 = 0.26\left(\frac{H}{L}\right)^2$$

AN EXAMPLE of TBT

One element discretization using the CIE element

$$\frac{2EI}{\mu L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2\Sigma & 3L & L^2\Theta \\ -6 & 3L & 6 & 3L \\ -3L & L^2\Theta & 3L & 2L^2\Sigma \end{bmatrix}\begin{Bmatrix} w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\Sigma = 1.0 + 3\Lambda \ , \ \Theta = 1.0 - 6\Lambda \ , \ \Lambda = \frac{E\,I}{G\,A\,K_s L^2}, \ \mu = 1 + 12\Lambda$$

Condensed equations for the unknown displacements

$$\frac{2EI}{\mu L^3}\begin{bmatrix} 6 & 3L \\ 3L & 2\Sigma L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow U_3 = \frac{\mu L^3}{2EI}\frac{2FL^2\Sigma}{\left(12L^2\Sigma - 9L^2\right)} = \frac{\mu F L^3 \Sigma}{EI(12\Sigma - 9)}$$

AN EXAMPLE (TBT) (continued)

When $\Lambda = 0 \Rightarrow \Sigma = 1$ and $\mu = 1$; then $U_3 = \dfrac{\mu F L^3 \Sigma}{EI(12\Sigma - 9)} = \dfrac{FL^3}{3EI}$

When $\Lambda \neq 0$, $\quad U_3 = \dfrac{\mu F L^3 \Sigma}{EI(12\Sigma - 9)} = \dfrac{FL^3}{3EI}\dfrac{(1+3\Lambda)(1+12\Lambda)}{(1+12\Lambda)} = (1+3\Lambda)\dfrac{FL^3}{3EI}$

$$\Lambda = \frac{EI}{GAK_s L^2} = \frac{2(1+\nu)H^2}{12L^2 K_s} = \frac{(1+\nu)}{6K_s}\left(\frac{H}{L}\right)^2 = \frac{1.3}{5}\left(\frac{H}{L}\right)^2 = 0.26\left(\frac{H}{L}\right)^2$$

SUMMARY

In this lecture we have covered the following topics:

- **Derived the governing equations of the Euler-Bernoulli beam theory**
- **Derived the governing equations of the Timoshenko beam theory**
- **Developed Weak forms of EBT and TBT**
- **Developed Finite element models of EBT and TBT**
- **Discussed shear locking in Timoshenko beam finite element**
- **Discussed assembly of beam elements**
- **Discussed examples**